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THE EXISTENCE AND CONTROLLABILITY RESULTS FOR FRACTIONAL ORDER INTEGRO-DIFFERENTIAL INCLUSIONS IN FRÉCHET SPACES

M. BENCHOHRA, K. EZZINBI AND S. LITIMEIN

Abstract. In this paper, we prove the existence and controllability results for fractional integro-differential inclusions with state-dependent delay in Fréchet spaces. The results are obtained by using a recent nonlinear alternative for contractive multivalued maps in Fréchet spaces due to Frigon.

რეზიუმე. ნაშრომში ფრეშეს სივრცეებში, წილადური ინტეგრო-დიფერენციალური ჩართვისთვის დაგვიანებით, გამოკვლეულია არსებობისა და მართვადობის საკითხები. შედეგები მიღებულია ფრეშეს სივრცეში კუმშვადი მრავალსახა ასახვისათვის ფრიგონის უახლესი არაწრფივი ალტერნატივის გამოყენებით.

1. Introduction

Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, and so forth, and are widely studied by many authors (see [5, 42, 43, 45, 50, 52] and the references therein). For some recent developments on differential inclusions, we refer the reader to the references [2, 3, 46]. Moreover, the concept of controllability plays an important role both in many branches of physics and in technical sciences. In recent years, the problem of controllability for various kinds of functional differential and integro-differential systems, including delay systems in Banach spaces has been extensively studied by many researchers (see [14, 23], and the references therein).

The main object of this paper is to provide sufficient conditions for the existence of mild solutions for fractional integro-differential inclusions with

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a state-dependent delay in Fréchet spaces of the form

$$y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)ds \in F(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J = [0,\infty), \quad (1)$$

$$y_0 = \phi \in \mathcal{B}, \quad (2)$$

where $1 < \alpha < 2$ and $A : D(A) \subset E \to E$ is the generator of an integral resolvent family defined on a complex Banach space $(E, |\cdot|)$, the convolution integral in the equation is known as the Riemann-Liouville fractional integral and $f : [0,\infty) \times E \to \mathcal{P}(E)$ is a multivalued map. For any continuous function y defined on $\mathbb{R}$ and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state from each time $\theta \in (-\infty, 0]$ up to the present time t. We assume that the histories y_t belong to some abstract phase space $\mathcal{B}$, to be specified later.

For the past several years it has become apparent that equations with a state-dependent delay arise also in several areas such as classical electrodynamics [26, 27], population models [7, 10, 15, 21, 31, 51], models of commodity price fluctuations [16, 40], and in models of blood cell production [41]. The existence results among other things were derived recently for functional differential equations when the solution depends on the delay on a bounded interval for impulsive problems. For details, we refer the reader to the papers by Abada *et al.* [1], Ait Dads and Ezzinbi [6], Anguraj *et al.* [8, 11, 33, 24], Hernandez *et al.* [34] and Li *et al.* [38].

In the case where F is either a single or a multivalued map, the problem (1)–(2) has been investigated on compact intervals in the papers of Agarwal *et al.* [4], Benchohra *et al.* [18, 19]. On infinite intervals when F is a single map, the problem (1)–(2) was studied by Benchohra and Litimein by means of Schauder's fixed point theorem combined with the diagonalization process [17]. For the controllability of differential inclusions in Fréchet spaces see, for instance, the papers of Benchohra and Ouahab [20], Henderson and Ouahab [35] and the references cited therein.

The main aim of this paper is to establish the global uniqueness of solutions for problem (1)–(2) by applying the nonlinear alternative of Leray-Schauder type due to Frigon [30] for contractive multivalued maps in Fréchet spaces. The rest of this paper is organized as follows: in Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. The existence theorems for the problem (1)–(2) and their proofs are arranged in Section 3. Finally, in Section 4 an application of controllability result is given to illustrate the theory. The present results complement and extend to the Fréchet space setting those considered on the Banach spaces.

2. Preliminaries

In this section, we introduce the notations, definitions and preliminary facts from multivalued analysis which will be used throughout this paper.

Let $C([0,n];E)$, $n \in \mathbb{N}$ be the Banach space of all continuous functions from $J_n = [0,n]$ into E with the usual norm

$$\|y\|_n = \sup\{|y(t)| : 0 \le t \le n\}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : [0,\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [53]).

Let $L^1([0,\infty),E)$ denote the Banach space of continuous functions $y : [0,\infty) \to E$ which are Bochner integrable and have norm

$$\|y\|_{L^1} = \int_0^\infty |y(t)|dt \quad \text{for all} \quad y \in L^1([0,\infty),E).$$

Consider the space

$$B_{+\infty} = \{y : (-\infty,+\infty) \to E : y|_J \in C(J,E), y_0 \in \mathcal{B}\},$$

where $y|_J$ is the restriction of y to $J = [0,+\infty)$.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [32] and follow the terminology used in [36]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into E, and satisfying the following axioms:

(A_1) If $y : (-\infty,b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:
(i) $y_t \in \mathcal{B}$;
(ii) there exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;
(iii) there exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \le K(t)\sup\{\ |y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$.

