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STRONG HOMOLOGY GROUPS OF CONTINUOUS MAP

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ABSTRACT. In this paper coherent morphisms of chain maps and homology groups of this type of morphisms are defined. Using the obtained methods, the strong homology groups of continuous map of compact metric spaces are constructed. It is proved that for each continuous map $f : X \to Y$ there exists a long exact homological sequence. Besides, it is shown that for each inclusion $i : A \to X$ of compact metric spaces there exists the isomorphism $\overline{\mathbf{H}}_{\mathbf{n}}\left(i\right) \approx \overline{\mathbf{H}}_{\mathbf{n}}\left(X, A\right)$.

რეზიუმე. ნაშრომში განმარტებულია ჯაჭვური ასახვების კოჰერენტული მორფიზმები და ამ ტიპის მორფიზმების ჰომოლოგიის ჯგუფები. ჩვენს მიერ შემუშავებული მეთოდების გამოყენებით აგებულია კომპაქტურ მეატრიკულ სივრცეთა უწყვეტი ასახვების ძლიერი ჰომოლოგიის ჯგუფები. დამტკიცებულია, რომ ყოველი $f : X \to Y$ უწყვეტი ასახვისათვის არსებობს გრძელი ზუსტი ჰომოლოგიური მიმდევრობა. გარდა ამისა, ნაჩვენებია, რომ მეტრიკულ კომპაქტურ სივრცეებს შორის ჩადგმის $i : A \to X$ ასახვებისათვის არსებობს $\overline{\mathbf{H}}_{\mathbf{n}}\left(i\right) \approx \overline{\mathbf{H}}_{\mathbf{n}}\left(X, A\right)$ იზომორფიზმი.

INTRODUCTION

The idea of the expansion of a map into the inverse or direct system consisting of "good" maps has been successfully used by various mathematicians to solve various problems of general topology, geometric topology and algebraic topology [1], [2], [4], [5], [10], [11]. Using this idea in the papers [1], [2], [3] and [8], continuous maps are investigated from the point of view of homology and homotopy theories. Applying (co)shape properties of continuous maps functors from the category of maps of topological spaces to the category of long exact sequences of groups were constructed by V. Baladze [6]. The main aim of our work is to construct and investigate strong homology groups of a continuous map of compact metric spaces. For this aim we use the methods developed in fiber shape theory ([2], [6]) and strong shape theory [11]. In particular, as is known there exists the strong shape functor $S : H\left(\mathrm{Top}_{CM}\right) \to SSh\left(\mathrm{Top}_{CM}\right)$. Thus, to construct the homology functor on the category Top_{CM}, it suffices to construct the

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homology functor on the category $CH(\text{pro}-\text{Top}_{CM})$. In the same way, to define homology groups of a continuous map $f : X \to Y$, it suffices to define homology group of a coherent homotopy class of coherent mapping $[\mathbf{f}]$:$\mathbf{X} \to \mathbf{Y}$. On the other hand, with this end in view, we need to define and study coherent morphism of chain maps and its homology groups.

Throughout the paper, the use is made of the following notation [10–11]:
Top_{CM} is the category of compact metric spaces and continuous maps;
$SSh(\text{Top}_{CM})$ is the strong shape category of compact metric spaces;
$S : H(\text{Top}_{CM}) \to SSh(\text{Top}_{CM})$ is the strong shape functor;
$\mathbf{p}$:$X \to \mathbf{X}$ is the polyhedral expansion of the space X;
$\mathbf{f}$:$\mathbf{X} \to \mathbf{Y}$ is the coherent mapping of the inverse system;
$\mathbf{F}$:$\mathbf{X}\times\mathbf{I} \to \mathbf{Y}$ is the coherent homotopy;
$[\mathbf{f}]$:$\mathbf{X} \to \mathbf{Y}$ is the coherent homotopy class of the coherent mapping;
$C(\cdot)$ is the the coherence operator;
$C_*(f_\#)$ is the chain cone of the chain map $f_\# : L_* \to M_*$.

It should be noted that definition of chain cone of chain map is little differs from the usual case. In particular, in our case, $C_*(f_\#) = \left\{C_n(f_\#), \widetilde{\partial}\right\}$ is the chain complex, where $C_n(f_\#) \approx L_{n-1} \oplus M_n, \ \forall\, n \in N$ and $\widetilde{\partial}(l,m) = (\partial(l), -\partial(m) + f_\#(l)) \ \ \forall\, (l,m) \in \ C_n(f_\#)$.

1. Coherent Morphism of Chain Maps

Let $f_\# : L_* \to M_*$ and $g_\# : P_* \to Q_*$ be arbitrary chain maps. A system $\Phi = \left\{\left(\varphi^1, \varphi^2\right), \varphi^{1,2}\right\}$ is called coherent morphism of the chain maps $f_\#$ and $g_\#$, if $\varphi^1 : L_* \to P_*$ and $\varphi^2 : M_* \to Q_*$ are the chain maps and $\varphi^{1,2} : L_* \to Q_*$ is the chain homotopy of chain maps $\varphi^2 f_\#$ and $g_\#\varphi^1$. In this case, we will write $\Phi : f_\# \to g_\#$.

Lemma 1.1. *Each coherent morphism* Φ: $f_\# \to g_\#$ *induces the chain map*

$$\Phi_\#{:}C_*(f_\#) \to C_*(g_\#),$$

which is defined by the formula

$$\Phi_\#(l,m) = \left(\varphi^1(l), \varphi^2(m) + \varphi^{1,2}(l)\right).$$

Proof. Let $(l,m) \in C_n(f_\#)$, then

$$\begin{aligned}
\widetilde{\partial}\Phi_\#(l,m) &= \widetilde{\partial}\left(\varphi^1(l), \varphi^2(m) + \varphi^{1,2}(l)\right) = \\
&= \left(\partial\left(\varphi^1(l)\right), -\partial\left(\varphi^2(m) + \varphi^{1,2}(l)\right) + g_\#\left(\varphi^1(l)\right)\right) = \\
&= \left(\partial\left(\varphi^1(l)\right), -\partial\varphi^2(m) - \partial\varphi^{1,2}(l) + g_\#\varphi^1(l)\right),
\end{aligned}$$

$$\begin{aligned}
\Phi_\#\widetilde{\partial}(l,m) &= \Phi_\#\left(\partial(l), -\partial(m) + f_\#(l)\right) = \\
&= \left(\varphi^1\partial(l), \varphi^2\left(-\partial(m) + f_\#(l)\right) + \varphi^{1,2}\left(\partial(l)\right)\right) = \\
&= \left(\varphi^1\partial(l), -\varphi^2\partial(m) + \varphi^2 f_\#(l) + \varphi^{1,2}\partial(l)\ \right).
\end{aligned}$$

