Proceedings of A. Razmadze
Mathematical Institute
Vol. **161** (2013), 25–45

ON DIVERGENCE OF MULTIPLE FOURIER–WALSH AND FOURIER–HAAR SERIES OF BOUNDED FUNCTION OF SEVERAL VARIABLES ON SET OF MEASURE ZERO

K. BITSADZE

ABSTRACT. For arbitrary subset E of measure zero of n-dimensional cube $[0,1]^n$ there exists a bounded measurable function f given on $[0,1]^n$ such that the sequence of diagonal partial sums

$$\sum_{p_1,p_2,\dots,p_n=0}^{m} a_{p_1,p_2,\dots,p_n}(f)\omega_{p_1}(x_1)\omega_{p_2}(x_2)\cdots\omega_{p_n}(x_n),$$
$$m=0,1,2,\dots,$$

of n-fold Fourier–Walsh (Fourier–Walsh–Paley, Fourier–Walsh–Kaczmarz) series of f diverges for every $(x_1,x_2,\dots,x_n)\in E$.

Analogous result is true for the Fourier–Haar series.

რეზიუმე. n-განზომილებიანი $[0,1]^n$ კუბის ნული ზომის ყოველი E ქვესიმრავლისათვის არსებობს $[0,1]^n$-ზე განსაზღვრული, შემოსაზღვრული, ზომადი ფუნქცია, რომლის ფურიე-უოლშის (ფურიე-უოლშ-პელის, ფურიე-უოლშ-კაჩმაჟის) n-ჯერადი მწკრივის დიაგონალური კერძო ჯამების

$$\sum_{p_1,p_2,\dots,p_n=0}^{m} a_{p_1,p_2,\dots,p_n}(f)\omega_{p_1}(x_1)\omega_{p_2}(x_2)\cdots\omega_{p_n}(x_n),$$
$$m=0,1,2,\dots,$$

მიმდევრობა განშლადია ყოველი $(x_1,x_2,\dots,x_n)\in E$ წერტილისათვის.

ანალოგიური შედეგი სამართლიანია ფურიე-ჰაარის მწკრივისათვისაც.

1. Introduction

The problems of divergence for trigonometric series have been the subject of investigation for a long time [1]–[11].

2010 *Mathematics Subject Classification.* 42C10.
Key words and phrases. Fourier–Walsh, Fourier–Haar, divergence, function of several variables, measure zero.

Problems of divergence were investigated for Walsh and Haar series. Historical information and problems concerning Walsh and Haar series are given in the papers of B. I. Golubov [12], P. L. Uljanov [13] and W. Wade [14], [15].

The Fourier–Walsh, Fourier–Walsh–Paley or Fourier–Walsh–Kaczmarz series of a Lebesgue integrable function f (on $[0,1]$) has the form:

$$f(x) \sim \sum_{p=0}^{\infty} a_p(f)\omega_p(x), \quad x \in [0,1],$$

where

$$a_p(f) = \int\limits_0^1 f(t)\omega_p(t)dt, \quad p = 0, 1, 2, \dots,$$

and $\{\omega_p\}_{p=0}^{\infty}$ is the Walsh (cf. [16] or [17]), Walsh–Paley (cf. [16]) or Walsh–Kaczmarz (cf. [16]) system of functions.

E. Stein [18] constructed an integrable function whose Fourier–Walsh–Paley series diverges almost everywhere.

F. Schipp [19] constructed an integrable function whose Fourier–Walsh–Paley series diverges everywhere.

Sh. Kheladze [20] proved that for an arbitrary number p, $1 \le p < \infty$, and arbitrary set of measure zero there exists a function in $L^p(0,1)$ whose Fourier–Walsh–Paley series diverges on this set.

In this connection W. Wade noted in his survey article [14] on Walsh series that the analogous question for bounded and continuous functions was still open.

This problem for bounded functions was solved by V. Bugadze [21]: For arbitrary set of measure zero there exists a bounded measurable function whose Fourier–Walsh (Fourier–Walsh–Paley, Fourier–Walsh–Kaczmarz) series diverges on this set.

The n-fold Fourier–Walsh, Fourier–Walsh–Paley or Fourier–Walsh–Kaczmarz series of a Lebesgue integrable function f, defined on the n-dimensional cube $[0,1]^n$, has the form:

$$f(x_1, \dots, x_n) \sim \sum_{p_1,\dots,p_n=0}^{\infty} a_{p_1,\dots,p_n}(f) w_{p_1}(x_1) \cdots w_{p_n}(x_n), \qquad (x_1, \dots, x_n) \in [0,1]^n, \tag{1}$$

where

$$a_{p_1,\dots,p_n}(f) = \int\limits_{[0,1]^n} f(t_1, \dots, t_n) w_{p_1}(t_1) \cdots w_{p_n}(t_n)\, dt_1 \cdots dt_n, \qquad p_1, \dots, p_n = 0, 1, \dots,$$

and $\{w_{p_1}\cdots w_{p_n}\}_{p_1,\dots,p_n=0}^{\infty}$ is the Walsh, Walsh–Paley or Walsh–Kaczmarz n-fold system of functions ($n \geq 2$, $n \in N$).

Let $\{x_{m_1,\dots,m_n}\}_{m_1,\dots,m_n=1}^{\infty}$ is a sequence of real numbers and $\lambda \geq 1$ is any real number. Sequence $\{x_{m_1,\dots,m_n}\}_{m_1,\dots,m_n=1}^{\infty}$ is said to be λ-convergent to any real number c, if for arbitrary number $\varepsilon > 0$ there exists natural number p such that for arbitrary natural numbers $m_k > p$, $k = 1, 2, \dots, n$, that satisfy condition

$$\frac{1}{\lambda} < \frac{m_i}{m_j} < \lambda, \quad i, j = 1, 2, \dots, n, \tag{2}$$

the following inequality holds

$$|x_{m_1,\dots,m_n} - c| < \varepsilon.$$

If the last inequality holds without conditions (2), then sequence $\{x_{m_1,\dots,m_n}\}_{m_1,\dots,m_n=1}^{\infty}$ is said to be convergent to c in the sense of Pringsheim.

R. Getsadze [22] constructed a continuous function whose multiple Fourier–Walsh–Paley series diverges almost everywhere.

The theorem 1 proved below, which was announced in [23], transfers the result of V. Bugadze [21] to the multiple Fourier–Walsh (Fourier–Walsh–Paley, Fourier–Walsh–Kaczmarz) series of the functions of several variables (for the convergence in the sense of Pringsheim; moreover for λ-convergence).

The Fourier–Haar series of a Lebesgue integrable function f (on $[0, 1]$) has the form:

$$f(x) \sim \sum_{p=0}^{\infty} b_p(f)\chi_p(x), \quad x \in [0, 1],$$

where

$$b_p(f) = \int_0^1 f(t)\chi_p(t)\, dt, \quad p = 0, 1, 2, \dots,$$

and $\{\chi_p\}_{p=0}^{\infty}$ is Haar system of functions (cf. [24]).

A. Haar [25] proved that for every function continuous on $[0, 1]$ Fourier–Haar series of the function converges to the function uniformly on $[0, 1]$. He also proved that for every Lebesgue integrable function its Fourier–Haar series converges to this function on $[0, 1]$ almost everywhere.

V. I. Prokhorenko [26] proved that for every set of measure zero there exists a function in $\bigcap_{p\geq 1} L^p$ whose Fourier–Haar series diverges on this set.

M. A. Lunina [27] proved that if φ is an even function nondecreasing on $[0, \infty)$ with $\varphi(0) = 0$, $\lim_{x\to\infty} \varphi(x) = \varphi(\infty) = +\infty$, than for each set E of type $\tilde{G}_\delta$ and of measure zero there exists a function $L \cap \varphi(L)$ whose Fourier–Haar series diverges unboundedly on E and converges on $[0, 1] \setminus E$.

V. Bugadze [28] proved that for every set of measure zero there exists a bounded measurable function whose Fourier–Haar series diverges on this set.