Remark 2.1. 1. (ii) is equivalent to $|\phi(0)| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ satisfy the condition that $\|\phi - \psi\|_{\mathcal{B}} = 0$ not necessarily with $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.

3. From the equivalence of the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$, we necessarily have $\phi(0) = \psi(0)$.

We now present some examples of phase spaces. For other details, we refer, for instance, to the book by Hino *et al* [36].

Example 2.2. Let:

BC be the space of bounded continuous functions defined from $(-\infty, 0]$ to E;

BUC be the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;

$C^\infty := \left\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) \text{ exist in } E\right\}$;

$C^0 := \left\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) = 0\right\}$, endowed with the uniform norm

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \le 0\}.$$

We have that the spaces BUC, C^∞ and C^0 satisfy the conditions (A_1)–(A_3). However, BC satisfies $(A_1), (A_3)$ but (A_2) is not satisfied.

Example 2.3. The spaces C_g, UC_g, C_g^∞ and C_g^0.

Let g be a positive continuous function on $(-\infty, 0]$. We define:

$C_g := \left\{\phi \in C((-\infty, 0], E) : \dfrac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0]\right\}$;

$C_g^0 := \left\{\phi \in C_g : \lim_{\theta\to-\infty} \dfrac{\phi(\theta)}{g(\theta)} = 0\right\}$, endowed with the uniform norm

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \ \theta \le 0\right\}.$$

Then we have that the spaces C_g and C_g^0 satisfy the conditions (A_3). We consider the following condition on the function g:

(g_1) For all $a > 0$, $\sup_{0\le t\le a} \sup\left\{\dfrac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty$.

They satisfy the conditions (A_1) and (A_2) if (g_1) holds.

Example 2.4. The space C_γ.

For any real positive constant γ, we define the functional space C_γ by

$$C_\gamma := \left\{\phi \in C((-\infty, 0], E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\right\},$$

endowed with the norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \le 0\}.$$

Then the axioms (A_1)–(A_3) in the space C_γ are satisfied.

The Laplace transformation of a function $f \in L^1_{loc}(\mathbb{R}_+, E)$ is defined by

$$\mathcal{L}(f)(\lambda) :=: \widehat{a}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega,$$

if the integral is absolutely convergent for $\mathrm{Re}(\lambda) > \omega$. In order to define the mild solution of the problem (1)–(2), we recall the following definition:

Definition 2.5. Let A be a closed and linear operator with the domain $D(A)$ defined on a Banach space E. We call A the generator of an integral resolvent if there exists $\omega > 0$ and a strongly continuous function $S : \mathbb{R}^+ \to B(E)$ such that

$$\left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S(t)x dt, \quad \mathrm{Re}\,\lambda > \omega, \quad x \in E.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

The following result is a direct consequence of [44] (Proposition 3.1 and Lemma 2.2).

Proposition 2.6. *Let $\{S(t)\}_{t\geq 0} \subset B(E)$ be an integral resolvent family with generator A. Then the following conditions are satisfied:*

a) *$S(t)$ is strongly continuous for $t \geq 0$ and $S(0) = I$;*
b) *$S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \geq 0$;*
c) *for every $x \in D(A)$ and $t \geq 0$,*

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)x ds.$$

d) *Let $x \in D(A)$. Then $\int_0^t a(t-s)S(s)x ds \in D(A)$ and*

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)x ds.$$

In particular, $S(0) = a(0)$.

Remark 2.7. The uniqueness of the resolvent is well known (see Prüss [48]).

If an operator A with the domain $D(A)$ is the infinitesimal generator of an integral resolvent family $S(t)$ and $a(t)$ is a continuous, positive and nondecreasing function satisfying $\lim_{t\to 0^+} \dfrac{\|S(t)\|_{B(E)}}{a(t)} < \infty$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - a(t)x}{(a * a)(t)},$$

see ([39], Theorem 2.1). For example, the case $a(t) \equiv 1$ corresponds to the generator of a C_0-semigroup (see [13]) and $a(t) = t$ actually corresponds to the generator of a sine family (see [9]). A characterization of generators of the integral resolvent families, analogous to the Hille-Yosida Theorem for C_0-semigroups, can be directly deduced from ([39], Theorem 3.4). More information on the C_0-semigroups and sine families can be found in [13, 28, 29, 47].

Definition 2.8. A resolvent family of bounded linear operators, $\{S(t)\}_{t>0}$, is called uniformly continuous if

$$\lim_{t\to s} \|S(t) - S(s)\|_{B(E)} = 0.$$

Throughout this paper, we will use the following notation:
$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ bounded}\}$,
$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ compact}\}$.
Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\big\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$ and $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [42]).

Definition 2.9. A multivalued map $G : J \to \mathcal{P}_{cl}(E)$ is said to be measurable if for each $y \in E$ the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

belongs to $L^1(J, \mathbb{R})$.