On the other hand, φ^1 and φ^2 are the chain maps and $\varphi^{1,2}$ is the chain homotopy, and hence

$$\partial\varphi^1(l) = \varphi^1\partial(l), \partial\varphi^2(m) = \varphi^2\partial(m),$$
$$-\partial\varphi^{1,2}(l) + g_\#\varphi^1(l) = \varphi^2 f_\#(l) + \varphi^{1,2}\partial(l).$$

Therefore,

$$\widetilde{\partial}\Phi_\#(l,m) = \Phi_\#\widetilde{\partial}(l,m),$$

which means that $\Phi_\#$ is the chain map. □

Let $\Phi = \left\{\left(\varphi^1, \varphi^2\right), \varphi^{1,2}\right\}$ and $\Psi = \left\{\left(\psi^1, \psi^2\right), \psi^{1,2}\right\}$ be coherent morphisms of chain maps. A system $\mathrm{D} = \left\{\left(D^1, D^2\right), D^{1,2}\right\}$ is called coherent homotopy of coherent morphisms Φ and Ψ, if D^1 is a chain homotopy of φ^1 and ψ^1, D^2 is also a chain homotopy of φ^2 and ψ^2, and $D^{1,2} : L_* \to Q_*$ is a chain map of degree two, which satisfies the condition

$$\partial D^{1,2} - D^{1,2}\partial = g_\# D^1 - D^2 f_\# - \psi^{1,2} + \varphi^{1,2}.$$

In this case, we write $\mathrm{D} : \Phi{\approx}\Psi$.

Lemma 1.2. *Each coherent homotopy* D *of coherent morphisms* Φ *and* Ψ *induces the chain homotopy*

$$D_\# : C_*\left(f_\#\right) \to C_*(g_\#),$$

of chain maps

$$\Phi_\#,\ \Psi_\# : C_*\left(f_\#\right) \to C_*\left(g_\#\right).$$

Proof. Let for each $n \in N$, $D_\# : C_\mathrm{n}\left(f_\#\right) \to C_{n+1}(g_\#)$ be defined by the formula

$$D_\#(l,m) = \left(D^1(l), -D^2(m) + D^{1,2}(l)\right).$$

In this case, for each $(l,m) \in C_n(f_\#)$, we have

$$\begin{aligned}
&\big(\widetilde{\partial}D_\# + D_\#\widetilde{\partial}\big)(l,m) = \widetilde{\partial}D_\#(l,m) + D_\#\widetilde{\partial}(l,m) = \\
&\quad = \widetilde{\partial}\left(D^1(l), -D^2(m) + D^{1,2}(l)\right) + D_\#\left(\partial(l), -\partial(m) + f_\#(l)\right) = \\
&\quad = \left(\partial D^1(l), \partial D^2(m) - \partial D^{1,2}(l) + g_\# D^1(l)\right) + \\
&\qquad + \left(D^1\partial(l), D^2\partial(m) - D^2 f_\#(l) + D^{1,2}\partial(l)\right) = \\
&\quad = \left(\partial D^1(l) + D^1\partial(l), \left(\partial D^2(m) + D^2\partial(m)\right) - \right.\\
&\qquad \left. - \left(\partial D^{1,2}(l) - D^{1,2}\partial(l) - g_\# D^1(l) + D^2 f_\#(l)\right)\right) = \\
&\quad = \left(\psi^1(l) + \varphi^1(l), \left(\psi^2(m) - \varphi^2(m)\right) - \left(\varphi^{1,2}(l) - \psi^{1,2}(l)\right)\right) = \\
&\quad = \left(\psi^1(l) + \varphi^1(l), \left(\psi^2(m) - \varphi^2(m)\right) + \left(\psi^{1,2}(l) - \varphi^{1,2}(l)\right)\right) = \\
&\quad = \left(\psi^1(l), \psi^2(m) + \psi^{1,2}(l)\right) - \left(\varphi^1(l), \varphi^2(m) + \varphi^{1,2}(l)\right) = \\
&\quad = \Psi_\#(m,l) - \Phi_\#(m,l).
\end{aligned}$$

□

2. Homology Group of Coherent Mapping

Let $\mathbf{X} = (X_i, p_{i,i+1}, N)$ be an inverse sequence of topological spaces. Let for each $i \in N$, $S_*(X_i) = (S_n(X_i), \partial\}$ be a singular chain complex of the topological space X_i and $p^{\#}_{i,i+1} : \ S_*(X_{i+1}) \to S_*(X_i)$ be a chain map induced by the projection $p_{i,i+1} : X_{i+1} \to X_i$ of the inverse sequence $\mathbf{X}$. Thus we obtain the inverse sequence $\mathbf{S}_*(\mathbf{X}) = (S_*(X_i), p^{\#}_{i,i+1}, N)$ of chain complexes. Let for each $n \in N$

$$\mathbf{K_n}\left(\mathbf{X}\right) = \prod_{i=1}^{\infty} S_n(X_i)$$

and $\partial_n : \mathbf{K_n}(\mathbf{X}) \to \mathbf{K_{n-1}}(\mathbf{X})$ be given by the formula

$$\partial_n\left(c^n\right) = \partial_n\left\{c_i^n\right\} = \left\{\partial_n(c_i^n)\right\}, \ \forall\ c^n \in \mathbf{K_n}\left(\mathbf{X}\right).$$

