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SOME NOTES ON TRIGONOMETRIC APPROXIMATION OF (α,ψ)-DIFFERENTIABLE FUNCTIONS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. Improved Bernstein type inequality obtained and some inequalities of simultaneous approximation by trigonometric polynomials are proved. Also we proved an inverse theorem for functions having (α,ψ) derivatives in weighted variable exponent Lebesgue spaces.

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1. Introduction

We define required notations. Let the function $\omega : \boldsymbol{T}\to[0,\infty]$ be a weight on $\boldsymbol{T}$. We suppose that $\mathcal{P}$ is the class of Lebesgue measurable functions $p(x) : \boldsymbol{T} \to (1,\infty)$ such that $1 < p_* := essinf_{x\in\boldsymbol{T}} p(x) \leq p^* := esssup_{x\in\boldsymbol{T}} p(x) < \infty$. In this case we define the class $L^{p(x)}$ of 2π-periodic measurable functions $f : \boldsymbol{T} \to \mathbb{R}$ satisfying

$$\int_{\boldsymbol{T}} |f(x)|^{p(x)}\, dx < \infty$$

for $p \in \mathcal{P}$. It is known that the class $L^{p(x)}$ is a Banach space with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\alpha > 0 : \int_{\boldsymbol{T}} \left|\frac{f(x)}{\alpha}\right|^{p(x)} dx \leq 1\right\}.$$

By $L^{p(\cdot)}_{\omega}$ we will denote the class of Lebesgue measurable functions $f : \boldsymbol{T} \to \mathbb{R}$ satisfying the condition $\omega f \in L^{p(\cdot)}$. The weighted variable exponent Lebesgue space $L^{p(\cdot)}_{\omega}$ is a Banach space with the norm $\|f\|_{p(\cdot),\omega} := \|\omega f\|_{p(\cdot)}$.

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For given $p \in \mathcal{P}$ the class of weights ω satisfying the condition [3]

$$\left\| \omega \chi_Q \right\|_{p(\cdot)} \left\| \omega^{-1} \chi_Q \right\|_{p'(\cdot)} \leq C \left| Q \right|$$

for all balls Q in $\boldsymbol{T}$ will be denoted by $A_{p(\cdot)}$. Here $p'(x) := p(x) / (p(x) - 1)$ is the conjugate exponent of $p(x)$. The variable exponent $p(x)$ is said to be satisfy *log-Hölder continuous* on $\boldsymbol{T}$ if there exists a constant $c \geq 0$ such that

$$\left| p(x_1) - p(x_2) \right| \leq \frac{c}{\log \left(e + 1/\left| x_1 - x_2 \right| \right)} \quad \text{for all } x_1, x_2 \in \boldsymbol{T}.$$

We will denote by $\mathcal{P}^{\log}(\boldsymbol{T})$ the class of those exponents $p \in \mathcal{P}$ such that $1/p : \boldsymbol{T} \longrightarrow [0,1]$ is *log-Hölder continuous* on $\boldsymbol{T}$.

If $p \in \mathcal{P}^{\log}(\boldsymbol{T})$ and $f \in L_\omega^{p(\cdot)}$, then it was proved in [3] that the Hardy-Littlewood maximal function $\mathcal{M}$ is norm bounded in $L_\omega^{p(\cdot)}$ if and only if $\omega \in A_{p(\cdot)}$.

We set $f \in L_\omega^{p(\cdot)}$ and

$$\mathcal{A}_h f(x) := \frac{1}{h} \int\limits_{x-h/2}^{x+h/2} f(t)\, dt, \quad x \in \boldsymbol{T}.$$

If $p \in \mathcal{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $\mathcal{A}_h$ is bounded in $L_\omega^{p(\cdot)}$. Consequently if $x, h \in \boldsymbol{T}$, $0 \leq r$, then we define, via Binomial expansion, that

$$\sigma_h^r f(x) := (I - \mathcal{A}_h)^r f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} (\mathcal{A}_h)^k$$

where $f \in L_\omega^{p(\cdot)}$, Γ is Gamma function and I is the identity operator.

For $0 \leq r$ we define the *fractional moduli of smoothness* for $p \in \mathcal{P}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}$ and $f \in L_\omega^{p(\cdot)}$ as

$$\Omega_r(f, \delta)_{p(\cdot),\omega} := \sup_{0 < h_i, t \leq \delta} \left\| \prod_{i=1}^{[r]} (I - \mathcal{A}_{h_i}) \sigma_t^{\{r\}} f \right\|_{p(\cdot),\omega}, \quad \delta \geq 0,$$

where $\Omega_0(f, \delta)_{p(\cdot),\omega} := \|f\|_{p(\cdot),\omega}$; $\prod\limits_{i=1}^{0} (I - \mathcal{A}_{h_i}) \sigma_t^r f := \sigma_t^r f$ for $0 < r < 1$; $[r]$ denotes the integer part of the real number r and $\{r\} := r - [r]$.