Definition 2.10. A multivalued map $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is an L^1_{loc} - Carathéodory if

(i) $t \mapsto F(t, y)$ is measurable for each $y \in \mathcal{B}$, and
(ii) $y \mapsto F(t, y)$ is continuous for almost all $t \in J$.
(iii) for every positive constant k there exists $h_k \in L^1_{loc}(J, \mathbb{R}^+)$ such that

$$\|F(t, y)\| \le h_k(t) \ \text{ for all } \|y\|_{\mathcal{B}} \le k \ \text{ and for almost all } t \in J.$$

For each $y \in B_{+\infty}$, define the set of *selections* for F by

$$S_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y_{\rho(t,y_t)}) \ \text{ for a.e. } t \in J\}.$$

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$, we say that F is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that

$$\|y\|_n \le M_n \qquad \text{for all } y \in Y.$$

Proposition 2.11. ([22], *Proposition* III.4) *If Γ_1 and Γ_2 are compact valued measurable multifunctions, then the multifunction $t \to \Gamma_1(t) \cap \Gamma_2(t)$ is measurable. If (Γ_n) is a sequence of compact valued measurable multifunctions, then $t \to \cap\Gamma_n(t)$ is measurable, and if $\overline{\cup\Gamma_n(t)}$ is compact, then $\overline{t \to \cup\Gamma_n(t)}$ is measurable.*

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: for every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of subset X^n and define $Y^n = \{[x]_n : x \in Y\}$. We denote by $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|x\|_1 \le \|x\|_2 \le \ \|x\|_3 \le \cdots \qquad \text{for every } x \in X.$$

Definition 2.12 ([30])**.** A multivalued map $F : X \to P_{cl}(E)$ is called an admissible contraction with the constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $k_n \in [0,1)$ such that

(i) $H_d(F(x), F(y)) \le k_n\|x-y\|_n$ for all $x, y \in X$;
(ii) For every $x \in X$ and every $\epsilon \in (0,\infty)^n$, there exists $y \in F(x)$ such that

$$\|x-y\|_n \le \|x - F(x)\|_n + \epsilon_n \ \text{ for every } n \in \mathbb{N}.$$

Theorem 2.13 (Nonlinear Alternative [30])**.** *Let X be a Fréchet space and U be an open neighborhood of the origin in X and let $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements holds:*

(C1) *N has a fixed point;*
(C2) *there exist $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.*

For more details on multivalued maps see the books of Aubin and Cellina [12], Deimling [25] and Hu and Papageorgiou [37].

3. Main Results

In this section, we are concerned with the existence of solutions for the problem (1)–(2).

Definition 3.1. We say that the function $y : (-\infty, +\infty) \to E$ is a mild solution of (1)–(2) if $y(t) = \phi(t)$ for all $t \leq 0$, the restriction of $y(.)$ to the interval $[0, \infty)$ is continuous and there exists $v(.) \in L^1(J, E)$, such that $v(t) \in f(t, y_{\rho(t,y_t)})$ a.e. $t \in [0, \infty)$, and y satisfies the following integral equation:

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)\ v(s)\,ds \quad \text{for each } t \in [0, +\infty). \tag{3}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2. Related to the condition (H_ϕ), we point out here that this condition is frequently verified by the functions continuous and bounded. In fact, if the space $\mathcal{B}$ verifies axiom C_2 in the nomenclature of [36], then there exists a constant $L > 0$ such that $\|\phi\|_{\mathcal{B}} \leq L \sup_{\theta \leq 0} \|\phi(\theta)\|$ for every $\phi \in \mathcal{B}$ continuous and bounded, (see [36], Proposition 7.1.1.) for details. Consequently, $\|\phi_t\|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0} \|\phi(\theta)\|}{\|\phi\|_{\mathcal{B}}} \|\phi\|_{\mathcal{B}}$ for every continuous and bounded function $\phi \in \mathcal{B}\ \{0\}$ and every $t \leq 0$.

Lemma 3.3. ([34], *Lemma* 2.4) *If* $y : (-\infty, +\infty) \to E$ *is a function such that* $y_0 = \phi$, *then*

$$\|y_s\|_{\mathcal{B}} \leq (M_n + L^\phi)\|\phi\|_{\mathcal{B}} + K_n \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Proof. The assertion follows from the inequalities

$$\|y_s\|_{\mathcal{B}} = \|\phi_s\|_{\mathcal{B}} \leq L^\phi(s)\|\phi\|_{\mathcal{B}},\quad s \in \mathcal{R}(\rho^-),$$

and

$$\|y_s\|_{\mathcal{B}} \leq M(s)\|\phi\|_{\mathcal{B}} + K(s)\|y\|_s,\quad s \in J.$$ □

To establish our main result concerning the existence of the problem (1)–(2), we list the following hypotheses:

(H1) The operator solution $S(t)_{t \in J}$ is compact for $t > 0$.