The n-fold Fourier–Haar series of a Lebesgue integrable function f, defined on the n-dimensional cube $[0,1]^n$, has the form:

$$f(x_1,\dots,x_n) \sim \sum_{p_1,\dots,p_n=0}^{\infty} b_{p_1,\dots,p_n}(f)\chi_{p_1}(x_1)\cdots\chi_{p_n}(x_n), \qquad (3)$$
$$(x_1,\dots,x_n)\in[0,1]^n,$$

where

$$b_{p_1,\dots,p_n}(f) = \int\limits_{[0,1]^n} f(t_1,\dots,t_n)\chi_{p_1}(t_1)\cdots\chi_{p_n}(t_n)\,dt_1\cdots dt_n,$$
$$p_1,\dots,p_n = 0,1,\dots,$$

and $\{\chi_{p_1}\cdots\chi_{p_n}\}_{p_1,\dots,p_n=0}^{\infty}$ is n-fold Haar system of functions ($n\geq 2$, $n\in N$).

O. Dzagnidze [29] proved that for some $f\in L[0,1]^2$ double Fourier–Haar series of f can be divergent almost everywhere, but it λ-converges (for every $\lambda>1$) to f almost everywhere. Moreover for $f\in L\log^+ L[0,1]^2$ double Fourier–Haar series converges to that function almost everywhere.

Author [30] proved that for arbitrary subset E of measure zero of n-dimensional cube $[0,1]^n$ there exists a bounded measurable function f given on $[0,1]^n$ whose n-fold Fourier–Haar series does not λ-converge on E for arbitrary $\lambda>1$.

Below it is proved that analogous statement (cf. Theorem 2) is true for $\lambda=1$.

2. Definitions and Lemmas

We denote by R the set of real numbers, by Z the set of integers, by N the set of natural numbers and by N_0 the set of nonnegative integers.

The intervals

$$\Delta_k^{(m)} \equiv [k\cdot 2^{-m},(k+1)\cdot 2^{-m}),$$

where $k=0,1,\dots,2^m-1$, $m\in N_0$, are called as dyadic intervals. The number m is the rank of the interval $\Delta_k^{(m)}$.

For a point $x\in[0,1)$, $\Delta_x^{(m)}$ denotes a dyadic interval of rank m, which contains the point x.

Numbers of the form $k\cdot 2^{-m}$, where $k\in Z$, $m\in N$, are called as dyadic rationals. All other real numbers are called as dyadic irrationals.

We denote by μA the Lebesgue measure of any Lebesgue measurable set A.

Below every where we mean that $n\in N$ and $n\geq 2$.

We define the system $\{\Gamma_k^{(i)}\}_{i,k=1}^{\infty}$ of half open n-dimensional dyadic cubes. For that consider n pieces of arbitrary systems of dyadic intervals

$$\big\{[\alpha_{j,k}^{(i)}, \beta_{j,k}^{(i)})\big\}_{i,k=1}^{\infty}, \quad j = 1, 2, \ldots, n,$$

which have the following property: for arbitrary pair $(i, k)$of natural numbers the ranks of the intervals $[\alpha_{j,k}^{(i)}, \beta_{j,k}^{(i)})$, $j = 1, 2, \ldots, n$, are equal. Note that the rank of the interval $[\alpha_{j,k}^{(i)}, \beta_{j,k}^{(i)})$ is

$$r_{j,k}^{(i)} = \log_2(\beta_{j,k}^{(i)} - \alpha_{j,k}^{(i)})^{-1} \equiv r_k^{(i)}, \quad j = 1, 2, \ldots, n.$$

Fix the arbitrary numbers $i, k \in N$. On basis of definition of dyadic interval we have

$$[\alpha_{j,k}^{(i)}, \beta_{j,k}^{(i)}) \equiv \Delta^{(r_k^{(i)})}_{2^{r_k^{(i)}} \cdot \alpha_{j,k}^{(i)}}, \quad j = 1, 2, \ldots, n. \tag{4}$$

For arbitrary dyadic interval the following presentation holds (cf. [14], [16], [17]):

$$\Delta_k^{(m)} = \bigcup_{u^{(s)}=0}^{2^s-1} \Delta^{(m+s)}_{2^s \cdot k + u^{(s)}}, \quad s = 0, 1, \ldots,$$

therefore (cf. (4))

$$\Delta^{(r_k^{(i)})}_{2^{r_k^{(i)}} \cdot \alpha_{j,k}^{(i)}} = \bigcup_{u_j^{(s)}=0}^{2^s-1} \Delta^{(r_k^{(i)}+s)}_{2^{r_k^{(i)}+s} \cdot \alpha_{j,k}^{(i)} + u_j^{(s)}}, \quad s \in N_0, \quad j = 1, 2, \ldots, n. \tag{5}$$

We denote (cf. (4), (5)):

$$\Gamma^{(i)}_{k;s;u_1^{(s)},\ldots,u_n^{(s)}} \equiv \prod_{j=1}^{n} \Delta^{(r_k^{(i)}+s)}_{2^{r_k^{(i)}+s} \cdot \alpha_{j,k}^{(i)} + u_j^{(s)}},$$
$$s \in N_0, \quad u_j^{(s)} = 0, 1, \ldots, 2^s - 1, \quad j = 1, 2, \ldots, n; \tag{6}$$
$$\Gamma_k^{(i)} \equiv \Gamma^{(i)}_{k;0;0,\ldots,0}.$$

It is obvious, that (cf. (5), (6))

$$\Gamma_k^{(i)} = \bigcup_{u_1^{(s)},\ldots,u_n^{(s)}=0}^{2^s-1} \Gamma^{(i)}_{k;s;u_1^{(s)},\ldots,u_n^{(s)}}, \quad s \in N_0, \tag{7}$$

and

$$\Gamma^{(i)}_{k;s;u_1^{(s)},\ldots,u_n^{(s)}} = \bigcup_{i_1,\ldots,i_n=0}^{1} \Gamma^{(i)}_{k;s+1;2u_1^{(s)}+i_1,\ldots,2u_n^{(s)}+i_n}, \tag{8}$$
$$s \in N_0, \quad u_j^{(s)} = 0, 1, \ldots, 2^s - 1, \quad j = 1, 2, \ldots, n.$$

It follows from (6) that

$$\mu\Gamma^{(i)}_{k;s;u_1^{(s)},\dots,u_n^{(s)}} = 2^{-nr_k^{(i)}-ns}, \tag{9}$$

$$s\in N_0,\ \ u_j^{(s)}=0,1,\dots,2^s-1,\ \ j=1,2,\dots,n.$$

We denote also

$$\dot{\Gamma}^{(i)}_{k;s;u_1^{(s)},\dots,u_n^{(s)}} \equiv$$

$$\equiv\prod_{j=1}^{n}\Big(\alpha_{j,k}^{(i)}+2^{-r_k^{(i)}-s}\cdot u_j^{(s)}-2^{-r_k^{(i)}-s-12},\alpha_{j,k}^{(i)}+2^{-r_k^{(i)}-s}\cdot u_j^{(s)}+2^{-r_k^{(i)}-s}\Big), \tag{10}$$

where $s\in N_0$, $u_j^{(s)}=0,1,\dots,2^s-1$, $j=1,2,\dots,n$.

For any set A we denote by $(\mathrm{A})'$ the part of boundary of set A, which belongs to A and by $\mathring{\mathrm{A}}$ the set $\mathrm{A}\setminus(\mathrm{A})'$.

The partial sums of the n-fold Fourier–Walsh and Fourier–Haar series (cf. (1), (3)) of a function f are defined by the equalities (correspondingly):

$$W_{m_1,\dots,m_n}(f,x_1,x_2,\dots,x_n) = \\ = \sum_{p_1,\dots,p_n=0}^{m_1-1,\dots,m_n-1} a_{p_1,\dots,p_n}(f)w_{p_1}(x_1)\cdots w_{p_n}(x_n), \tag{11}$$

$$H_{m_1,\dots,m_n}(f,x_1,x_2,\dots,x_n) = \\ = \sum_{p_1,\dots,p_n=0}^{m_1-1,\dots,m_n-1} b_{p_1,\dots,p_n}(f)\chi_{p_1}(x_1)\cdots \chi_{p_n}(x_n), \tag{12}$$

where $m_1,\dots,m_n=1,2,\dots$, $(x_1,\dots,x_n)\in[0,1]^n$.