It is clear that $\mathbf{K}_*\left(\mathbf{X}\right) = \{\mathbf{K_n}\left(\mathbf{X}\right), \partial\}$ is a chain complex. Consider the map $\mathbf{p}_{\#} : \mathbf{K}_*(\mathbf{X}) \to \mathbf{K}_*(\mathbf{X})$ defined by

$$\mathbf{p}_{\#}\left(c^n\right) = \mathbf{p}_{\#}\left\{c_i^n\right\} = \left\{p^{\#}_{i,i+1}\left(c^n_{i+1}\right) - c_i^n\right\}, \ \forall\ c^n \in \mathbf{K_n}\left(\mathbf{X}\right).$$

Lemma 2.1. *For each inverse sequence $\mathbf{X} = (X_i, p_{i,i+1}, N)$ of topological spaces the map $\mathbf{p}_{\#} : \mathbf{K}_*(\mathbf{X}) \to \mathbf{K}_*(\mathbf{X})$ is the chain map.*

Proof. Let $c^n \in \mathbf{K_n}\left(\mathbf{X}\right)$, then

$$\partial_n \mathbf{p}_{\#}\left(c^n\right) = \partial_n \mathbf{p}_{\#}\left\{c_i^n\right\} = \partial_n\left\{p^{\#}_{i,i+1}\left(c^n_{i+1}\right) - c_i^n\right\} =$$

$$= \left\{\partial_n p^{\#}_{i,i+1}\left(c^n_{i+1}\right) - \partial_n(c_i^n)\right\} = \left\{p^{\#}_{i,i+1}\partial_n\left(c^n_{i+1}\right) - \partial_n(c_i^n)\right\} =$$

$$= \mathbf{p}_{\#}\left\{\partial_n(c_i^n)\right\} = \mathbf{p}_{\#}\partial_n\left(c^n\right). \qquad \square$$

Lemma 2.2. *Each coherent mapping $\mathbf{f} = \left\{f_i^0, f^1_{i,i+1}\right\} : \mathbf{X} \to \mathbf{Y}$ of inverse sequences induces coherent morphism $\mathbf{F} = \left\{\left(\mathbf{f}^{\mathbf{0}}_{\#}, \mathbf{f}^{\mathbf{0}}_{\#}\right), \mathbf{f}^{\mathbf{1}}_{\#}\right\} : \mathbf{p}_{\#} \to \mathbf{q}_{\#}$ of chain maps.*

Proof. By Lemma 3.3 in [10], we can assume that

$$f_i^0 : X_i \to Y_i, \quad \forall\ i \in N$$

$$f^1_{i,i+1} : X_{i+1} \times I \to Y_i, \quad \forall\ i \in N$$

are the maps which satisfy the following properties:

$$f^1_{i,i+1}(x,0) = \left(f_i^0 p_{i,i+1}\right)(x), \quad \forall\ x \in X_{i+1}, \tag{1}$$

$$f^1_{i,i+1}(x,1) = \left(q_{i,i+1} f^0_{i+1}\right)(x), \quad \forall\ x \in X_{i+1}, \tag{2}$$

$$\left(f^1_{i,i+1}\left(p_{i+1,i+2} \times id\right)\right) * \left(q_{i,i+1} f^1_{i+1,i+2}\right) = f^1_{i,i+2} \ \left(rel\left\{0,1\right\}\right). \tag{3}$$

Let $f_i^{\#} : S_*(X_i) \to S_*(Y_i)$ be the chain map induced by the map $f_i^0 : X_i \to Y_i$ and $f^{\#}_{i,i+1} : S_*(X_{i+1}) \to S_*(Y_i)$ be the chain map of degree one induced by $f^1_{i,i+1} : X_{i+1} \times I \to Y_i$. By (1), (2) and (3), it is the clear that $f^{\#}_{i,i+1}$ is chain homotopy of chain maps $\left(f^0_i p_{i,i+1}\right)_{\#}$, $\left(q_{i,i+1} f^0_{i+1}\right)_{\#}$: $S_*(X_{i+1}) \to S_*(Y_i)$. Let

$$\begin{aligned} \mathbf{f^0_{\#}} &: \mathbf{K}_*(\mathbf{X}) \to \mathbf{K}_*(\mathbf{Y}), \\ \mathbf{f^1_{\#}} &: \mathbf{K}_*(\mathbf{X}) \to \mathbf{K}_*(\mathbf{Y}) \end{aligned}$$

be the maps defined by the formulas

$$\begin{aligned} \mathbf{f^0_{\#}}\left(c^n\right) &= \mathbf{f^0_{\#}}\left\{c^n_i\right\} = \left\{f^{\#}_i(\ c^n_i)\right\}, \quad \forall\ c^n \in \mathbf{K}_*(\mathbf{X}), \\ \mathbf{f^1_{\#}}\left(c^n\right) &= \mathbf{f^1_{\#}}\left\{c^n_{i+1}\right\} = \left\{f^{\#}_{i,i+1}(\ c^n_{i+1})\right\}, \quad \forall\ c^n \in \mathbf{K}_*\left(\mathbf{X}\right). \end{aligned}$$

In this case, $\mathbf{f^0_{\#}}$ is the chain map and $\mathbf{f^1_{\#}}$ is the chain homotopy of chain maps $\mathbf{f^0_{\#}p_{\#}}$ and $\mathbf{q_{\#}f^0_{\#}}$ and, therefore, the system $\mathbf{F}=\left\{\left(\mathbf{f^0_{\#}}, \mathbf{f^0_{\#}}\right), \mathbf{f^1_{\#}}\right\}$ is the coherent morphism of chain maps $\mathbf{p}_{\#}$ and $\mathbf{q}_{\#}$. Indeed, let $c^n \in K_n(X)$, then