(H2) The multifunction $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is L^1_{loc}-Carathéodory with compact and convex values, and there exist a function $p \in L^1_{loc}(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : J \to (0, \infty)$ such that

$$\|F(t,u)\|_{\mathcal{P}(E)} \leq p(t)\psi(\|u\|_{\mathcal{B}}) \quad \text{for a.e } \ t \in J \ \text{ and each } u \in \mathcal{B}.$$

(H3) For all $R > 0$, there exists $l_R \in L^1_{loc}(J, \mathbb{R}_+)$ such that

$$H_d(F(t,u), F(t,v)) \leq l_R(t)\|u - v\|_{\mathcal{B}}$$

for each $t \in J$ and for all $u, v \in \mathcal{B}$ with $\|u\|_{\mathcal{B}} \leq R$ and $\|v\|_{\mathcal{B}} \leq R$ and

$$d(0, F(t,0)) \leq l_R(t) \ \text{ a.e. } t \in J.$$

For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the family of semi-norms by

$$\|y\|_n := \sup \{ \ e^{-\tau \ L^*_n(t)} \ |y(t)| : t \in [0,n] \ \},$$

where $L^*_n(t) = \int\limits_0^t \bar{l}_n(s)\, ds$, $\bar{l}_n(t) = K_n M l_n(t)$ and l_n is the function from $(H3)$.

Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$. In what follows, we will choose $\tau > 1$.

Theorem 3.4. *Suppose that hypotheses* (H1)–(H3) *are satisfied and, moreover,*

$$\int\limits_0^{+\infty} \frac{ds}{\psi(s)} > K_n M \int\limits_0^n p(s)\, ds \quad \text{for} \ \ n \in \mathbb{N}. \tag{4}$$

Then the problem (1)–(2) *has a mild solution on* $(-\infty, +\infty)$.

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the multivalued operator $N : B_{+\infty} \to \mathcal{P}(B_{+\infty})$ defined by: $N(h) = \{h \in B_{+\infty}\}$ with

$$h(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ S(t)\ \phi(0) + \int\limits_0^t S(t-s)\ v(s)\, ds, \quad v \in S_{F,y_{\rho(s,y_s)}}, & \text{if } t \in J. \end{cases} \tag{5}$$

For $\phi \in \mathcal{B}$, we define the function $x : (-\infty, \infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \leq 0; \\ S(t)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in J; \\ z(t), & \text{if } t \in J. \end{cases}$$

If y satisfies (3), we can decompose it as $y(t) = \overline{z}(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$, for every $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t S(t-s)\ v(s)\, ds, \quad t \in J,$$

where $v(s) \in S_{F, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}}$.

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0 \in \mathcal{B}\}.$$

For any $z \in B^0_{+\infty}$ we have

$$\|z\|_{+\infty} = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \le s < +\infty\} = \sup\{|z(s)| : 0 \le s < +\infty\}.$$

Thus $(B^0_{+\infty}, \|\cdot\|_{+\infty})$ is a Banach space. We define the operator $P : B^0_{+\infty} \to \mathcal{P}(B^0_{+\infty})$ by: $P(z) := \{h \in B^0_{+\infty}\}$ with

$$h(t) = \int_0^t S(t-s)\ v(s) ds, \quad v(s) \in S_{F, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}}, \quad t \in J. \quad (6)$$

Obviously, the operator N has a fixed point, equivalent to P, so it remains to prove that P has a fixed point. Let $z \in B^0_{+\infty}$ be a possible fixed point of the operator P. Given $n \in \mathbb{N}$, then z should be a solution of the inclusion $z \in \lambda P(z)$ for some $\lambda \in (0,1)$ and there exists $v \in S_{F,z}$ such that for each $t \in [0,n]$, we have

$$\begin{aligned} |z(t)| &\le \int_0^t \|S(t-s)\|_{B(E)}\ |v(s)|\, ds \le \\ &\le M \int_0^t p(s)\ \psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right)\ ds \le \\ &\le M \int_0^t p(s)\ \psi\left(K_n|z(s)| + (M_n + L^\phi + K_n M H)\|\phi\|_{\mathcal{B}}\right)\ ds. \end{aligned}$$

Set

$$c_n := (M_n + L^\phi + K_n M H)\|\phi\|_{\mathcal{B}}.$$

Then we have

$$|z(t)| \leq M \int_0^t p(s)\ \psi\left(K_n|z(s)| + c_n\right)\, ds.$$

Thus

$$K_n|z(t)| + c_n \leq c_n + K_n M \int_0^t p(s)\ \psi\left(K_n|z(s)| + c_n\right)\, ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ K_n|z(s)| + c_n\ :\ 0 \leq s \leq t\ \}, \quad t \in J.$$

Let $t^\star \in [0,t]$ be such that

$$\mu(t) = K_n|z(t^\star)| + c_n\|\phi\|_{\mathcal{B}}.$$

By the previous inequality, we have

$$\mu(t) \leq c_n + K_n\ M \int_0^t p(s)\ \psi(\mu(s))\, ds \quad \text{for} \quad t \in [0,n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \ \text{ for all } t \in [0,n].$$

From the definition of v, we have

$$v(0) = c_n \quad \text{and} \quad v'(t) = K_n M p(t)\ \psi(\mu(t)) \quad \text{a.e. } \ t \in [0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq K_n M\ p(t)\ \psi(v(t)) \ \text{ a.e. } t \in [0,n].$$

Using the condition (4), this implies that for each $t \in [0,n]$ we have

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \leq K_n M \int_0^t p(s)\, ds \leq K_n M \int_0^n p(s)\, ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda$.