The following two lemmas (Lemma 1 and Lemma 2) are analogs of correspondingly the classical results (cf. [17], [25]):

Lemma 1. *For any point* $(x_1,\dots,x_n)\in[0,1]^n$

$$W_{2^{m_1},\dots,2^{m_n}}(f,x_1,\dots,x_n) = \\ = \frac{1}{\mu\big(\prod\limits_{i=1}^{n}\Delta_{x_i}^{(m_i)}\big)}\int\limits_{\prod\limits_{i=1}^{n}\Delta_{x_i}^{(m_i)}} f(t_1,\dots,t_n)\,dt_1\cdots dt_n, \tag{13}$$

$$m_1,\dots,m_n=0,1,\dots.$$

Proof. By the definition of the partial sums of the n-fold Fourier–Walsh series (cf. (11))

$$W_{2^{m_1},\dots,2^{m_n}}(f,x_1,\dots,x_n)=$$
$$=\sum_{p_1,\dots,p_n=0}^{2^{m_1}-1,\dots,2^{m_n}-1} a_{p_1,\dots,p_n}(f)w_{p_1}(x_1)\cdots w_{p_n}(x_n)=$$
$$=\int\limits_{[0,1]^n} f(t_1,\dots,t_n)K_{m_1},\dots,K_{m_n}(x_1,\dots,x_n;t_1,\dots,t_n)\,dt_1\cdots dt_n, \quad (14)$$

where

$$K_{m_1,\dots,m_n}(x_1,\dots,x_n;t_1,\dots,t_n)=\prod_{i=1}^{n}\Bigg[\sum_{p_i=0}^{2^{m_i}-1} w_{p_i}(x_i)w_{p_i}(t_i)\Bigg], \quad (15)$$

$m_1,\dots,m_n=0,1,\dots,$ $(x_1,\dots,x_n)\in[0,1]^n$.

We prove by induction that for fixed natural number i, $1\le i\le n$, the following equality is true

$$\sum_{p_i=0}^{2^{m_i}-1} w_{p_i}(x_i)w_{p_i}(t_i)=\begin{cases}2^{m_i}, & t_i\in\Delta_{x_i}^{(m_i)}\\ 0, & t_i\in[0,1]\setminus\Delta_{x_i}^{(m_i)}\end{cases}. \quad (16)$$

It is obvious that equality (16) is true for $m_i=0$.

Let $m_i=1$. Then we have

$$\sum_{p_i=0}^{1} w_{p_i}(x_i)w_{p_i}(t_i)=1+w_1(x_i)w_1(t_i)=\begin{cases}2, & t_i\in\Delta_{x_i}^{(1)}\\ 0, & t_i\in[0,1]\setminus\Delta_{x_i}^{(1)}\end{cases},$$

i.e. (16) is true.

Let equality (16) be valid for (m_i-1) and prove that it also holds for m_i. We have

$$\sum_{p_i=0}^{2^{m_i}-1} w_{p_i}(x_i)w_{p_i}(t_i)=$$
$$=\sum_{p_i=0}^{2^{m_i-1}-1} w_{p_i}(x_i)w_{p_i}(t_i)+\sum_{p_i=2^{m_i-1}}^{2^{m_i}-1} w_{p_i}(x_i)w_{p_i}(t_i). \quad (17)$$

It is known that (cf. [14], [16], [17]) for the Walsh function w_p ($2^m\le p<2^{m+1}$, $m\in N_0$) the intervals $\Delta_k^{(m)}$ ($0\le k<2^m$) are intervals of change of sign and the intervals $\Delta_k^{(m+1)}$ ($0\le k<2^{m+1}$) are the intervals of constancy. In addition, for any number $p=2^m,2^m+1,\dots,2^{m+1}-1$, $m\in N_0$, we have

$w_p(x) = w_{2^m}(x) w_{p-2^m}(x)$, $x \in [0,1]$ (cf. [31]). By induction and this facts we have

$$\sum_{p_i=2^{m_i-1}}^{2^{m_i}-1} w_{p_i}(x_i) w_{p_i}(t_i) =$$

$$= w_{2^{m_i-1}}(x_i) w_{2^{m_i-1}}(t_i) \cdot \sum_{p_i=2^{m_i-1}}^{2^{m_i}-1} w_{p_i-2^{m_i-1}}(x_i) w_{p_i-2^{m_i-1}}(t_i) =$$

$$= w_{2^{m_i-1}}(x_i) w_{2^{m_i-1}}(t_i) \cdot \sum_{\nu=0}^{2^{m_i-1}-1} w_\nu(x_i) w_\nu(t_i) =$$

$$= \begin{cases} \sum\limits_{\nu=0}^{2^{m_i-1}-1} w_\nu(x_i) w_\nu(t_i) = 2^{m_i-1}, & t_i \in \Delta_{x_i}^{(m_i)} \\ -\sum\limits_{\nu=0}^{2^{m_i-1}-1} w_\nu(x_i) w_\nu(t_i) = -2^{m_i-1}, & t_i \notin \Delta_{x_i}^{(m_i)},\ t_i \in \Delta_{x_i}^{(m_i-1)} \\ 0, & t_i \notin \Delta_{x_i}^{(m_i-1)} \end{cases}.$$

By induction and last equality we have (cf. (17))

$$\sum_{p_i=0}^{2^{m_i}-1} w_{p_i}(x_i) w_{p_i}(t_i) = \begin{cases} 2^{m_i}, & t_i \in \Delta_{x_i}^{(m_i)} \\ 0, & t_i \notin \Delta_{x_i}^{(m_i)} \end{cases}.$$

By obtained equality it follows from (15) that

$$K_{m_1,\dots,m_n}(x_1,\dots,x_n;t_1,\dots,t_n) =$$

$$= \begin{cases} 2^{\sum\limits_{i=1}^{n} m_i}, & (t_1,\dots,t_n) \in \prod\limits_{i=1}^{n} \Delta_{x_i}^{(m_i)} \\ 0, & (t_1,\dots,t_n) \in [0,1]^n \setminus \prod\limits_{i=1}^{n} \Delta_{x_i}^{(m_i)} \end{cases}.$$

By the last equality it follows from (14) that the equality (13) is true. □

Lemma 2. *For any point* $(x_1,\dots,x_n) \in [0,1]^n$

$$W_{2^{m_1},\dots,2^{m_n}}(f,x_1,\dots,x_n) = H_{2^{m_1},\dots,2^{m_n}}(f,x_1,\dots,x_n), \qquad (18)$$
$$m_1,\dots,m_n = 0,1,\dots .$$

Proof. It is known that for partial sums (12) of the n-fold Fourier–Haar series the following equality is true

$$H_{2^{m_1},\dots,2^{m_n}}(f,x_1,\dots,x_n)= \\ =\frac{1}{\left|\prod\limits_{i=1}^{n}\Delta_{x_i}^{(m_i)}\right|}\int\limits_{\prod\limits_{i=1}^{n}\Delta_{x_i}^{(m_i)}} f(t_1,\dots,t_n)\,dt_1\cdots dt_n, \quad (19)$$

where $m_1,\dots,m_n=0,1,\dots,$ $(x_1,\dots,x_n)\in[0,1]^n$.

It follows from equalities (13) and (19) that the equality (18) is true. □

Lemma 3. *For arbitrary subset E of measure zero of n-dimensional cube $[0,1]^n$ there exists a sequence of open sets G_i, $i=1,2,\dots,$ satisfying the following conditions:*

(a) $G_{i+1}\subset G_i\subset G_1=[0,1)^n\equiv\Gamma_1^{(1)}$, $i=2,3,\dots$.