$$\begin{aligned} \left(\partial \mathbf{f}^1_{\#} + \mathbf{f}^1_{\#}\partial\right)\left(c^n\right) &= \partial \mathbf{f}^1_{\#}\left(c^n\right) + \mathbf{f}^1_{\#}\partial\left(c^n\right) = \partial\left\{f^{\#}_{i,i+1}\left(c^n_{i+1}\right)\right\} + \mathbf{f}^1_{\#}\left\{\partial\left(c^n_{i+1}\right)\right\} = \\ &= \left\{\partial f^{\#}_{i,i+1}\left(c^n_{i+1}\right)\right\} + \left\{f^{\#}_{i,i+1}\partial\left(c^n_{i+1}\right)\right\} = \left\{\partial f^{\#}_{i,i+1}\left(c^n_{i+1}\right) + f^{\#}_{i,i+1}\partial\left(c^n_{i+1}\right)\right\} = \\ &= \left\{\left(\partial f^{\#}_{i,i+1} + f^{\#}_{i,i+1}\partial\right)\left(c^n_{i+1}\right)\right\} = \left\{\left(q^{\#}_{i,i+1}f^{\#}_{i+1} - f^{\#}_i p^{\#}_{i,i+1}\right)\left(c^n_{i+1}\right)\right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left(\mathbf{q}_{\#}\mathbf{f}^0_{\#} - \mathbf{f}^0_{\#}\mathbf{p}_{\#}\right)\left(c^n\right) &= \left(\mathbf{f}^0_{\#}\mathbf{q}_{\#}\right)\left(c^n\right) - \mathbf{p}_{\#}\mathbf{f}^0_{\#}\left(c^n\right) = \\ &= q_{\#}\left\{f^{\#}_{i+1}\left(c^n_{i+1}\right)\right\} - \mathbf{f}^0_{\#}\left\{p_{i,i+1}\left(c^n_{i+1}\right) - c^n_i\right\} = \\ &= \left\{q^{\#}_{i,i+1}f^{\#}_{i+1}\left(c^n_{i+1}\right) - f^{\#}_i\left(c^n_i\right)\right\} - \left\{f^{\#}_i p_{i,i+1}\left(c^n_{i+1}\right) - f^{\#}_i\left(c^n_i\right)\right\} = \\ &= \left\{\left(q^{\#}_{i,i+1}f^{\#}_{i+1} - f^{\#}_i p^{\#}_{i,i+1}\right)\left(c^n_{i+1}\right)\right\}. \end{aligned}$$

□

By Lemma 1.1, the coherent morphism $\mathbf{F} : \mathbf{p}_{\#} \to \mathbf{q}_{\#}$ induced by the coherent mapping $\mathbf{f}:\mathbf{X} \to \mathbf{Y}$ induces the chain map $\mathbf{F}_{\#} : C_*\left(\mathbf{p}_{\#}\right) \to C_*\left(\mathbf{q}_{\#}\right)$. On the other hand, $\mathbf{F}_{\#} : C_*\left(\mathbf{p}_{\#}\right) \to C_*\left(\mathbf{q}_{\#}\right)$ induces the homomorphism $\mathbf{F}_* : H_n\left(C_*\left(\mathbf{p}_{\#}\right)\right) \to H_n\left(C_*\left(\mathbf{q}_{\#}\right)\right)$.

Let $C_*\left(\mathbf{F}_{\#}\right)$ be the chain cone of the chain map $\mathbf{F}_{\#}$ and

$$\sigma : C_*\left(\mathbf{q}_{\#}\right) \to C_*\left(\mathbf{F}_{\#}\right), \quad \partial : C_*\left(\mathbf{F}_{\#}\right) \to C_*\left(\mathbf{p}_{\#}\right)$$

be the maps, which for each $n \in N$, $\sigma : C_n\left(\mathbf{q}_{\#}\right) \to C_n\left(\mathbf{F}_{\#}\right)$ and ∂ : $C_n\left(\mathbf{F}_{\#}\right) \to C_{n-1}\left(\mathbf{p}_{\#}\right)$ are defined by

$$\begin{aligned} \sigma(m) &= (0, m), \quad \forall\ m \in C_n\left(\mathbf{q}_{\#}\right), \\ \partial(l, m) &= l, \quad \forall\ (l, m) \in C_n\left(\mathbf{F}_{\#}\right). \end{aligned}$$

In this case, there exists the short exact sequence

$$0 \to C_*\left(\mathbf{q}_{\#}\right) \xrightarrow{\sigma} C_*\left(\mathbf{F}_{\#}\right) \xrightarrow{\partial} C_*\left(\mathbf{p}_{\#}\right) \to 0. \tag{4}$$

On the other hand, (4) induces the long exact homological sequence

$$\cdots \longrightarrow \mathrm{H}_{\mathrm{n}+1}\left(C_*\left(\mathbf{q}_{\#}\right)\right) \xrightarrow{\sigma_*} \mathrm{H}_{\mathrm{n}+1}\left(C_*\left(\mathbf{F}_{\#}\right)\right) \xrightarrow{\partial_*} \mathrm{H}_{\mathrm{n}}\left(C_*\left(\mathbf{p}_{\#}\right)\right)$$
$$\xrightarrow{E} \mathrm{H}_{\mathrm{n}}\left(C_*\left(\mathbf{q}_{\#}\right)\right) \longrightarrow \cdots . \tag{5}$$

It is known that the homomorphism $E : \mathrm{H}_{\mathrm{n}}\left(C_*\left(\mathbf{p}_{\#}\right)\right) \to \mathrm{H}_{\mathrm{n}}\left(C_*\left(\mathbf{q}_{\#}\right)\right)$ in the sequence (5) coincides with the homomorphism $\mathbf{F}_* : \mathrm{H}_{\mathrm{n}}\left(C_*\left(\mathbf{p}_{\#}\right)\right) \to \mathrm{H}_{\mathrm{n}}\left(C_*\left(\mathbf{q}_{\#}\right)\right)$ [9]. Our aim is to show that the given sequence does not depend on the type of the coherent homotopy of the coherent mapping $\mathbf{f}{:}\mathbf{X} \to \mathbf{Y}$. On the other hand, towards this end we need the following