Set

$$U = \left\{\ z \in B^0_{+\infty}\ :\ \sup_{0 \leq t \leq n} |z(t)| < \Lambda_n + 1 \ \text{ for all } \ n \in \mathbb{N}\right\}.$$

Clearly, U is an open subset of $B^0_{+\infty}$. We shall show that $P : \overline{U} \to \mathcal{P}(B^0_{+\infty})$ is a contraction and an admissible operator. First, we prove that P is a contraction. Let $z, \overline{z} \in B^0_{+\infty}$ and $h \in P(z)$. Then there exists $v(t) \in F(t, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$ such that

$$h(t) = \int_0^t S(t-s)v(s)ds$$

for each $t \in [0, n]$. It follows from $(H3)$ that

$$\begin{aligned} H_d(F(t, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})) \leq \\ \leq l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}. \end{aligned}$$

Hence, there is $w \in F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$ such that

$$|v(t) - w| \leq l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}, \quad t \in [0, n].$$

Consider $U_* : [0, n] \to \mathcal{P}(E)$ given by

$$U_*(t) = \{w \in E : |v(t) - w| \leq l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\}.$$

Since the multivalued operator $V_*(t) = U_*(t) \cap F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$ is measurable (see Proposition 2.11), there exists a function $\overline{v}(t)$, which is a measurable selection for V_*. So, $\overline{v}(t) \in F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$, and using $(A1)$, for each $t \in [0, n]$, we obtain

$$\begin{aligned} |v(t) - \overline{v}(t)| &\leq l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \leq \\ &\leq l_n(t)[K(t)|z(t) - \overline{z}(t)| + M(t)\|z_0 - \overline{z}_0\|_{\mathcal{B}}] \leq \\ &\leq l_n(t)K_n|z(t) - \overline{z}(t)|. \end{aligned}$$

For each $t \in [0, n]$, let us define

$$\overline{h}(t) = \int_0^t S(t-s)\overline{v}(s)ds.$$

Then we have

$$\begin{aligned}|h(t)-\overline{h}(t)| &\leq \int_0^t \|S(t-s)\||v(s)-\overline{v}(s)|ds \leq \\ &\leq \int_0^t MK_n l_n(s)|z(s)-\overline{z}(s)|ds \leq \\ &\leq \int_0^t [\bar{l}_n(s)e^{\tau\ L_n^*(s)}][e^{-\tau\ L_n^*(s)}|z(s)-\overline{z}(s)|]ds \leq \\ &\leq \int_0^t \left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]' ds\ \|z-\overline{z}\|_n \leq \\ &\leq \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n.\end{aligned}$$

Therefore,

$$\|h-\overline{h}\|_n \leq \frac{1}{\tau}\ \|z-\overline{z}\|_n.$$

By an analogous relation obtained by interchanging the roles of z and $\overline{z}$, it follows that

$$H_d(P(z),P(\overline{z})) \leq \frac{1}{\tau}\ \|z-\overline{z}\|_n.$$

So, P is a contraction for all $n \in \mathbb{N}$. Now we shall show that P is an admissible operator. Let $z \in B^0_{+\infty}$. Set, for every $n \in \mathbb{N}$, the space

$$B_n^0 = \left\{y : (-\infty, n] \to E : y|_{[0,n]} \in C([0,n],E), y_0 \in \mathcal{B}\right\},$$

and let us consider the multivalued operator

$$P(z) = \left\{h \in B_n^0 : h(t) = \int_0^t S(t-s)v(s)ds,\ t \in [0,n]\right\},$$

where $v \in S^n_{F,y} = \{v \in L^1([0,n],E) : v(t) \in F(t, y_{\rho(t,y_t)})\ \text{ for a.e. } t \in [0,n]\}$.

