Proceedings of A. Razmadze
Mathematical Institute
Vol. **161** (2013), 47–53

AN OPTIMAL CONTROL PROBLEM FOR HELMHOLTZ EQUATIONS WITH BITSADZE–SAMARSKIĬ BOUNDARY CONDITIONS

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ABSTRACT. The paper deals with optimal control problems whose behavior is described by Helmholtz equations with Bitsadze–Samarskiĭ nonlocal boundary conditions. The theorem about a necessary and sufficient optimality condition is given. The existence and uniqueness of a solution of the conjugate problem are proved. A numerical method of the solution of an optimal problem by means of the Mathcad package is presented.

რეზიუმე. ნაშრომში განხილულია ოპტიმალური მართვის ამოცანა ჰელმჰოლცის განტოლებისათვის ბიწაძე-სამარსკის არალოკალური სასაზღვრო პირობებით. მოყვანილია თეორემა ოპტიმალობის აუცილებელი და საკმარისი პირობების შესახებ. დამტკიცებულია შეუღლებული განტოლების ამონახსნის არსებობა და ერთადერთობა. წარმოდგენილია Mathcad-ის საშუალებით ოპტიმალური მართვის ამოცანის ამოხსნის რიცხვითი მეთოდი.

Introduction. Nonlocal boundary value problems are a very interesting generalization of classical problems and at the same time they are obtained in a natural manner when constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology and so on [1]–[3]. The Bitsadze–Samarskiĭ nonlocal boundary value problem [4] arose in connection with mathematical modeling of processes occurring in plasma physics. Intensive studies of Bitsadze–Samarskiĭ nonlocal problems [5] and its various generalizations began in the 80s of the last century [4]–[8].

The present paper deals with optimal control problems whose behavior is described by Helmholtz equations with Bitsadze–Samarskiĭ nonlocal boundary conditions. Necessary optimality conditions are established by using the approach worked out in [9], [11] for controlled systems of general type. To investigate the conjugate problem we use the algorithm reducing nonlocal boundary value problems to a sequence of Dirichlet problems.

2010 *Mathematics Subject Classification.* 49J20.
Key words and phrases. Helmholtz equation, nonlocal boundary value problem, Bitsadze–Samarskiĭ problem, optimal control, optimality condition.

Such a method makes it possible not only to solve the problem numerically, but also to prove the existence of its solution [5].

In the paper: the Bitsadze–Samarskiĭ boundary value problem for Helmholtz equations is considered. An optimal control problem is stated for a nonlocal boundary value problem with an integral quality test. A theorem about a necessary and sufficient optimality condition is formulated. The existence and uniqueness of a solution of the conjugate problem are proved. A numerical method is presented for solving an optimal problem by means of the Mathcad package.

1. **Statement of the Optimal Control Problem.** Let $\overline{G}$ be the rectangle, $\overline{G} = [0,1] \times [0,1]$, Γ be the boundary of the domain G, $0 < x_0 < 1$, $\gamma_0 = \{(x_0, y) : \ 0 \le y \le 1\}$, $\gamma = \{(1, y) : \ 0 \le y \le 1\}$, $a(x,y), b(x,y), c(x,y), d(x,y) \in L_p(\overline{G})$, $p > 2$, $0 \le q(x,y) \in L_\infty(\overline{G})$.

Let U be an arbitrary bounded set from R. Every function $\omega(x,y)$: $G \to U$ will be called a control. The set U is called the control domain. A function $\omega(x,y)$ is called an admissible control if $\omega(x,y) \in L_p(G)$, $p > 2$. The set of all admissible controls is denoted by Ω.

For each fixed $\omega(x,y) \in \Omega$ in the domain $\overline{G}$ let us consider the following Bitsadze–Samarskiĭ boundary value problem for Helmholtz equations:

$$\begin{gathered}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(x,y)u = a(x,y)\omega(x,y) + b(x,y), \quad (x,\, y) \in G, \\
u(x,y) = 0, \quad (x,y) \in \Gamma \setminus \gamma, \\
u(1,y) = \sigma u(x_0, y), \quad 0 \le y \le 1, \quad 0 < \sigma < 1.
\end{gathered} \tag{1.1}$$

The solution of the problem (1.1) exists, is unique and belongs to the space $W_2^2(G) \cap \overset{\circ}{W}{}_2^1(G)$.