(b) *If $i\geq 2$, $i\in N$, then G_i is not more than countable union of half open n-dimensional dyadic cubes $\Gamma_k^{(i)}$, which in pairs do not have common inner point.*

(c) *For any number $i\in N$ any point of set $E\cap(G_i\setminus G_{i+1})$ belong to one of the sets $(\Gamma_k^{(i)})'$.*

(d) *For any number $i\in N$ each cube $\Gamma_k^{(i+1)}$ is entirely contained in one of the cubes $\overset{\circ}{\Gamma}{}_{k_1}^{(i)}$, $k_1\in N$.*

(e) *For any natural numbers i, k and s and numbers $u_j^{(s)}$, $j=1,2,\dots,n$, where $u_{j_1}^{(s)}=u_{j_2}^{(s)}=\dots=u_{j_q}^{(s)}=0$, $j_1,j_2,\dots,j_q=1,2,\dots,n$, $j_1<j_2<\dots<j_q$, $q=1,2,\dots,n-1$ and $u_j^{(s)}=1,2,\dots,2^s-1$, $j=1,2,\dots,n$ and in this last equalities j does not equal to anyone of the following numbers $j_1,j_2,\dots,j_q$, the following inequality holds*

$$\mu G_{i+1}\cap\Gamma^{(i)}_{k;s;u_1^{(s)},u_2^{(s)},\dots,u_n^{(s)}}<2^{-nr_k^{(i)}-ns-8n-1} \quad (20)$$

and for any number $s\in N_0$

$$\mu G_{i+1}\cap\Gamma^{(i)}_{k;s;0,0,\dots,0}<2^{-nr_k^{(i)}-ns-8n-1}. \quad (21)$$

Proof. Note that for construction of sets G_i, $i\in N$, we use the system of half open n-dimensional dyadic cubes (which is defined above) and for the dyadic intervals which are used in construction of this system we suppose that $[\alpha_{j,1}^{(1)},\beta_{j,1}^{(1)})\equiv[0,1)$, $j=1,2,\dots,n$ (cf. (4)–(6)).

Suppose we have sets $G_1,G_2,\dots,G_m$, $m\geq 2$, satisfying conditions (a), (c), (d), (e), when $i=1,2,\dots,m-1$ and condition (b), when $i=1,2,\dots,m$. We shall construct the set G_{m+1}.

Let $k \in N$. We represent the cube $\overset{\circ}{\Gamma}{}_k^{(m)}$ in the form (cf. (7)):

$$\overset{\circ}{\Gamma}{}_k^{(m)} = \Gamma_{k;1;1,\dots,1}^{(m)} \bigcup \bigcup_{s=2}^{\infty} \Bigg[\bigcup_{u_2^{(s)},\dots,u_n^{(s)}=1}^{2^s-1} \Gamma_{k;s;1,u_2^{(s)},\dots,u_n^{(s)}}^{(m)} \bigcup \bigcup_{u_1^{(s)}=2}^{2^s-1} \Big(\bigcup_{u_3^{(s)},u_4^{(s)},\dots,u_n^{(s)}=1}^{2^s-1} \Gamma_{k;s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}}^{(m)} \Big)$$

$$\bigcup \bigcup_{u_1^{(s)},u_2^{(s)}=2}^{2^s-1} \Big(\bigcup_{u_4^{(s)},u_5^{(s)},\dots,u_n^{(s)}=1}^{2^s-1} \Gamma_{k;s;u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)}}^{(m)} \Big) \bigcup \cdots \bigcup \bigcup_{u_1^{(s)},\dots,u_{n-1}^{(s)}=2}^{2^s-1} \Gamma_{k;s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}^{(m)} \Bigg]. \qquad (22)$$

Let us consider the n-tuples

$(1,\dots,1)$ for $s=1$, and for any natural number $s \geq 2$,

$(1,u_2^{(s)},\dots,u_n^{(s)})$, where $u_2^{(s)},\dots,u_n^{(s)} = 1,\dots,2^s-1$,

$(u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)})$,
where $u_1^{(s)} = 2,\dots,2^s-1$; $u_3^{(s)},\dots,u_n^{(s)} = 1,\dots,2^s-1$,

$(u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)})$,
where $u_1^{(s)},u_2^{(s)} = 2,\dots,2^s-1$; $u_4^{(s)},\dots,u_n^{(s)} = 1,\dots,2^s-1$,

....

$(u_1^{(s)},u_2^{(s)},\dots,u_{n-1}^{(s)},1)$, where $u_1^{(s)},\dots,u_{n-1}^{(s)} = 2,\dots,2^s-1$.

Since $\mu E = 0$ there exist open sets, correspondingly,

$$F_{k;1;1,\dots,1}^{(m)}, F_{k;s;1,u_2^{(s)},\dots,u_n^{(s)}}^{(m)}, F_{k;s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}}^{(m)},$$

$$F_{k;s;u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)}}^{(m)}, \dots, F_{k;s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}^{(m)}$$

such that (cf. (10)

$$
\begin{aligned}
&E\cap\Gamma^{(m)}_{k;1;1,\dots,1}\subset F^{(m)}_{k;1;1,\dots,1}\subset\dot{\Gamma}^{(m)}_{k;1;1,\dots,1},\\
&E\cap\Gamma^{(m)}_{k;s;1,u_2^{(s)},\dots,u_n^{(s)}}\subset F^{(m)}_{k;s;1,u_2^{(s)},\dots,u_n^{(s)}}\subset\dot{\Gamma}^{(m)}_{k;s;1,u_2^{(s)},\dots,u_n^{(s)}},\\
&E\cap\Gamma^{(m)}_{k;s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}}\subset F^{(m)}_{k;s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}}\subset\dot{\Gamma}^{(m)}_{k;s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}},\\
&E\cap\Gamma^{(m)}_{k;s;u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)}}\subset\\
&\qquad\subset F^{(m)}_{k;s;u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)}}\subset\dot{\Gamma}^{(m)}_{k;s;u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)}},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&E\cap\Gamma^{(m)}_{k;s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}\subset F^{(m)}_{k;s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}\subset\dot{\Gamma}^{(m)}_{k;s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
\mu F^{(m)}_{k;1;1,\dots,1}&<2^{nr_k^{(m)}-11n},\\
\mu F^{(m)}_{k;s;1,u_2^{(s)},\dots,u_n^{(s)}}&<2^{nr_k^{(m)}-ns-10n},\\
\mu F^{(m)}_{k;s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}}&<2^{nr_k^{(m)}-ns-10n},\\
\mu F^{(m)}_{k;s;u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)}}&<2^{nr_k^{(m)}-ns-10n},\\
\dots\dots\dots\dots\dots\dots&\dots\dots\dots\dots\\
\mu F^{(m)}_{k;s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}&<2^{nr_k^{(m)}-ns-10n}.
\end{aligned}
\tag{24}
$$

Let

$$
\begin{aligned}
G_{m+1}=\bigcup_{k=1}^{\infty}\Bigg\{F^{(m)}_{k;1;1,\dots,1}\bigcup\bigcup_{s=2}^{\infty}\Bigg[&\bigcup_{u_2^{(s)},\dots,u_n^{(s)}=1}^{2^s-1}F^{(m)}_{k;s;1,u_2^{(s)},\dots,u_n^{(s)}}\bigcup\\
&\bigcup_{u_1^{(s)}=2}^{2^s-1}\Bigg(\bigcup_{u_3^{(s)},\dots,u_n^{(s)}=1}^{2^s-1}F^{(m)}_{k;s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}}\Bigg)\\
\bigcup\bigcup_{u_1^{(s)},u_2^{(s)}=2}^{2^s-1}&\Bigg(\bigcup_{u_4^{(s)},\dots,u_n^{(s)}=1}^{2^s-1}F^{(m)}_{k;s;u_1^{(s)},u_2^{(s)},1,u_4^{(s)},\dots,u_n^{(s)}}\Bigg)\bigcup\cdots\bigcup\\
&\bigcup_{u_1^{(s)},\dots,u_{n-1}^{(s)}=2}^{2^s-1}F^{(m)}_{k;s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}\Bigg]\Bigg\}.
\end{aligned}
\tag{25}
$$

It is obvious that G_{m+1} is open set. Therefore, it can be represented as not more then countable union of half open n-dimensional dyadic cubes. Let

$$G_{m+1}=\bigcup_{k=1}^{\infty}\Gamma_k^{(m+1)}$$

be one of such representation (we mean that the cube $\Gamma_k^{(m+1)}$are disjoint in pairs).