From (H1)–(H3) and since P is a multivalued map with compact values, we can prove that for every $z \in B_n^0$, $P(z) \in \mathcal{P}_{cp}(B_n^0)$ and there exist $z_* \in B_n^0$ such that $z_* \in P(z_*)$. Let $h \in B_n^0$, $\overline{z} \in \overline{U}$ and $\epsilon > 0$. Assume that $z_* \in P(\overline{z})$, then we have

$$\begin{aligned}|\overline{z}(t)-z_*(t)| &\leq |\overline{z}(t)-h(t)| + |z_*(t)-h(t)| \leq \\ &\leq e^{\tau\ L_n^*(t)}\|\overline{z}-P(\overline{z})\|_n + \|z_*-h\|.\end{aligned}$$

Since h is arbitrary, we may suppose that $h \in B(z_*, \epsilon) = \{h \in B_n^0 : \|h - z_*\|_n \leq \epsilon\}$. Therefore,

$$\|\overline{z} - z_*\|_n \leq \|\overline{z} - P(\overline{z})\|_n + \epsilon.$$

If z is not in $P(\overline{z})$, then $\|z_* - P(\overline{z})\| \neq 0$. Since $P(\overline{z})$ is compact, there exists $x \in P(\overline{z})$ such that $\|z_* - P(\overline{z})\| = \|z_* - x\|$. Then we have

$$\begin{aligned} |\overline{z}(t) - x(t)| &\leq |\overline{z}(t) - h(t)| + |x(t) - h(t)| \leq \\ &\leq e^{\tau\ L_n^*(t)}\|\overline{z} - P(\overline{z})\|_n + \|x(t) - h(t)\|. \end{aligned}$$

Therefore,

$$\|\overline{z} - x\|_n \leq \|\overline{z} - P(\overline{z})\|_n + \epsilon.$$

Thus, P is an admissible operator contraction. From the choice of U, there is no $z \in \partial U$ such that $z = \lambda P(z)$ for some $\lambda \in (0, 1)$. Then the statement $(C2)$ in Theorem 2.13 does not hold. As a consequence of the nonlinear alternative, we deduce that the operator P has a fixed point z^*. Then $y^* = z^* + x$, is a fixed point of the operator N, which is a mild solution of the problem (1)–(2). □

4. Controllability Results

As an application of Theorem 3.4, we consider the following controllability for fractional integro-differential inclusions with a state-dependent delay in a complex Banach space $(E, |.|)$:

$$y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)ds \in F(t, y_{\rho(t,y_t)}) + (Bu)(t), \tag{7}$$

$$\text{a.e. } \ t \in J = [0, \infty),$$

$$y_0 = \phi \in \mathcal{B}, \tag{8}$$

where α, F, A, ϕ are as in (1)–(2), B is a bounded linear operator from U into E and the control parameter $u(.)$ belongs to $L^2(J, U)$, a space of admissible controls, and U is a Banach space.

Similar to the Definition 3.1, the mild solution of (7)–(8) is given by

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)\ v(s)\,ds + \\ + \int_0^t (Bu)(s)\,ds \quad \text{for each } \ t \in [0, +\infty). \tag{9}$$

Definition 4.1. The system (7)–(8) is said to be controllable if for any continuous function $\phi \in \mathcal{B}$ and any $x_1 \in E$ and for each $n \in \mathbb{N}$ there exists a control $u \in L^2([0,n], E)$ such that the mild solution y of (7)–(8) satisfies $y(n) = x_1$.

Let us introduce the following hypotheses:

(H4) For each $n > 0$, the linear operator $W : L^2([0,n], U) \to E$ defined by

$$Wu = \int_0^n S(n-s)Bu(s)ds,$$

has an invertible operator W^{-1} which takes values in $L^2([0,n], U)/\mathrm{Ker}\, W$, and there exist positive constants M_1, M_2 such that $\|B\| \leq M_1$ and $\|W^{-1}\| \leq M_2$.

Remark 4.2. The question of the existence of the operator W and of its inverse is discussed in the paper by Quinn and Carmichael [49].

Then $B_{+\infty}$ is a Fréchet space with those families of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$. Consider the operator $N_1 : B_{+\infty} \to \mathcal{B}_{+\infty}$ defined by: $N_1(h) = \{h \in B_{+\infty}\}$ with

$$h(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ S(t)\ \phi(0) + \displaystyle\int_0^t S(t-s)\ v(s)\, ds & \\ \quad + \displaystyle\int_0^t S(t-s)Bu_y^n(s)ds, \quad v \in S_{F,y_{\rho(s,y_s)}}, & \text{if } t \in J. \end{cases} \tag{10}$$

Using the assumption (H4), for an arbitrary function $y(.)$ we define the control

$$u_y^n(t) = W^{-1}\left[y_1 - S(t)\ \phi(0) - \int_0^t S(n-s)\ v(s)\, ds\right](t).$$

Next, we will prove that $\widetilde{N}$ has a fixed point. Let $x(.) : \mathbb{R} \to E$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \leq 0; \\ S(t)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y(\cdot)$ satisfies (9), we can decompose it as $y(t) = \overline{z}(t) + x(t)$, $t \geq 0$, which implies $y_t = z_t + x_t$ for every $t \in J$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t S(t-s)v(s)ds + \int_0^t Bu^n_{z+x}(s)ds, \quad t \in J,$$

where $v(s) \in S_{F, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}}$.