Lemma 2.3. *Each coherent homotopy $\mathbf{H} : \mathbf{X}{\times}\mathbf{I} \to \mathbf{Y}$ of coherent mappings $\mathbf{f}=\left\{f_i^0, f_{i,i+1}^1\right\}$, $\mathbf{f}'=\left\{f_i'^{\,0}, f_{i,i+1}'^{\,1}\right\} : \mathbf{X} \to \mathbf{Y}$ of inverse sequences induces coherent homotopy $\mathbf{D}=\left\{\left(\mathbf{D^1}, \mathbf{D^1}\right), \mathbf{D^{1,2}}\right\}{:}\mathbf{F} \approx \mathbf{F}'$ of coherent morphisms $\mathbf{F}=\left\{\left(\mathbf{f}_{\#}^{\mathbf{0}}, \mathbf{f}_{\#}^{\mathbf{0}}\right), \mathbf{f}_{\#}^{\mathbf{1}}\right\}$, $\mathbf{F}'=\left\{\left(\mathbf{f}_{\#}^{\mathbf{0}}, \mathbf{f}_{\#}^{\mathbf{0}}\right), \mathbf{f}_{\#}^{\mathbf{1}}\right\} : \mathbf{p}_{\#} \to \mathbf{q}_{\#}$ of chain maps.*

Proof. To prove the lemma, we will need several facts from the singular homology theory. For each topological space X and each integer $n \in N$, consider the homomorphism $S_{\#} : S_n(X) \to S_{n+1}(X \times I)$ defined by the formula

$$S_{\#}(\sigma) = \sum_{i=0}^{n} (-1)^n s_i(\sigma), \quad \forall\, \sigma \in S_n(X),$$

where $s_i(\sigma) : \Delta^{n+1} \to X \times I$ is the singular simplex defined by

$$s_i(\sigma)(e_j) = \begin{cases} (\sigma(e_j), 0), & \text{if } j \le i, \\ (\sigma(e_{j-1}), 1), & \text{if } j > i. \end{cases}$$

Let $i_{0\#},\ i_{1\#} : S_n(X) \to S_{n+1}(X \times I)$ be the chain maps induced by the inclusions $i_0, i_1 : X \to X \times I$. It is known that in this case $S_{\#} : S_*(X) \to S_*(X \times I)$ is the chain homotopy of the chain maps $i_{0\#}$ and $i_{1\#}$. Therefore, for the maps

$$i_{0\#}, i_{1\#} : S_n(X) \to S_n(X \times I),$$

we have

$$\partial S_{\#} = i_{1\#} - i_{0\#} - S_{\#}\partial. \tag{6}$$

This fact implies that if $H : X \times I \to Y$ is homotopy of the continuous maps $f_0, f_1 : X \to Y$, then $D = H_{\#}S_{\#}{:}\mathrm{S}_*(\mathrm{X}) \to S_*(\mathrm{Y})$ is the chain homotopy of the chain maps $f_{0\#},\ f_{1\#} : S_n(X) \to S_n(Y)$, where $H_{\#}{:}\mathrm{S}_*(\mathrm{X} \times \mathrm{I}) \to S_*(\mathrm{Y})$ is the chain map induced by H. Thus,

$$\partial D + D\partial = f_{1\#} - f_{0\#}. \tag{7}$$

Let $j_0, j_1 : X \times I \to X \times I \times I$ be inclusions defined by

$$j_1(x,t) = (x,t,1), \quad j_0(x,t) = (x,t,0), \quad \forall\,(x,t) \in X \times I.$$

In this case, for the maps

$$j_{0\#}, j_{1\#} : S_{n+1}(X \times I) \to S_{n+1}(X \times I \times I), \tag{8}$$

we have

$$\partial S_{\#} = j_{1\#} - j_{0\#} - S_{\#}\partial.$$

In this case, by virtue of (6) and (8), we obtain

$$\begin{aligned}\partial S_{\#}S_{\#} &= (j_{1\#} - j_{0\#} - S_{\#}\partial)\, S_{\#} = j_{1\#}S_{\#} - j_{0\#}S_{\#} - S_{\#}\partial S_{\#} = \\ &= j_{1\#}S_{\#} - j_{0\#}S_{\#} - S_{\#}\,(i_{1\#} - i_{0\#} - S_{\#}\partial) = \\ &= j_{1\#}S_{\#} - j_{0\#}S_{\#} - S_{\#}i_{1\#} + S_{\#}i_{0\#} + S_{\#}S_{\#}\partial.\end{aligned}$$

On the other hand, if $g_0, g_1 : X \times I \to X \times I \times I$ are inclusions defined by

$$g_1(x,t) = (x,1,t), \quad g_0(x,t) = (x,0,t), \quad \forall\,(x,t) \in X \times I,$$

then

$$g_{1\#}S_{\#} = S_{\#}i_{1\#}, \quad g_{0\#}S_{\#} = S_{\#}i_{0\#}.$$

Therefore, for the maps

$$g_{0\#}, g_{1\#}, j_{0\#}, j_{1\#} : S_n(X) \to S_{n+1}(X \times I \times I)$$

we have

$$\partial S_{\#}S_{\#} = j_{1\#}S_{\#} - j_{0\#}S_{\#} - g_{1\#}S_{\#} + g_{0\#}S_{\#} + S_{\#}S_{\#}\partial. \tag{9}$$

On the other hand, (9) implies that each continuous map $F : X \times I \times I \to Y$ induces the map $D^2 = F_{\#}S_{\#}S_{\#}$:S$_*$(X) $\to$ S_*(Y) which satisfies the following condition

$$\partial D^2 - D^2\partial = F_{1\#}S_{\#} - F_{2\#}S_{\#} - F_{3\#}S_{\#} + F_{4\#}S_{\#}, \tag{10}$$

where $F_1 = Fj_1$, $F_2 = Fj_0$, $F_3 = Fg_1$ and $F_4 = Fg_0$.