If $p \in \mathcal{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $\omega^{p(\cdot)} \in L^1(\boldsymbol{T})$. This implies that the set of trigonometric polynomials is dense [5] in the space $L_\omega^{p(\cdot)}$. On the other hand if $p \in \mathcal{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $L_\omega^{p(\cdot)} \subset L^1(\boldsymbol{T})$.

For given $f \in L_\omega^{p(\cdot)}$, let

$$f(x) \backsim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f) \cos kx + b_k(f) \sin kx \right) = \sum_{k=1}^{\infty} A_k(x, f)$$

and

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} (a_k(f)\sin kx - b_k(f)\cos kx)$$

be the *Fourier* and the *conjugate Fourier series* of f, respectively.

We will say that a function $f \in L_{\omega}^{p(\cdot)}$, $p \in \mathcal{P}$, $\omega \in A_{p(\cdot)}$, has a (α, ψ)-derivative f_{α}^{ψ} if, for a given sequence $\psi(k)$, $k = 1, 2, \dots$, and a number $\alpha \in \mathbb{R}$, the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k(f) \cos k \left(x + \frac{\alpha\pi}{2k} \right) + b_k(f) \sin k \left(x + \frac{\alpha\pi}{2k} \right) \right)$$

is the Fourier series of function f_{α}^{ψ}.

Let $\mathfrak{M}$ be the set of functions $\psi(v)$ convex downwards for any $v \geq 1$ and satisfying the condition $\lim_{v\to\infty} \psi(v) = 0$.

We associate every function $\psi \in \mathfrak{M}$ with a pair of functions $\eta(t) = \psi^{-1}(\psi(t)/2)$ and $\mu(t) = t/(\eta(t) - t)$. We set $\mathfrak{M}_0 := \{\psi \in \mathfrak{M} : 0 < \mu(t) \leq K\}$. We start with proving an improved Bernstein inequality.

Theorem 1.1. *Let $p \in \mathcal{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $r \in \mathbb{R}^+$, $f \in L_{\omega}^{p(\cdot)}$, T_n is the best approximating trigonometrical polynomial for the function f, $\psi(k)$, $k \in \mathbb{N}$, be a nonincreasing sequence of non-negative numbers such that $\psi(k) \to 0$ as $k \to \infty$ and $\frac{1}{\psi(k)k^r}$ be nondecreasing. Then for any $n = 1, 2, 3, \dots$ the following inequality holds:*

$$\psi(n) \| (T_n)_r^{\psi} \|_{p(\cdot),\omega} \leq c\Omega_{r/2}(T_n, 1/n)_{p(\cdot),\omega}.$$

Proof of Theorem 1.1. By definition

$$\left\| (T_n)_r^{\psi} \right\|_{p(\cdot),\omega} = \left\| \sum_{k=1}^{n} \frac{1}{\psi(k)} A_k \left(x + \frac{r\pi}{2k}, T_n \right) \right\|_{p(\cdot),\omega} =$$

$$= \left\| \sum_{k=1}^{n} \frac{1}{\psi(k)} (\cos(r\pi/2) A_k(x, T_n) - \sin(r\pi/2) A_k(x, \widetilde{T_n})) \right\|_{p(\cdot),\omega} \leq$$

$$\leq \left\| \sum_{k=1}^{n} \frac{1}{\psi(k)} \cos(r\pi/2) A_k(x, T_n) \right\|_{p(\cdot),\omega} +$$

$$+ \left\| \sum_{k=1}^{n} \frac{1}{\psi(k)} \sin(r\pi/2) A_k(x, \widetilde{T_n}) \right\|_{p(\cdot),\omega} =$$

$$= n^r \left\| \sum_{k=1}^{n} \frac{1}{\psi(k)k^r} \cos(r\pi/2) \left(\frac{(\frac{k}{n})^2}{(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}})} \right)^{r/2} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}} \right)^{r/2} A_k(x, T_n) \right\|_{p(\cdot),\omega} +$$

$$+ n^r \left\| \sum_{k=1}^{n} \frac{1}{\psi(k)k^r} \sin(r\pi/2) \left(\frac{(\frac{k}{n})^2}{(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}})} \right)^{r/2} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}} \right)^{r/2} A_k(x, \widetilde{T_n}) \right\|_{p(\cdot),\omega}.$$

Using Marcinkiewicz multiplier theorem [4] for weighted variable exponent Lebesgue spaces we obtain

$$\left\| (T_n)_r^{\psi} \right\|_{p(\cdot),\omega} \le \frac{c}{\psi(n)} \left\| \sum_{k=1}^{n} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r/2} A_k(x, T_n) \right\|_{p(\cdot),\omega} +$$
$$+ \frac{c}{\psi(n)} \left\| \sum_{k=1}^{n} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r/2} A_k\left(x, \widetilde{T_n}\right) \right\|_{p(\cdot),\omega} =$$
$$= \frac{c}{\psi(n)} \left\| \sum_{k=1}^{n} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r/2} A_k(x, T_n) \right\|_{p(\cdot),\omega} +$$
$$+ \frac{c}{\psi(n)} \left\| \left(\sum_{k=1}^{n} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r/2} A_k(x, T_n) \right)^{\sim} \right\|_{p(\cdot),\omega}.$$