It is obvious that G_{m+1} satisfies condition (b) for $i=m+1$ and conditions (a) and (d) for $i=m$ (cf. (22), (23), (25)).

We have (cf. (22), (23), (25))

$$E\cap(G_m\setminus G_{m+1})=\bigcup_{k=1}^{\infty}\big[(\Gamma_k^{(m)}\setminus G_{m+1})\cap E\big] \text{ and } E\cap\mathring{\Gamma}_k^{(m)}\subset G_{m+1}.$$

Therefore

$$E\cap(G_m\setminus G_{m+1})\subset\bigcup_{k=1}^{\infty}\Big\{\big[(\Gamma_k^{(m)})'\cup(\mathring{\Gamma}_k^{(m)}\setminus E)\big]\cap E\Big\}\subset\bigcup_{k=1}^{\infty}(\Gamma_k^{(m)})'.$$

Thus also holds condition (c) for $i=m$. Condition (e) remains to be verified.

It follows from (22), (23) and (25) that:

1) For any natural number $s\ge 2$ and for numbers $u_j^{(s)}$, $j=1,2,\dots,n$, where

$$u_{j_1}^{(s)}=u_{j_2}^{(s)}=\dots=u_{j_q}^{(s)}=0$$
$$(j_1,j_2,\dots,j_q=1,2,\dots,n,\;\; j_1<j_2<\dots<j_q,\;\; q=1,2,\dots,n-1)$$

and

$$u_j^{(s)}=1,2,\dots,2^s-2\;\;(j=1,2,\dots,n,\;\; j\ne j_1,j_2,\dots,j_q),$$

the following inclusion is true

$$G_{m+1}\cap\Gamma^{(m)}_{k;s;u_1^{(s)},\dots,u_n^{(s)}}\subset$$
$$\subset\bigcup_{\nu=0}^{\infty}\Bigg[\bigcup_{\substack{d_j^{(\nu)}=0\\ j=1,\dots,n;\; j\ne j_1,\dots,j_q}}^{2^\nu}\Bigg(\bigcup_{\substack{d_{j_2}^{(\nu)},d_{j_3}^{(\nu)},\dots,d_{j_q}^{(\nu)}=1\\ d_{j_1}^{(\nu)}=1}}^{2^\nu}F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}}$$
$$\bigcup\bigcup_{\substack{d_{j_1}^{(\nu)},d_{j_3}^{(\nu)},\dots,d_{j_q}^{(\nu)}=1\\ d_{j_2}^{(\nu)}=1}}^{2^\nu}F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}}\bigcup\cdots\bigcup$$

$$\bigcup \bigcup_{\substack{d_{j_1}^{(\nu)}, d_{j_2}^{(\nu)}, \dots, d_{j_{q-1}}^{(\nu)}=1 \\ d_{j_q}^{(\nu)}=1}}^{2^\nu} F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}}\Bigg)\Bigg];$$

2) For any natural number $s \geq 2$ and for numbers $u_j^{(s)}$, $j = 1, 2, \dots, n$, where

$$u_{j_1}^{(s)} = u_{j_2}^{(s)} = \dots = u_{j_q}^{(s)} = 0$$
$$(j_1, j_2, \dots, j_q = 1, 2, \dots, n,\ \ j_1 < j_2 < \dots < j_q,\ \ q = 1, 2, \dots, n-2),$$
$$u_{l_1}^{(s)} = u_{l_2}^{(s)} = \dots = u_{l_p}^{(s)} = 2^s - 1$$
$$(l_1, l_2, \dots, l_p = 1, \dots, n,\ \ l_1 < l_2 < \dots < l_p,$$
$$l_1, \dots, l_p \neq j_1, \dots, j_q,\ \ p = 1, 2, \dots, n-q-1),$$
$$u_j^{(s)} = 1, 2, \dots, 2^s - 2\ \ (j = 1, 2, \dots, n,\ \ j \neq j_1, j_2, \dots, j_q, l_1, l_2, \dots, l_p)$$

(note that this case hold only for $n \geq 3$, $n \in N$) the following inclusion is true

$$G_{m+1} \cap \Gamma^{(m)}_{k;s;u_1^{(s)},\dots,u_n^{(s)}} \subset$$
$$\subset \bigcup_{\nu=0}^{\infty} \Bigg[\bigcup_{\substack{d_j^{(\nu)}=0 \\ j=1,\dots,n; j\neq j_1,\dots,j_q,l_1,\dots,l_p}}^{2^\nu} \bigcup_{d_{l_1}^{(\nu)}, d_{l_2}^{(\nu)}, \dots, d_{l_p}^{(\nu)}=0}^{2^\nu-1} \bigcup$$
$$\Bigg(\bigcup_{\substack{d_{j_2}^{(\nu)}, d_{j_3}^{(\nu)}, \dots, d_{j_q}^{(\nu)}=1 \\ d_{j_1}^{(\nu)}=1}}^{2^\nu} F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}}$$
$$\bigcup \bigcup_{\substack{d_{j_1}^{(\nu)}, d_{j_3}^{(\nu)}, \dots, d_{j_q}^{(\nu)}=1 \\ d_{j_2}^{(\nu)}=1}}^{2^\nu} F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}} \bigcup \cdots \bigcup$$
$$\bigcup_{\substack{d_{j_1}^{(\nu)}, \dots, d_{j_{q-1}}^{(\nu)}=1 \\ d_{j_q}^{(\nu)}=1}}^{2^\nu} F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}}\Bigg)\Bigg];$$

3) For any natural number $s \geq 1$ and for numbers $u_j^{(s)}$, $j = 1, 2, \dots, n$, where

$$u_{j_1}^{(s)} = u_{j_2}^{(s)} = \dots = u_{j_q}^{(s)} = 0$$
$$(j_1, j_2, \dots, j_q = 1, 2, \dots, n,\ \ j_1 < j_2 < \dots < j_q,\ \ q = 1, 2, \dots, n-1)$$

and

$$u_j^{(s)} = 2^s - 1 \ \ (j = 1, 2, \dots, n, \ \ j \neq j_1, j_2, \dots, j_q),$$

the following inclusion is true

$$G_{m+1} \cap \Gamma^{(m)}_{k;s;u_1^{(s)},\dots,u_n^{(s)}} \subset$$

$$\subset \bigcup_{\nu=0}^{\infty} \Bigg[\bigcup_{\substack{d_j^{(\nu)}=0 \\ j=1,\dots,n; j\neq j_1,\dots,j_q}}^{2^\nu-1} \Bigg(\bigcup_{\substack{d_{j_2}^{(\nu)}, d_{j_3}^{(\nu)},\dots,d_{j_q}^{(\nu)}=1 \\ d_{j_1}^{(\nu)}=1}}^{2^\nu} F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}}$$

$$\bigcup \bigcup_{\substack{d_{j_1}^{(\nu)}, d_{j_3}^{(\nu)},\dots,d_{j_q}^{(\nu)}=1 \\ d_{j_2}^{(\nu)}=1}}^{2^\nu} F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}} \bigcup \cdots \bigcup$$

$$\bigcup \bigcup_{\substack{d_{j_1}^{(\nu)}, d_{j_2}^{(\nu)},\dots,d_{j_{q-1}}^{(\nu)}=1 \\ d_{j_q}^{(\nu)}=1}}^{2^\nu} F^{(m)}_{k;s+\nu;2^\nu u_1^{(s)}+d_1^{(\nu)},\dots,2^\nu u_n^{(s)}+d_n^{(\nu)}} \Bigg) \Bigg];$$

4) For any numbers $s \in N_0$, $u_j^{(s)} = 0$ $(j = 1, 2, \dots, n)$ the following inclusion is true

$$G_{m+1} \cap \Gamma^{(m)}_{k;s;0,\dots,0} \subset \bigcup_{\nu=0}^{\infty} \Bigg(\bigcup_{d_2^{(\nu)}, d_3^{(\nu)},\dots,d_n^{(\nu)}=1}^{2^\nu} F^{(m)}_{k;s+\nu;1,d_2^{(\nu)},\dots,d_n^{(\nu)}}$$

$$\bigcup \bigcup_{d_1^{(\nu)}, d_3^{(\nu)},\dots,d_n^{(\nu)}=1}^{2^\nu} F^{(m)}_{k;s+\nu;d_1^{(\nu)},1,d_3^{(\nu)},\dots,d_n^{(\nu)}} \bigcup \cdots \bigcup$$

$$\bigcup_{d_1^{(\nu)},\dots,d_{n-1}^{(\nu)}=1}^{2^\nu} F^{(m)}_{k;s+\nu;d_1^{(\nu)},\dots,d_{n-1}^{(\nu)},1} \Bigg).$$

Therefore by inequalities (24) we have that condition (e) also holds.