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0 \in \mathcal{B}\}.$$

For any $z \in B^0_{+\infty}$, we have

$$\|z\|_{+\infty} = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s < +\infty\} = \sup\{|z(s)| : 0 \leq s < +\infty\}.$$

Thus $(B^0_{+\infty}, \|\cdot\|_{+\infty})$ is a Banach space. We define the operator $P_1 : B^0_{+\infty} \to \mathcal{P}(B^0_{+\infty})$ by: $P_1(z) := \{h \in B^0_{+\infty}\}$ with

$$h(t) = \int_0^t S(t-s)\ v(s)ds + \int_0^t S(t-s)Bu_{z+x}(s)ds, \tag{11}$$
$$v(s) \in S_{F, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}}, \quad t \in J.$$

Let $z \in B^0_{+\infty}$ be a possible fixed point of the operator P_1. Given $n \in \mathbb{N}$, then z should be a solution of the inclusion $z \in \lambda P_1(z)$ for some $\lambda \in (0,1)$ and there exists $v \in S_{F,z}$ such that, for each $t \in [0,n]$, we have

$$\begin{aligned}
|z(t)| \leq & \int_0^t \|S(t-s)\|_{B(E)}\ |v(s)|\ ds + \\
& + \int_0^t \|S(t-s)\|_{B(E)}\ \|B\|\ \|u^n_{z+x}(s)\|\, ds \leq \\
& \leq M \int_0^t p(s)\ \psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\, ds + \\
& + MM_1M_2 \int_0^t \left[\|y_1\| + M\|\phi\| + M \int_0^n p(\tau)\psi(\|z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\|)\, d\tau \right] ds \leq
\end{aligned}$$

$$\leq M \int_0^t p(s)\ \psi\left(K_n|z(s)| + (M_n + L^\phi + K_n MH)\|\phi\|_{\mathcal{B}}\right)\ ds +$$
$$+ nMM_1M_2[\|y_1\| + M\|\phi\|] +$$
$$+ nM^2\ M_1M_2 \int_0^n p(s)\ \psi\left(K_n|z(s)| + (M_n + L^\phi + K_n MH)\|\phi\|_{\mathcal{B}}\right)\ ds.$$

Set

$$c_n := (M_n + L^\phi + K_n MH)\|\phi\|_{\mathcal{B}}.$$

Then we have

$$|z(t)| \leq M \int_0^t p(s)\ \psi\left(K_n|z(s)| + c_n\right)\ ds + nMM_1M_2[\|y_1\| + M\|\phi\|] +$$
$$+ nM^2M_1M_2 \int_0^n p(s)\ \psi\left(K_n|z(s)| + c_n\right) ds.$$

Then

$$K_n|z(t)| + c_n \leq c_n + K_nM \int_0^t p(s)\ \psi\left(K_n|z(s)| + c_n\right)\ ds +$$
$$+ K_n nMM_1M_2[\|y_1\| + M\|\phi\|] +$$
$$+ K_nM^2M_1M_2n \int_0^n p(s)\ \psi\left(K_n|z(s)| + c_n\right) ds.$$

Set

$$\beta_n = K_n nMM_1M_2[\|y_1\| + M\|\phi\|] + c_n.$$

Thus

$$K_n|z(t)| + c_n \leq \beta_n + K_nM \int_0^t p(s)\psi\left(K_n|z(s)| + c_n\right)\ ds +$$
$$+ K_nM^2M_1M_2n \int_0^n p(s)\psi\left(K_n|z(s)| + c_n\right) ds.$$

Let

$$\mu(t) := \sup\ \{\ K_n|z(s)| + c_n\ :\ 0 \leq s \leq t\ \}, \quad t \in J.$$

By the previous inequality, we have

$$\mu(t) \leq \beta_n + K_n M \int_0^t p(s)\psi(\mu(s))\, ds +$$
$$+ K_n M^2 M_1 M_2 n \int_0^n p(s)\psi(\mu(s)) ds \quad \text{for} \quad t \in [0, n].$$

Let us take the right-hand side of the above inequality as $v(t)$. We have

$$\mu(t) \leq v(t) \quad \text{for all } t \in [0, n].$$

From the definition of v, we have

$$v(0) = \beta_n + K_n M^2 M_1 M_2 n \int_0^n p(s)\psi(\mu(s)) ds,$$

and

$$v'(t) = K_n M p(t)\psi(\mu(t)) \quad \text{a.e.} \quad t \in [0, n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq K_n M p(t)\psi(v(t)) \quad \text{a.e. } t \in [0, n].$$

Using the condition (4), this implies that for each $t \in [0, n]$, we have

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \leq K_n M \int_0^t p(s)\, ds \leq K_n M \int_0^n p(s)\, ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

Thus, for every $t \in [0, n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$.