Let $\mathbf{H}$=$\{H_i, H_{i+1}\} : \mathbf{X}\times\mathbf{I} \to \mathbf{Y}$ be coherent homotopy of coherent mappings $\mathbf{f}$, $\mathbf{f}^{'} : \mathbf{X} \to \mathbf{Y}$. In this case, $H_i : X_i{\times}I \to Y_i$ and $H_{i,i+1} : X_i{\times}I{\times}I \to Y_i$ are the continuous maps for which

$$H_i(x,0) = f_i(x), \quad H_i(x,1) = f_i'(x), \tag{11}$$

$$H_{i,i+1}(x,0,t) = f_{i,i+1}(x,t), \tag{12}$$

$$H_{i,i+1}(x,1,t) = f'_{i,i+1}(x,t), \tag{13}$$

$$H_{i,i+1}(x,s,0) = H_i(p_{i,i+1} \times id)(x,s), \tag{14}$$

$$H_{i,i+1}(x,s,1) = q_{i,i+1}H_i(x,s). \tag{15}$$

Let $H_{i\#} : S_*\,(X_i{\times}I) \to S_*(Y_i)$ be the chain map induced by the continuous map $H_i : X_i{\times}I \to Y_i$. Let the composition $H_{i\#}S_{\#}$ be defined by D_i^1. Then owing to (7) and (11), it will be the chain homotopy of the chain maps $f_{i\#}, f'_{i\#} : S_*(X_i) \to S_*(Y_i)$ induced by the maps $f_i, f_i' : X_i \to Y_i$. Let $\mathbf{D^1} = \{D_i^1\}$, $\mathbf{f^0_\#}$=$\{f_{i\#}\}$ and $\mathbf{f'^0_\#}$=$\{f'_{i\#}\}$, then $\mathbf{D^1}$:$\mathbf{K}_*(\mathbf{X}) \to \mathbf{K}_*(\mathbf{Y})$ will be the chain homotopy of the chain maps $\mathbf{f^0_\#}$, $\mathbf{f'^0_\#}$:$\mathbf{K}_*\,(\mathbf{X}) \to \mathbf{K}_*\,(\mathbf{Y})$. Thus we have

$$\partial\mathbf{D^1}{+}\mathbf{D^1}\partial = \mathbf{f'^0_\#} - \mathbf{f^0_\#}.$$

In the same way, we can construct the chain map of degree two $\mathbf{D^{1,2}}=\left\{D_i^{1,2}\right\}:\mathbf{K}_*(\mathbf{X})\to\mathbf{K}_*(\mathbf{Y})$**,** where $D_i^{1,2}=H_{i,i+1\#}S_\# S_\#$. To prove the theorem it remains to show that $\mathbf{D}=\left\{\left(\mathbf{D^1},\mathbf{D^1}\right),\mathbf{D^{1,2}}\right\}$ is the coherent homotopy of coherent morphisms $\mathbf{F}=\left\{\left(\mathbf{f^0_\#},\mathbf{f^0_\#}\right),\mathbf{f^1_\#}\right\}$, $\mathbf{F}'=\left\{\left(\mathbf{f'^0_\#},\mathbf{f'^0_\#}\right),\mathbf{f'^1_\#}\right\}:\mathbf{p}_\#\to\mathbf{q}_\#$**.** On the other hand, by (10), (12), (13), (14) and (15), we have

$$\begin{aligned}
\partial\mathbf{D^{1,2}}-&\mathbf{D^{1,2}}\partial=\partial\left(\left\{D_i^{1,2}\right\}\right)-\left(\left\{D_i^{1,2}\right\}\right)\partial=\\
&=\partial\left(\{H_{i,i+1\#}S_\# S_\#\}\right)-\left(\{H_{i,i+1\#}S_\# S_\#\}\right)\partial=\\
&=\left\{\partial\left(H_{i,i+1\#}S_\# S_\#\right)-\left(H_{i,i+1\#}S_\# S_\#\right)\partial\right\}=\\
&=\left\{\left(H_{i,i+1}j_1\right)_\# S_\#\right\}-\left\{\left(H_{i,i+1}j_0\right)_\# S_\#\right\}-\\
&\qquad-\left\{\left(H_{i,i+1}g_1\right)_\# S_\#\right\}+\left\{\left(H_{i,i+1}g_0\right)_\# S_\#\right\}=\\
&=\left\{\left(q_{i,i+1}H_i\right)_\# S_\#\right\}-\left\{\left(H_i\left(p_{i,i+1}\times id\right)\right)_\# S_\#\right\}-\\
&\qquad-\left\{\left(f'_{i,i+1}\right)_\# S_\#\right\}+\left\{\left(f_{i,i+1}\right)_\# S_\#\right\}=\\
&=\mathbf{q}_\#\left\{H_{i\#}S_\#\right\}-\mathbf{p}_\#\left\{H_{i\#}S_\#\right\}-\left\{\left(f'_{i,i+1}\right)_\# S_\#\right\}+\left\{\left(f_{i,i+1}\right)_\# S_\#\right\}=\\
&=\mathbf{q}_\#\left\{D_i^1\right\}-\mathbf{p}_\#\left\{D_i^1\right\}-\mathbf{f'^1_\#}+\mathbf{f^1_\#}=\mathbf{q}_\#\mathbf{D^1}-\mathbf{p}_\#\mathbf{D^1}-\mathbf{f'^1_\#}+\mathbf{f^1_\#}. \qquad \square
\end{aligned}$$

Corollary 2.4. *For each coherent homotopic coherent mappings* $\mathbf{f}$, $\mathbf{f}':\mathbf{X}\to\mathbf{Y}$ *of inverse sequences we have*

$$\mathbf{F}_*=\mathbf{F'}_*:\mathrm{H_n}\left(C_*\left(\mathbf{p}_\#\right)\right)\to\mathrm{H_n}\left(C_*\left(\mathbf{q}_\#\right)\right),$$
$$\mathrm{H_n}\left(C_*\left(\mathbf{F}_\#\right)\right)\approx\mathrm{H_n}\left(C_*\left(\mathbf{F'}_\#\right)\right).$$

3. Strong Homology Groups of Continuous Map

Let $X\in\mathrm{Top}_{CM}$ be the compact metric space and $\mathbf{X}=(X_i,p_{i,i+1},N)$ be the corresponding polyhedral expansion (inverse sequence of polyhedra). The following theorem below characterizes the relation of homology group $\mathrm{H_{n+1}}\left(C_*\left(\mathbf{p}_\#\right)\right)$ and the strong homology group $\overline{\mathbf{H}}_\mathbf{n}(X)$ (Steenrod homology, total homology) of X [11], [12], [13]. In particular, we have