Thus we have a sequence G_i $(i = 1, 2, \dots)$ satisfying conditions (a)–(e). □

3. Basic Result

Theorem 1. *For arbitrary subset E of measure zero of n-dimensional cube $[0,1]^n$ there exists a bounded measurable function f given on $[0,1]^n$*

such that the sequence of diagonal partial sums

$$\sum_{p_1,p_2,\dots,p_n=0}^{m} a_{p_1,p_2,\dots,p_n}(f)\omega_{p_1}(x_1)\omega_{p_2}(x_2)\cdots\omega_{p_n}(x_n),\quad m=0,1,2,\dots,$$

of n-fold Fourier–Walsh (Fourier–Walsh–Paley, Fourier–Walsh–Kaczmarz) series of f diverges for every $(x_1,x_2,\dots,x_n)\in E$.

Proof. Let $E\subset[0,1]^n$, $\mu E=0$. By Lemma 3 there exists a sequence of open sets G_i, $i=1,2,\dots$, satisfying the condicionts (a)–(e).

We now define the functions f_i, $i=1,2,\dots$. For that denote by (cf. (7))

$$\begin{aligned} A_i \equiv \bigcup_{k=1}^{\infty}\bigcup_{s=1}^{\infty}\Bigg(& \bigcup_{u_2^{(s)},u_3^{(s)},\dots,u_n^{(s)}=1}^{2^{2s-1}} \Gamma^{(i)}_{k;2s;1,u_2^{(s)},\dots,u_n^{(s)}} \bigcup \\ & \bigcup_{u_1^{(s)}=2}^{2^{2s-1}}\Bigg(\bigcup_{u_3^{(s)},u_4^{(s)},\dots,u_n^{(s)}=1}^{2^{2s-1}} \Gamma^{(i)}_{k;2s;u_1^{(s)},1,u_3^{(s)},\dots,u_n^{(s)}}\Bigg)\bigcup\cdots\bigcup \\ & \bigcup \bigcup_{u_1^{(s)},u_2^{(s)},\dots,u_{n-2}^{(s)}=2}^{2^{2s-1}}\Bigg(\bigcup_{u_n^{(s)}=1}^{2^{2s-1}}\Gamma^{(i)}_{k;2s;u_1^{(s)},\dots,u_{n-2}^{(s)},1,u_n^{(s)}}\Bigg)\bigcup \\ & \bigcup_{u_1^{(s)},\dots,u_{n-1}^{(s)}=2}^{2^{2s-1}} \Gamma^{(i)}_{k;2s;u_1^{(s)},\dots,u_{n-1}^{(s)},1}\Bigg),\quad i=1,2,\dots. \end{aligned} \tag{26}$$

Note that for any i, $i=1,2,\dots$, the set A_i is union of pair wise disjoint half open n-dimensional dyadic cubes.

Let

$$f_i(t_1,\dots,t_n)=\begin{cases} a_i, & (t_1,\dots,t_n)\in A_i\\ 0, & (t_1,\dots,t_n)\in[0,1]^n\setminus A_i\end{cases},\quad i=1,2,\dots, \tag{27}$$

where $a_i=\begin{cases}1 & \text{if } i \text{ is odd}\\ 2 & \text{if } i \text{ is even}\end{cases}$.

We now define the function f. Let

$$f(t_1,\dots,t_n)=\begin{cases} f_i(t_1,\dots,t_n), & (t_1,\dots,t_n)\in G_i\setminus G_{i+1},\\ & \qquad i=1,2,\dots\\ 0, & (t_1,\dots,t_n)\in\bigcap_{i=1}^{\infty}G_i\end{cases}. \tag{28}$$

By conditions (a) and (b) the definition of the function is correct.

Let s is odd number, $s\in N_0$, and

$$B^{(i)}_{k,s} \equiv \Gamma^{(i)}_{k;s+1;1,\dots,1} \bigcup \bigcup_{\nu=2}^{\infty} \Bigg(\bigcup_{d_2^{(\nu)},\dots,d_n^{(\nu)}=1}^{2^{2\nu-1}-1} \Gamma^{(i)}_{k;s+2\nu-1;1,d_2^{(\nu)},\dots,d_n^{(\nu)}} \bigcup$$
$$\bigcup_{d_1^{(\nu)}=2}^{2^{2\nu-1}-1} \Bigg(\bigcup_{d_3^{(\nu)},\dots,d_n^{(\nu)}=1}^{2^{2\nu-1}-1} \Gamma^{(i)}_{k;s+2\nu-1;d_1^{(\nu)},1,d_3^{(\nu)},\dots,d_n^{(\nu)}} \Bigg)$$
$$\bigcup \cdots \bigcup \bigcup_{d_1^{(\nu)},\dots,d_{n-2}^{(\nu)}=2}^{2^{2\nu-1}-1} \Bigg(\bigcup_{d_n^{(\nu)}=1}^{2^{2\nu-1}-1} \Gamma^{(i)}_{k;s+2\nu-1;d_1^{(\nu)},\dots,d_{n-2}^{(\nu)},1,d_n^{(\nu)}} \Bigg) \bigcup$$
$$\bigcup_{d_1^{(\nu)},\dots,d_{n-1}^{(\nu)}=2}^{2^{2\nu-1}-1} \Gamma^{(i)}_{k;s+2\nu-1;d_1^{(\nu)},\dots,d_{n-1}^{(\nu)},1} \Bigg), \quad i,k=1,2,\dots.$$

Then it follows from equality's (9), (26) and (27) that for any numbers $i,k \in N$

$$\int\limits_{\Gamma^{(i)}_{k;s;0,\dots,0}} f_i(t_1,\dots,t_n)\,dt_1\cdots dt_n =$$
$$= \int\limits_{B^{(i)}_{k,s}} f_i(t_1,\dots,t_n)\,dt_1\cdots dt_n = a_i\Bigg(2^{-nr_k^{(i)}-ns-n} +$$
$$+ n(n-1)2^{-nr_k^{(i)}-ns+n-1}\Big(\frac{16\cdot 2^{-4n}}{1-4\cdot 2^{-2n}} - \frac{3\cdot 2^{-4n}}{1-2^{-2n}}\Big)\Bigg). \quad (29)$$

Let s is even number, $s \in N_0$, and

$$B^{(i)}_{k,s} \equiv \bigcup_{\nu=1}^{\infty} \Bigg(\bigcup_{d_2^{(\nu)},\dots,d_n^{(\nu)}=1}^{2^{2\nu}-1} \Gamma^{(i)}_{k;s+2\nu;1,d_2^{(\nu)},\dots,d_n^{(\nu)}} \bigcup$$
$$\bigcup_{d_1^{(\nu)}=2}^{2^{2\nu}-1} \Bigg(\bigcup_{d_3^{(\nu)},\dots,d_n^{(\nu)}=1}^{2^{2\nu}-1} \Gamma^{(i)}_{k;s+2\nu;d_1^{(\nu)},1,d_3^{(\nu)},\dots,d_n^{(\nu)}} \Bigg) \bigcup \cdots \bigcup$$
$$\bigcup_{d_1^{(\nu)},\dots,d_{n-2}^{(\nu)}=2}^{2^{2\nu}-1} \Bigg(\bigcup_{d_n^{(\nu)}=1}^{2^{2\nu}-1} \Gamma^{(i)}_{k;s+2\nu;d_1^{(\nu)},\dots,d_{n-2}^{(\nu)},1,d_n^{(\nu)}} \Bigg) \bigcup$$
$$\bigcup_{d_1^{(\nu)},\dots,d_{n-1}^{(\nu)}=2}^{2^{2\nu}-1} \Gamma^{(i)}_{k;s+2\nu;d_1^{(\nu)},\dots,d_{n-1}^{(\nu)},1} \Bigg), \quad i,k=1,2,\dots.$$