Set

$$U_1 = \left\{ z \in B^0_{+\infty} \ : \ \sup_{0 \leq t \leq n} |z(t)| < \Lambda_n + 1 \quad \text{for all} \quad n \in \mathbb{N} \right\}.$$

Clearly, U is an open subset of $B^0_{+\infty}$. Let us how that $P_1 : \overline{U}_1 \to \mathcal{P}(B^0_{+\infty})$ is a contraction and an admissible operator. First, we prove that P_1 is a contraction. Let $z, \overline{z} \in B^0_{+\infty}$ and $h \in P_1(z)$. Then there exists $v(t) \in F(t, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$ such that for each $t \in [0, n]$,

$$h(t) = \int_0^t S(t-s)v(s) ds + \int_0^t S(t-s)Bu^n_{z+x}(s) ds.$$

From $(H3)$, it follows that

$$H_d(F(t, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})) \leq$$
$$\leq l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}.$$

Hence, there is $w \in F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$ such that

$$|v(t) - w| \le l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \quad t \in [0, n].$$

Consider $U_* : [0, n] \to \mathcal{P}(E)$, given by

$$U_*(t) = \left\{w \in E : |v(t) - w| \le l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right\}.$$

Since the multivalued operator $V_*(t) = U_*(t) \cap F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$ is measurable (see Proposition 2.11), there exists a function $\overline{v}(t)$, which is a measurable selection for V_*. So, $\overline{v}(t) \in F(t, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})$, and using $(A1)$, for each $t \in [0, n]$, we obtain

$$\begin{aligned}|v(t) - \overline{v}(t)| &\le l_n(t)\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \le \\ &\le l_n(t)[K(t)|z(t) - \overline{z}(t)| + M(t)\|z_0 - \overline{z}_0\|_{\mathcal{B}}] \le \\ &\le l_n(t)K_n|z(t) - \overline{z}(t)|.\end{aligned}$$

For each $t \in [0, n]$, let us define

$$\overline{h}(t) = \int_0^t S(t-s)\overline{v}(s)ds + \int_0^t S(t-s)B\ u^n_{\overline{z}+x}(s)ds.$$

Then we have

$$\begin{aligned}|h(t) - \overline{h}(t)| &\le \int_0^t \|S(t-s)\||v(s) - \overline{v}(s)|ds + \\ &+ \int_0^t \|S(t-s)\|[(Bu^n_{z+x})(s) - (Bu^n_{\overline{z}+x})(s)]ds \le \\ &\le \int_0^t MK_n l_n(s)|z(s) - \overline{z}(s)|ds + \\ &+ MM_1 \int_0^t \left| W^{-1}\left[y_1 - S(n)\phi(0) - \int_0^n S(n-s)v(\tau)d\tau\right] - \right. \\ &- \left. W^{-1}\left[y_1 - S(n)\phi(0) - \int_0^n S(n-s)\overline{v}(\tau)d\tau\right]\right| ds \le \\ &\le \int_0^t MK_n l_n(s)|z(s) - \overline{z}(s)|ds + \\ &+ MM_1M_2 \int_0^t M \int_0^n |v(\tau) - \overline{v}(\tau)|d\tau\, ds \le\end{aligned}$$

$$\leq \int_0^t MK_n l_n(s)|z(s)-\overline{z}(s)|ds+$$
$$+ M^2M_1M_2 n\int_0^t K_n l_n(s)|z(s)-\overline{z}(s)|ds \leq$$
$$\leq \int_0^t [\bar{l}_n(s)|z(s)-\overline{z}(s)|ds \leq$$
$$\leq \int_0^t [\bar{l}_n(s)e^{\tau\ \overline{L}_n(s)}][e^{-\tau\ \overline{L}_n(s)}|z(s)-\overline{z}(s)|]ds \leq$$
$$\leq \int_0^t \left[\frac{e^{\tau\ \overline{L}_n(s)}}{\tau}\right]' ds\ \|z-\overline{z}\|_n \leq$$
$$\leq \frac{1}{\tau}\ e^{\tau\ \overline{L}_n(t)}\ \|z-\overline{z}\|_n.$$

Therefore,

$$\|h-\overline{h}\|_n \leq \frac{1}{\tau}\ \|z-\overline{z}\|_n.$$

By an analogous relation, obtained by interchanging the roles of z and $\overline{z}$, it follows that

$$H_d(P_1(z),P_1(\overline{z})) \leq \frac{1}{\tau}\ \|z-\overline{z}\|.$$

So, P_1 is a contraction for all $n\in\mathbb{N}$, and as in Theorem 3.4, we can prove that P_1 is an admissible multivalued map. From the choice of U_1 there is no $z\in\partial U_1$ such that $z=\lambda P_1(z)$for some $\lambda\in(0,1)$. As a consequence of Theorem 2.13 we deduce that the operator P_1 has a fixed point z^*. Then $y^*=z^*+x$ is a fixed point of the operator N_1, which is a mild solution of the problem (7)–(8).

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Authors' addresses:

M. Benchohra and S. Litimein

Laboratoire de Mathématiques, Université de Sidi Bel-Abbès
BP 89, 22000 Sidi Bel-Abbès, Algérie
E-mail: benchohra@univ-sba.dz; sara_litimein@yahoo.fr

K. Ezzinbi

Département de Mathématiques, Faculté des Sciences
Semlalia, B.P.2390, Marrakech, Morocco
E-mail: ezzinbi@ucam.ac.ma