We consider the functional

$$I(\omega) = \iint\limits_G \Big[c(x,y)u(x,y) + d(x,y)\omega(x,y)\Big]\, dx\, dy \tag{1.2}$$

and state the following optimal control problem: Find a function $\omega_0(x,y) \in \Omega$, for which the solution of problem (1.1) gives functional (1.2) a minimal value. A function $\omega_0(x,y) \in \Omega$ will be called an optimal control, and the corresponding solution $u_0(x,y)$ an optimal solution.

Theorem. *Let ψ_0 be the solution of the adjoint problem*

$$\begin{gathered}
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - q(x,y)\psi = -c(x,y), \quad (x,y) \in G \setminus \gamma_0, \\
\psi(x,y) = 0, \quad (x,y) \in \Gamma, \\
\frac{\partial \psi(x_0^+, y)}{\partial x} - \frac{\partial \psi(x_0^-, y)}{\partial x} = \sigma\, \frac{\partial \psi(1,y)}{\partial x}\,, \quad 0 \le y \le 1.
\end{gathered} \tag{1.3}$$

Then for (u_0, ω_0) *to be optimal it is necessary and sufficient that the principle of minimum*

$$\inf_{\omega \in U} \big[d(x,y) + a(x,y)\psi_0(x,y)\big]\omega = \big[d(x,y) + a(x,y)\psi_0(x,y)\big]\omega_0 \qquad (1.4)$$

be fulfilled almost everywhere on G [10].

2. **Existence and Uniqueness of a Solution of the Conjugate Problem.** We write a solution of the conjugate problem (1.3) in the form $\psi = w + w^*$, where w^* is the solution of the Dirichlet problem

$$\begin{gathered} \frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} - q(x,y)w^* = -c(x,y), \ \ (x,y) \in G, \\ w^*(x,y) = 0, \ \ (x,y) \in \Gamma, \end{gathered} \qquad (2.1)$$

and w is the solution of the nonclassical boundary value problem

$$\begin{gathered} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - q(x,y)w = 0, \ \ (x,y) \in G \setminus \gamma_0, \\ w(x,y) = 0, \ \ (x,y) \in \Gamma, \\ \frac{\partial w(x_0^+, y)}{\partial x} - \frac{\partial w(x_0^-, y)}{\partial x} = \sigma\, \frac{\partial w(1,y)}{\partial x} + \sigma\, \frac{\partial w^*(1,y)}{\partial x}\,, \ \ 0 \le y \le 1. \end{gathered} \qquad (2.2)$$

As is known [12], problem (2.1) has a unique solution that belongs to the space $W_2^2(G) \cap \overset{\circ}{W}{}_2^1(G)$. Therefore it remains for us to investigate problem (2.2). To this end, we consider the iteration process

$$\begin{gathered} \frac{\partial^2 w^{k+1}}{\partial x^2} + \frac{\partial^2 w^{k+1}}{\partial y^2} - q(x,y)w^{k+1} = 0, \ \ (x,y) \in G \setminus \gamma_0, \\ w^{k+1}(x,y) = 0, \ \ (x,y) \in \Gamma, \\ \frac{\partial w^{k+1}(x_0^+, y)}{\partial x} - \frac{\partial w^{k+1}(x_0^-, y)}{\partial x} = \\ = \sigma\, \frac{\partial w^k(1,y)}{\partial x} + \phi(y), \ \ 0 \le y \le 1, \ \ k = 0,1,2,\dots, \end{gathered} \qquad (2.3)$$

where $\phi(y) = \sigma\, \frac{\partial w^*(1,y)}{\partial x}$, $w^0(x,y)$ is the initial approximation which can be assumed to be equal to zero. Let $z^{(k)}(x,y) = w^{(k+1)}(x,y) - w^{(k)}(x,y)$, then from equalities (2.3) we obtain

$$\begin{gathered} \frac{\partial^2 z^k}{\partial x^2} + \frac{\partial^2 z^k}{\partial y^2} - q(x,y)z^k = 0, \ \ (x,y) \in G \setminus \gamma_0, \\ z^k(x,y) = 0, \ \ (x,y) \in \Gamma, \\ \frac{\partial z^k(x_0^+, y)}{\partial x} - \frac{\partial z^k(x_0^-, y)}{\partial x} = \\ = \sigma\, \frac{\partial z^{k-1}(1,y)}{\partial x}\,, \ \ 0 \le y \le 1, \ \ k = 1,2,3,\dots. \end{gathered} \qquad (2.4)$$