Then it follows from equality's (9), (26) and (27) that for any numbers $i, k \in N$

$$\int\limits_{\Gamma^{(i)}_{k;s;0,\dots,0}} f_i(t_1,\dots,t_n)\,dt_1\cdots dt_n = \int\limits_{B^{(i)}_{k,s}} f_i(t_1,\dots,t_n)\,dt_1\cdots dt_n =$$

$$= a_i n(n-1)2^{-nr_k^{(i)}-ns-1}\Big(\frac{32\cdot 2^{-4n}}{1-4\cdot 2^{-2n}} - \frac{3\cdot 2^{-4n}}{1-2^{-2n}}\Big). \quad (30)$$

Let $x=(x_1,\dots,x_n)\in E$. By conditions (a) and (c) either

I. $x \in (\Gamma_k^{(i)})'$ for any numbers $i, k \in N$ or

II. $x \in \bigcap\limits_{i=1}^{\infty} G_i$.

In the latter case for each $i \in N$ there exists number $m_i \in N$ such that $x \in \mathring{\Gamma}^{(i)}_{m_i}$.

Let us first consider Case I. For the coordinates x_j, $j=1,2,\dots,n$, of a point x we have:

1) For any natural number j_0, $1 \le j_0 \le n$,

$$x_{j_0} = \alpha^{(i)}_{j_0,k}.$$

2) There exists a number q_0, $0 \le q_0 \le n-1$ and even number $s_0 \in N_0$ that:

a) For any natural numbers $l_1, l_2, \dots, l_{q_0}$, $l_1 < l_2 < \cdots < l_{q_0}$, $1 \le l_1, l_2, \dots, l_{q_0} \le n$, which are different from the number j_0 and for any number $s \in N_0$ there exist the numbers $u^{(s_0+s)}_{l_m}$, $u^{(s_0+s)}_{l_m} = 0,1,\dots,2^{s_0+s}-1$, such that

$$x_{l_m} \in \Big[\alpha^{(i)}_{l_m,k} + 2^{-r_k^{(i)}-s_0-s}\cdot u^{(s_0+s)}_{l_m},$$
$$\alpha^{(i)}_{l_m,k} + 2^{-r_k^{(i)}-s_0-s}\cdot u^{(s_0+s)}_{l_m} + 2^{-r_k^{(i)}-s_0-s}\Big), \quad m=1,2,\dots,q_0$$

(if $q_0=0$, we mean that the point x have not the coordinates of this type).

b) For each number j, $j=1,2,\dots,n$, which is different from the numbers $l_1, l_2, \dots, l_{q_0}$ and j_0 there exist numbers $u_j^{(s_0)}$, $u_j^{(s_0)} = 0,1,\dots,2^{s_0}-1$, such that

$$x_j = \alpha^{(i)}_{j,k} + 2^{-r_k^{(i)}-s_0} u_j^{(s_0)}$$

(if $q_0 = n-1$, we mean that the point x have not the coordinates of this type).

It is obvious that (cf. (6)) for any number $s \in N_0$

$$x \in \Gamma^{(i)}_{k;s_0+s;v_1^{(s_0+s)},\dots,v_n^{(s_0+s)}},$$

where

$$v_{j_0}^{(s_0+s)} = 0;$$
$$v_{l_m}^{(s_0+s)} = u_{l_m}^{(s_0+s)}, \;\; m = 1, 2, \dots, q_0;$$
$$v_j^{(s_0+s)} = 2^s u_j^{(s_0)}, \;\; j = 1, 2, \dots, n, \;\; j \neq j_0, l_1, l_2, \dots, l_{q_0}.$$

Let $p_s = r_k^{(i)} + s_0 + 2s$, $s = 1, 2, \dots$. Consider the difference (cf. (13)):

$$W_{2^{p_s+1},\dots,2^{p_s+1}}(f, x_1, \dots, x_n) - W_{2^{p_s},\dots,2^{p_s}}(f, x_1, \dots, x_n) =$$
$$= 2^{np_s}\Bigg(2^n \int\limits_{\Gamma^{(i)}_{k;s_0+2s+1;v_1^{(s_0+2s+1)},\dots,v_n^{(s_0+2s+1)}}} f(t_1, \dots, t_n)\, dt_1 \cdots dt_n -$$
$$- \int\limits_{\Gamma^{(i)}_{k;s_0+2s;v_1^{(s_0+2s)},\dots,v_n^{(s_0+2s)}}} f(t_1, \dots, t_n)\, dt_1 \cdots dt_n\Bigg), \;\; s = 1, 2, \dots. \tag{31}$$

By (20), (26), (27), (28) and (29), for the first integral we have

$$\int\limits_{\Gamma^{(i)}_{k;s_0+2s+1;v_1^{(s_0+2s+1)},\dots,v_n^{(s_0+2s+1)}}} f(t_1, \dots, t_n)\, dt_1 \cdots dt_n \geq$$
$$\geq \int\limits_{\Gamma^{(i)}_{k;s_0+2s+1;v_1^{(s_0+2s+1)},\dots,v_n^{(s_0+2s+1)}}} f_i(t_1, \dots, t_n)\, dt_1 \cdots dt_n -$$
$$- \int\limits_{\Gamma^{(i)}_{k;s_0+2s+1;v_1^{(s_0+2s+1)},\dots,v_n^{(s_0+2s+1)}}} \big|f(t_1, \dots, t_n) - f_i(t_1, \dots, t_n)\big|\, dt_1 \cdots dt_n \geq$$
$$\geq \int\limits_{\Gamma^{(i)}_{k;s_0+2s+1;0,\dots,0}} f_i(t_1, \dots, t_n)\, dt_1 \cdots dt_n -$$
$$-\mu G_{i+1} \cap \Gamma^{(i)}_{k;s_0+2s+1;v_1^{(s_0+2s+1)},\dots,v_n^{(s_0+2s+1)}} \geq a_i\Bigg(2^{-np_s-2n} +$$
$$+ n(n-1)2^{-np_s-1}\left(\frac{16 \cdot 2^{-4n}}{1 - 4 \cdot 2^{-2n}} - \frac{3 \cdot 2^{-4n}}{1 - 2^{-2n}}\right) - 2^{-np_s-9n-1}\Bigg), \tag{32}$$
$$s = 1, 2, \dots.$$

By (20), (26), (27), (28) and (30)

$$\left| \int\limits_{\Gamma^{(i)}_{k;s_0+2s;v_1^{(s_0+2s)},\dots,v_n^{(s_0+2s)}}} f(t_1,\dots,t_n)\,dt_1\cdots dt_n \right| \le$$

$$\le \int\limits_{\Gamma^{(i)}_{k;s_0+2s;0,\dots,0}} f_i(t_1,\dots,t_n)\,dt_1\cdots dt_n + \mu G_{i+1}\cap\Gamma^{(i)}_{k;s_0+2s;v_1^{(s_0+2s)},\dots,v_n^{(s_0+2s)}} \le$$

$$\le a_i\biggl(n(n-1)2^{-np_s-1}\Bigl(\frac{32\cdot 2^{-4n}}{1-4\cdot 2^{-2n}} - \frac{3\cdot 2^{-4n}}{1-2^{-2n}}\Bigr) + 2^{-np_s-8n-1}\biggr), \quad s=1,2,\dots. \tag{33}$$

It follows from (31), (32) and (33) that

$$\bigl|W_{2^{p_s+1},\dots,2^{p_s+1}}(f,x_1,\dots,x_n) - W_{2^{p_s},\dots,2^{p_s}}(f,x_1,\dots,x_n)\bigr| > $$
$$> 2^{-2n}, \quad s=1,2,\dots. \tag{34}$$

Let us first consider Case II.