Theorem 3.1. *For each compact metric space* $X\in\mathrm{Top}_{CM}$ *the* $(n+1)-$***d****imensional homology group* $H_{n+1}\left(C_*\left(\mathbf{p}_\#\right)\right)$ *of the chain cone* $C_*\left(\mathbf{p}_\#\right)$ *of the chain map* $\mathbf{p}_\#:\mathbf{K}_*(\mathbf{X})\to\mathbf{K}_*(\mathbf{X})$ *is isomorphic to the* $n-$*dimensional strong homology group* $\overline{\mathbf{H}}_\mathbf{n}(X)$ *of* X*.*

Proof. As is known, for the compact metric space X the strong homology group $\overline{\mathbf{H}}_\mathbf{n}(X)$ is the homology group of the chain complex $\overline{\mathrm{T}}^*(X)=\{\overline{T}^*(X),\partial\}$, which is constructed as follows [11]: let $\mathbf{X}=(X_i,p_{i,i+1},N)$ be the given corresponding polyhedral expansion of X. Consider the tower of the chain complexes $\mathbf{S}_*\left(\mathbf{X}\right)=(S_*(X_i),p^\#_{i,i+1},N\}$ and let the symbol $\overline{T}^n(X)$ denote the set of functions c^n defined on singleton i and on a pair $(i,i+1)$,

$i \in N$, where $c^n(i) = c_i^n \in S_n(X_i)$ and $c^n(i, i+1) = c_{i,i+1}^{n+1} \in S_{n+1}(X_i)$. The boundary operator $\overline{d} : \overline{T}^{\,n}(X) \to \overline{T}^{\,n-1}(X)$ is defined by the formula

$$\left(\overline{d}c^n\right)_i = \partial\left(c_i^n\right),$$
$$\left(\overline{d}c^n\right)_{i,i+1} = \partial\left(c_{i,i+1}^{n+1}\right) + (-1)^n\left(p_{i,i+1}^{\#}\left(c_{i+1}^n\right) - c_i^n\right).$$

Let the map $h_n : \overline{T}^{\,n}(X) \to C_{n+1}(\mathbf{p}_{\#})$ be defined by the formula

$$h_n\left(c^n\right) = \left((-1)^n\left\{c_i^n\right\}, -\left\{c_{i,i+1}^{n+1}\right\}\right), \quad \forall \quad c^n \in \overline{T}^{\,n}.$$

It is easy to see that h_n is an isomorphism. So, to complete the proof, it suffices to show that the diagram

$$\begin{array}{ccc} \overline{T}^{\,n}(X) & \xrightarrow{\overline{d}} & \overline{T}^{\,n-1}(X) \\ \downarrow h_n & & \downarrow h_{n-1} \\ C_{n+1}\left(\mathbf{p}_{\#}\right) & \xrightarrow{\widetilde{\partial}} & C_n\left(\mathbf{p}_{\#}\right) \end{array}$$

is commutative.

Let $c^n \in \overline{T}^{\,n}(X)$, then

$$\begin{aligned} h_{n-1}\overline{d}(c^n) &= h_{n-1}\Big(\big\{\overline{d}(c^n)_i\big\}, \big\{\overline{d}(c^n)_{i,i+1}\big\}\Big) = \\ &= \Big((-1)^{n-1}\big\{\overline{d}(c^n)_i\big\}, -\big\{\overline{d}(c^n)i, i+1\big\}\Big) \\ &\Big((-1)^{n-1}\big\{\partial(c_i^n)\big\}, -\big\{\partial\big(c_{i,i+1}^{n+1}\big) + (-1)^n\big(p_{i,i+1}^{\#}\big(c_{i+1}^n\big) - c_i^n\big)\big\}\Big) = \\ &= -\Big((-1)^n\big\{\partial(c_i^n)\big\}, \big\{\partial\big(c_{i,i+1}^{n+1}\big) + (-1)^n\big(p_{i,i+1}^{\#}\big(c_{i+1}^n\big) - c_i^n\big)\big\}\Big) = \\ &= -\Big(\big(-1\big)^n\partial\big\{c_i^n\big\}, \partial\big\{c_{i,i+1}^{n+1}\big\} + \big\{\big(-1\big)^n\mathbf{p}_{\#}\big\{c_i^n\big\}\big\}\Big) = \\ &= -\Big(\partial\big\{\big(-1\big)^n c_i^n\big\}\ , \partial\big\{c_{i,i+1}^{n+1}\big\} + \mathbf{p}_{\#}\big\{\big(-1\big)^n\big\{c_i^n\big\}\big\}\Big) = \\ &= -\widetilde{\partial}\Big(\big\{(-1)^n c_i^n\big\}, -\big\{c_{i,i+1}^{n+1}\big\}\Big) = -\widetilde{\partial}h_n(c^n). \end{aligned}$$

Thus, we find that the $n-$**d**imensional homology group of the chain complex $\overline{\mathrm{T}}^{\,*}(X)$ is isomorphic to the $(n+1)-$**d**imensional homology group of the chain cone $C_*\left(\mathbf{p}_{\#}\right)$. □

Let $f : X \to Y$ be the continuous map of compact metric spaces. Consider the corresponding strong shape map $F = S(f) : X \to Y$. So, $F = \{\mathbf{p}, \mathbf{q}, [\mathbf{f}]\}$, where $\mathbf{p}$:$X \to \mathbf{X}$ and $\mathbf{q}$:$Y \to \mathbf{Y}$ are the strong expansions and $[\mathbf{f}]$:$\mathbf{X} \to \mathbf{Y}$ is a coherent class of the coherent mapping for which

$$[\mathbf{f}]\, C\left(\mathbf{p}\right) = C\left(\mathbf{q}\right) C(f),$$

where $C\left(\cdot\right)$ is the coherence operator [11]. According to Theorem 4.1 and Corollary 3.4, we can denote the $(n+1)-$**d**imensional homology group of complexes $C_*\left(\mathbf{p}_{\#}\right)$, $C_*\left(\mathbf{q}_{\#}\right)$ and $C_*\left(\mathbf{F}_{\#}\right)$ by using the symbols $\overline{\mathbf{H}}_{\mathbf{n}}(X)$,