In the Space $W_2^2(G\setminus\gamma_0)\cap\overset{\circ}{W}{}_2^1(G)$ problem (2.4) is equivalent to the following problem [13], [14]

$$\begin{gathered}\frac{\partial^2 z^k}{\partial x^2}+\frac{\partial^2 z^k}{\partial y^2}-q(x,y)z^k=-\sigma\delta(x_0-x)\,\frac{\partial z^{k-1}(1,y)}{\partial x}\,,\ \ (x,y)\in G,\\ z^k(x,y)=0,\ \ (x,y)\in\Gamma,\ \ k=1,2,3,\dots,\end{gathered}\tag{2.5}$$

where $\delta(x_0-x)$ is the Dirac function. Let us introduce the notation

$$f(x,y)=-\sigma\delta(x_0-x)\,\frac{\partial z^{(k-1)}(1,y)}{\partial x}\,.$$

Since $f(x,y)\in W_2^{-1}(G)$, the solution of problem (2.5) exists, is unique and belongs to the space $\overset{\circ}{W}{}_2^1(G)$ [12]. Moreover, if $G(x,y,\xi,\eta)$ is the Green's function of problem (2.5), then the solution can be represented as follows

$$\begin{aligned}z^{(k)}(x,y)&=\iint\limits_G\sigma G(x,y,\xi,\eta)\delta(x_0-\xi)\,\frac{\partial z^{(k-1)}(1,\eta)}{\partial x}\,d\xi\,d\eta=\\ &=\int\limits_0^1\sigma G(x,y,x_0,\eta)\,\frac{\partial z^{(k-1)}(1,\eta)}{\partial x}\,d\eta.\end{aligned}\tag{2.6}$$

Furthermore, taking the property of the Green's function into account [13], [14], from equality (2.6) it follows that $z^{(k)}(x,y)\in W_2^2(G\setminus\gamma_0)\cap\overset{\circ}{W}{}_2^1(G)$, hence we define the trace of the function $\frac{\partial}{\partial x}\,(z^{(k)}(1,y))$ which belongs to the space $W_2^{1/2}(0,1)$. Thus, using (2.6) we can write

$$\frac{\partial z^{(k)}(1,y)}{\partial x}=\int\limits_0^1\sigma\,\frac{\partial G(1,y,x_0,\eta)}{\partial x}\cdot\frac{\partial z^{(k-1)}(1,\eta)}{\partial x}\,d\eta.\tag{2.7}$$

Next, using the Cauchy–Bunyakovsky inequality, from equality (2.7) we obtain

$$\begin{aligned}&\int\limits_0^1\Big[\frac{\partial z^{(k)}(1,y)}{\partial x}\Big]^2\,dy\le\\ &\qquad\le\sigma^2\int\limits_0^1\int\limits_0^1\Big[\frac{\partial G(1,y,x_0,\eta)}{\partial x}\Big]^2\,dx\,dy\cdot\int\limits_0^1\Big[\frac{\partial z^{(k-1)}(1,\eta)}{\partial x}\Big]^2\,d\eta.\end{aligned}\tag{2.8}$$

Denote

$$\theta=\sigma\Big\|\frac{\partial G(1,y,x_0,\eta)}{\partial x}\Big\|_{L_2([0,1]\times[0,1])},$$

then from (2.8) we obtain the estimate

$$\Big\|\frac{\partial z^{(k)}(1,y)}{\partial x}\Big\|_{L_2[0,1]} \le \theta \Big\|\frac{\partial z^{(k-1)}(1,y)}{\partial x}\Big\|_{L_2[0,1]}. \tag{2.9}$$

Let $\sigma < 1/\|\frac{\partial}{\partial x}(G(1,y,x_0,\eta))\|$, then $\theta < 1$ and estimate (2.9) implies that the series $\sum\limits_{k=1}^{\infty} \frac{\partial}{\partial x}\,(z^{(k)}(1,y))$ converges. This means that the sequence $\big\{\frac{\partial w^{(k)}(1,y)}{\partial x}\big\}$ converges, where

$$\frac{\partial w^{(k)}(1,y)}{\partial x} = \sum_{i=0}^{k-1}\left[\frac{\partial w^{(i+1)}(1,y)}{\partial x} - \frac{\partial w^{(i)}(1,y)}{\partial x}\right] + \frac{\partial w^{(0)}(1,y)}{\partial x}$$

in the norm of the space $L_2[0,1]$. Therefore equality (2.6) implies the convergence of the series $\sum\limits_{k=1}^{\infty} z^{(k)}(x,y)$, and thereby the convergence of the iteration sequence $\{w^{(k)}(x,y)\}$:

$$\lim_{k\to\infty} w^{(k)}(x,y) = w(x,y),$$

where $w(x,y) \in W_2^2(G\setminus\gamma_0) \cap \overset{\circ}{W}{}_2^1(G)$. Further it is not difficult to prove that the function $w(x,y)$ is the solution of problem (2.2).