Let $d_i = r^{(i)}_{m_i}$, $i=1,2,\dots$. Then (cf. (13), (21), (26)–(28), (30))

$$\bigl|W_{2^{d_{2i}},\dots,2^{d_{2i}}}(f,x_1,\dots,x_n) - W_{2^{d_{2i-1}},\dots,2^{d_{2i-1}}}(f,x_1,\dots,x_n)\bigr| =$$

$$= \Biggl| 2^{nd_{2i}} \int\limits_{\Gamma^{(2i)}_{m_{2i}}} f(t_1,\dots,t_n)\,dt_1\cdots dt_n -$$

$$- 2^{nd_{2i-1}} \int\limits_{\Gamma^{(2i-1)}_{m_{2i-1}}} f(t_1,\dots,t_n)\,dt_1\cdots dt_n \Biggr| \ge$$

$$\ge \Biggl| 2^{nd_{2i}} \Bigl(\int\limits_{\Gamma^{(2i)}_{m_{2i}}} f_{2i}(t_1,\dots,t_n)\,dt_1\cdots dt_n - \mu G_{2i+1}\cap\Gamma^{(2i)}_{m_{2i}} \Bigr) -$$

$$- 2^{nd_{2i-1}} \Bigl(\int\limits_{\Gamma^{(2i-1)}_{m_{2i-1}}} f_{2i-1}(t_1,\dots,t_n)\,dt_1\cdots dt_n + \mu G_{2i}\cap\Gamma^{(2i-1)}_{m_{2i-1}} \Bigr) \Biggr| >$$

$$> 2^{-4n}, \quad i=1,2,\dots. \tag{35}$$

Thus the sequence of diagonal partial sums of n-fold Fourier–Walsh series of f diverges for every $(x_1,x_2,\dots,x_n)\in E$ (cf. (34), (35)).

By equality (18) the evaluations (34) and (35) are true for the partial sums of n-fold Fourier–Haar series. Consequently the following theorem is true. □

Theorem 2. *For arbitrary subset E of measure zero of n-dimensional cube $[0,1]^n$ there exists a bounded measurable function f given on $[0,1]^n$*

such that the sequence of diagonal partial sums

$$\sum_{p_1,p_2,\ldots,p_n=0}^{m} b_{p_1,p_2,\ldots,p_n}(f)\chi_{p_1}(x_1)\chi_{p_2}(x_2)\cdots\chi_{p_n}(x_n), \quad m=0,1,2,\ldots,$$

of n-fold Fourier–Haar series of f diverges for every $(x_1,x_2,\ldots,x_n)\in E$.

References

1. N. K. Bari, Trigonometric series. (Russian) With the editorial collaboration of P. L. Ul'janov. *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1961.
2. A. Zigmund, Trigonometric series. (Russian) *Izdat. Mir, Moscow*, 1965.
3. P. Du Bois-Reimond, Untersuchungen uber die Convergenz und Divergenz der Fourierschen Darstellungsformen. *Abh. Akad. Wiss. Munchen.* 1876. Bd. XII. 1–103.
4. N. Lusin, Über eine Potenzreihe. *Palermo Rend.* **32** (1911), 386–390.
5. S. B. Stechkin, On convergence and divergence of trigonometric series. (Russian) *Uspekhi Mat. Nauk*, **6** (1951), No. 2 (42), 148–149.
6. S. B. Stechkin, On trigonometric series that diverge at every point. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **21** (1957), 711–728.
7. A. Kolmogoroff, Une série de Fourier–Lebesgue divergente presque partout. (French) *Fundamenta math.* **4** (1923), 324–328.
8. A. Kolmogoroff, Une série de Fourier–Lebesgue divergente partout. (French) *C. R.* **183** (1926), 1327–1328.
9. L. V. Taikov, On the divergence of Fourier series with respect to a re-arranged trigonometrical system. (Russian) *Uspehi Mat. Nauk*, **18** (1963), No. 5 (113), 191–198.
10. J.-P. Kahane and Y. Katznelson, Sur les ensembles de divergence des series trigonometriques. (French) *Studia Math.* **26** (1966), 305–306.
11. V. V. Buzdalin, Unboundedly diverging trigonometric Fourier series of continuous functions. (Russian) *Mat. Zametki*, **7** (1970), 7–18.
12. B. I. Golubov, Series in the Haar system. (Russian) *Mathematical analysis, 1970 (Russian)*, pp. 109–146. (errata insert) *Akad. Nauk SSSR Vsesojuz. Inst. Nauchn. i Tehn. Informacii, Moscow*, 1971.
13. P. L. Uljanov, Haar series and related questions. *A. Haar memorial conference, Vol. I, II (Budapest, 1985)*, 57–96, Colloq. Math. Soc. Janos Bolyai, 49, *North-Holland, Amsterdam*, 1987.
14. W. R. Wade, Recent developments in the theory of Walsh series. *Internat. J. Math. Math. Sci.* **5** (1982), No. 4, 625–673.
15. W. R. Wade, Recent developments in the theory of Haar series. *Colloq. Math.* **52** (1987), No. 2, 213–238.
16. M. V. Balashov, On the geometric difference of multivalued mappings. (Russian) *Mat. Zametki*, **70** (2001), No. 2, 163–169; English transl.: *Math. Notes*, **70** (2001), No. 1-2, 147–153.
17. J. L. Walsh, A closed set of normal orthogonal functions. *Amer. J. Math.* **45** (1923), No. 1, 5–24.
18. E. M. Stein, On limits of seqences of operators. *Ann. of Math. (2)* **74** (1961), 140–170.
19. F. Schipp, Über die Grossenordnung der Partialsummen der Entwicklung integrierbarer Funktionen nach W-Systemen. (German) *Acta Sci. Math. (Szeged)*, **28** (1967), 123–134.
20. S. V. Kheladze, The divergence everywhere of Fourier–Walsh series. (Russian) *Sakharth. SSR Mecn. Akad. Moambe*, **77** (1975), 305–307.

21. V. M. Bugadze, Divergence of Fourier–Walsh series of bounded functions on sets of measure zero. (Russian) *Mat. Sb.* **185** (1994), No. 7, 119–127; English transl.: *Russian Acad. Sci. Sb. Math.* **82** (1995), No. 2, 365–372.
22. R. D. Getsadze, A continuous function with multiple Fourier series in the Walsh–Paley system that diverges almost everywhere. (Russian) *Mat. Sb., Nov. Ser.* **128(170)** (1985), No. 2(10), 269–286; English transl.: *Math. USSR, Sb.* **56** (1987), 262–278.
23. K. Bitsadze, On divergence of multiple Fourier-Walsh and Fourier-Haar series of bounded function of several variables on zero measure set. *Bull. Georgian Acad. Sci.* **170** (2004), No. 3, 458–461.
24. B. S. Kashin and A. A. Saakyan, Orthogonal series. (Russian) *Nauka, Moscow*, 1984.
25. A. Haar, Zur Theorie der orthogonalen Funktionensysteme. (German) *Math. Ann.* **69** (1910), No. 3, 331–371.
26. V. I. Prokhorenko, Divergent Fourier series with respect to Haar's system. *Izv. Vyssh. Uchebn. Zaved. Mat.* 1971, No. 1, 62–68.
27. M. A. Lunina, The set of points of unbounded divergence of series in the Haar system. (Russian) *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **31** (1976), No. 4, 13–20.
28. V. M. Bugadze, On the divergence of Fourier–Haar series of bounded functions on sets of measure zero. (Russian) *Mat. Zametki* **51** (1992), No. 5, 20–26; English transl.: *Math. Notes*, **51** (1992), No. 5-6, 437–441.
29. O. P. Dzagnidze, Representation of measurable functions of two variables by double series. (Russian) *Soobsc. Akad. Nauk Gruzin. SSR* **34** (1964), 277–282.
30. K. Bitsadze, On divergence of multiple Fourier-Haar series of bounded function of several variables on zero measure set. *Bull. Georgian Acad. Sci.* **154** (1996), No. 2, 181–182.
31. B. I. Golubov, A. V. Efimov and V. A. Skvortsov, Walsh series and transforms: theory and applications. (Russian) *Nauka, Moscow*, 1987.

(Received 13.04.2012)

Author's address:

Medical Management Companie "AVANTE"
2/6 Ljubljana St. Tbilisi 0159, Georgia
E-mail: kakha_bitsadze@yahoo.com