$\overline{\mathbf{H}}_{\mathbf{n}}(Y)$ and $\overline{\mathbf{H}}_{\mathbf{n}}(f)$, respectively. Consequently, the homomorphism $\mathbf{F}_* :$ $\mathrm{H}_{\mathrm{n+1}}\left(C_*\left(\mathbf{p}_{\#}\right)\right) \to \mathrm{H}_{\mathrm{n+1}}\left(C_*\left(\mathbf{q}_{\#}\right)\right)$ is denoted by $f_* : \overline{\mathbf{H}}_{\mathbf{n}}(X) \to \overline{\mathbf{H}}_{\mathbf{n}}(Y)$ and called an induced homomorphism of the continuous map $f : X \to Y$.

Theorem 3.2. *If $f,\ f' : X \to Y$ are the homotopic continuous maps or induce same strong shape morphism, then we have*

$$f_* = f'_* : \overline{\mathbf{H}}_{\mathbf{n}}(X) \to \overline{\mathbf{H}}_{\mathbf{n}}(Y), \quad \overline{\mathbf{H}}_{\mathbf{n}}(f) \approx \overline{\mathbf{H}}_{\mathbf{n}}(f').$$

Proof. It is the consequence of: definitions of strong homology groups of the map, definition of the induced homomorphism and Corollary 2.4. □

Theorem 3.3. *For each continuous map $f : X \to Y$ of compact metric spaces there exists the long exact homological sequence*

$$\cdots \to \overline{\mathbf{H}}_{\mathbf{n+1}}(Y) \xrightarrow{\sigma_*} \overline{\mathbf{H}}_{\mathbf{n+1}}(f) \xrightarrow{\partial_*} \overline{\mathbf{H}}_{\mathbf{n}}(X) \xrightarrow{f_*} \overline{\mathbf{H}}_{\mathbf{n}}(Y) \to \cdots .$$

Proof follows from definitions and long exact sequence (5).

Theorem 3.4. *For each inclusion $i : A \to X$ of compact metric spaces, there exists the isomorphism*

$$\varphi_* : \overline{\mathbf{H}}_{\mathbf{n}}(i) \to \overline{\mathbf{H}}_{\mathbf{n}}(X, A),$$

for which the diagram

$$\begin{array}{ccccccccc} \cdots \to & \overline{\mathbf{H}}_{\mathbf{n+1}}(Y) & \xrightarrow{\sigma_*} & \overline{\mathbf{H}}_{\mathbf{n+1}}(i) & \xrightarrow{\partial_*} & \overline{\mathbf{H}}_{\mathbf{n}}(X) & \xrightarrow{i_*} & \overline{\mathbf{H}}_{\mathbf{n}}(Y) & \to \cdots \\ & \downarrow id & & \downarrow \varphi_* & & \downarrow id & & \downarrow id & \\ \cdots \to & \overline{\mathbf{H}}_{\mathbf{n+1}}(Y) & \xrightarrow{j_*} & \overline{\mathbf{H}}_{\mathbf{n+1}}(X, A) & \xrightarrow{\partial} & \overline{\mathbf{H}}_{\mathbf{n}}(X) & \xrightarrow{i_*} & \overline{\mathbf{H}}_{\mathbf{n}}(Y) & \to \cdots \end{array} \tag{16}$$

is commutative.

Proof. It is known that if the chain map $f_{\#} : L_* \to M_*$ is monomorphic, then the chain map $\varphi : C_*(f_{\#}) \to M_*/L_*$, defined by the formula

$$\varphi(m, l) = m + f_{\#}(L_n), \quad \forall\ (m, l) \in C_n(f_{\#}),$$

induces the isomorphism $\varphi_* : H_n\left(C_*\left(f_{\#}\right)\right) \to H_n\left(M_*/L_*\right)$, and the diagram

$$\begin{array}{ccccccccc} \cdots \to & H_n(M_*) & \xrightarrow{\sigma_*} & H_n\left(C_*(f_{\#})\right) & \xrightarrow{\partial_*} & H_{n-1}(L_*) & \xrightarrow{i_*} & H_{n-1}(M_*) & \to \cdots \\ & \downarrow id & & \downarrow \varphi_* & & \downarrow id & & \downarrow id & \\ \cdots \to & H_n(L_*) & \xrightarrow{j_*} & H_n\left(M_*/L_*\right) & \xrightarrow{\partial} & H_{n-1}(L_*) & \xrightarrow{i_*} & H_{n-1}(M_*) & \to \cdots \end{array}$$

is commutative [9]. On the other hand, by Theorem 3.1, the strong homology sequence of the pair (X, A) of compact metric spaces can be obtained from the following short exact sequence:

$$0 \to C_*\left(\mathbf{p}_{\mathbf{A}\#}\right) \xrightarrow{i} C_*\left(\mathbf{p}_{\#}\right) \xrightarrow{j} C_*\left(\mathbf{p}_{\#}\right)/C_*\left(\mathbf{p}_{\mathbf{A}\#}\right) \to 0,$$

where $\mathbf{p}_{\mathbf{A}}$:A $\to \mathbf{A}$ is the restriction of the polyhedral expansion $\mathbf{p}$:X $\to \mathbf{X}$ and is itself the polyhedral expansion. Therefore, there exists the chain map

$\varphi : C_*(F_\#) \to C_*(\mathbf{p}_\#)/C_*(\mathbf{p}_{\mathbf{A}\#})$, where $F : A \to X$ is the strong shape morphism induced by the inclusion $A \to X$, which induces the isomorphism $\varphi_*{:}\overline{\mathbf{H}}_\mathbf{n}(f) \to \overline{\mathbf{H}}_\mathbf{n}(X, A)$ and diagram (16) is commutative. □

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