Let us prove the uniqueness of the solution. Assume that $w_1(x,y)$ and $w_2(x,y)$ are two generalized solutions of problem (2.2). Then their difference $z(x,y) = w_1(x,y) - w_2(x,y)$ is the solution of the problem

$$\begin{gathered}
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - q(x,y)z = 0, \ \ (x,y) \in G\setminus\gamma_0,\\
z(x,y) = 0, \ \ (x,y) \in \Gamma,\\
\frac{\partial z(x_0^+,y)}{\partial x} - \frac{\partial z(x_0^-,y)}{\partial x} = \sigma\,\frac{\partial z(1,y)}{\partial x}, \ \ 0 \le y \le 1.
\end{gathered} \tag{2.10}$$

By the same reasoning as above we obtain $z(x,y) = 0$. We have thereby proved the existence and uniqueness of the solution of problem (1.3).

3. **Numerical Method of the Solution of an Optimal Control Problem by Means of the Mathcad.** The scheme of the solution of an optimal control problem is as follows:

- to find a solution $\psi_0(x,y)$, we first solve the adjoint problem (1.3);
- using the function $\psi_0(x,y)$ from (1.4), we construct the optimal control $\omega_0(x,y)$;
- to find an optimal solution $u_0(x,y)$, we solve problem (1.1).

As we have shown, a solution of the adjoint problem can be written in the form $\psi = w + w^*$, where $w^*$is the solution of the Dirichlet problem (2.1), and w is the solution of the nonclassical boundary value problem (2.2). For the solution of problem (2.2) we consider the iteration process (2.3). Problem (2.3) is equivalent to the following problem

$$\begin{gathered}\frac{\partial^2 w^{k+1}}{\partial x^2}+\frac{\partial^2 w^{k+1}}{\partial y^2}-q(x,y)w^{k+1}= \\ =\delta(x_0-x)\Big(\sigma\frac{\partial w^k(1,\,y)}{\partial x}+\phi(y)\Big),\ \ (x,y)\in G, \\ w^{k+1}(x,y)=0,\ \ (x,y)\in\Gamma,\ \ k=0,1,2,\dots,\end{gathered}\tag{3.1}$$

where $\phi(y)=\sigma\,\frac{\partial w^*(1,y)}{\partial x}$, $w^0(x,y)$ is the initial approximation.

For the solution of problem (1.1) we consider the iteration process

$$\begin{gathered}\frac{\partial^2 u^{k+1}}{\partial x^2}+\frac{\partial^2 u^{k+1}}{\partial y^2}-q(x,y)u^{k+1}=a(x,y)\omega+b(x,y),\ \ (x,\,y)\in G, \\ u^{k+1}(x,y)=0,\ \ (x,y)\in\Gamma\setminus\gamma, \\ u^{k+1}(1,y)=\sigma u^k(x_0,y),\ \ 0\le y\le 1,\ \ 0<\sigma<1,\ \ k=0,1,2,\dots\,.\end{gathered}\tag{3.2}$$

To obtain a numerical solution of problems (3.1) and (3.2), at each iteration step we use the built-in function **relax(a, b, c, d, e, f, u, rjac)** in the Mathcad [15], [16].

The function **relax** returns the square matrix, where the position of an element in the matrix corresponds to its position inside the square domain, while the value corresponds to an approximate solution at this point.

The arguments of the function **relax** are as follows:

a, **b**, **c**, **d**, **e** are square matrices of one and the same size, containing the coefficients of a differential equation. In particular, for a Helmholtz equation the coefficients are $a_{i,j}=b_{i,j}=c_{i,j}=d_{i,j}=1$, $e_{i,j}=-4-q_{i,j}$, where $q_{i,j}$ are the values of $q(x,y)$ in the respective node inside the square domain;

f is a square matrix containing the values of the right-hand part of the equation in the respective node inside the square domain;

u is a square matrix containing the boundary values of the function at the domain edges, and also the initial approximation of the solution in the interior nodes of the square domain;

rjac is the parameter controlling the relaxation process convergence. It may vary in the range from 0 to 1, but an optimal value depends on the particulars of the problem.

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(Received 9.08.2012; revised 10.12.2012)

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