In the last equality we used the linearity of conjugate operator. Hence using boundedness of conjugate operator we have

$$\left\| (T_n)_r^{\psi} \right\|_{p(\cdot),\omega} \le \frac{c}{\psi(n)} \left\| \sum_{k=1}^{n} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r/2} A_k(x, T_n) \right\|_{p(\cdot),\omega} +$$
$$+ \frac{C}{\psi(n)} \left\| \sum_{k=1}^{n} \left(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r/2} A_k(x, T_n) \right\|_{p(\cdot),\omega} \le$$
$$\le \frac{c}{\psi(n)} \left\| \left(I - \sigma_{1/n}\right)^{r/2} T_n \right\|_{p(\cdot),\omega} =$$
$$= \frac{c}{\psi(n)} \left\| \left(I - \sigma_{1/n}\right)^{[r/2]+\{r/2\}} T_n \right\|_{p(\cdot),\omega} \le$$
$$\le \frac{c}{\psi(n)} \sup_{\substack{0<h_i,u\le 1/n \\ i=1,2,\ldots,[r/2]}} \left\| \prod_{i=1}^{[r/2]} \left(I - \sigma_{h_i}\right)\left(I - \sigma_u\right)^{\{r/2\}} T_n \right\|_{p(\cdot),\omega} \le$$
$$\le \frac{c}{\psi(n)} \Omega_{r/2}\left(T_n, 1/n\right)_{p(\cdot),\omega}.$$

Then we have the improved Bernstein inequality

$$\left\| (T_n)_r^{\psi} \right\|_{p(\cdot),\omega} < \frac{c}{\psi(n)} \Omega_{r/2}\left(T_n, 1/n\right)_{p(\cdot),\omega}. \qquad \square$$

The following Simultaneous approximation therem was proved in [1] but Professor V. Chaichenko informed us that there was a gap in its proof. He informed an example that the hypotesis on ψ of that theorem is not enough. Below we prove complately this theorem taking a stronger hypotesis on ψ, namely, " $\psi \in \mathfrak{M}_0$".

Theorem 1.2. *Let $p \in \mathcal{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $\alpha \in [0, \infty)$ and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi \in \mathfrak{M}_0$, then there exists a $T \in \mathcal{T}_n$, $n = 1, 2, 3, \ldots$ and a constant $c > 0$ depending only on α and p such that*

$$\left\| f_\alpha^\psi - T_\alpha^\psi \right\|_{p(\cdot),\omega} \leq c E_n \left(f_\alpha^\psi \right)_{p(\cdot),\omega}$$

holds.

Proof of Theorem 1.2. We set $W_n(f) := W_n(\cdot, f) := \frac{1}{n+1} \sum\limits_{\nu=n}^{2n} S_\nu(\cdot, f)$ for $n = 0, 1, 2, \ldots$. Since

$$W_n(\cdot, f_\alpha^\psi) = (W_n(\cdot, f))_\alpha^\psi$$

we have

$$\begin{aligned} \|f_\alpha^\psi(\cdot) - (S_n(\cdot, f))_\alpha^\psi\|_{p(\cdot),\omega} &\leq \|f_\alpha^\psi(\cdot) - W_n(\cdot, f_\alpha^\psi)\|_{p(\cdot),\omega} + \\ &+ \|(S_n(\cdot, W_n(f)))_\alpha^\psi - (S_n(\cdot, f))_\alpha^\psi\|_{p(\cdot),\omega} + \\ &+ \|(W_n(\cdot, f))_\alpha^\psi - (S_n(\cdot, W_n(f)))_\alpha^\psi\|_{p(\cdot),\omega} := I_1 + I_2 + I_3. \end{aligned}$$

In this case, from the boundedness of the operator S_n in $L_\omega^{p(\cdot)}$ we obtain the boundedness of operator W_n in $L_\omega^{p(\cdot)}$ and there hold

$$\begin{aligned} I_1 &\leq \left\| f_\alpha^\psi(\cdot) - S_n(\cdot, f_\alpha^\psi) \right\|_{p(\cdot),\omega} + \left\| S_n(\cdot, f_\alpha^\psi) - W_n(\cdot, f_\alpha^\psi) \right\|_{p(\cdot),\omega} \leq \\ &\leq c E_n \left(f_\alpha^\psi \right)_{p(\cdot),\omega} + \left\| W_n(\cdot, S_n(f_\alpha^\psi) - f_\alpha^\psi) \right\|_{p(\cdot),\omega} \leq c E_n \left(f_\alpha^\psi \right)_{p(\cdot),\omega}. \end{aligned}$$

From Bernstein Inequality of Corollary 2.1 in [1] we get

$$\begin{aligned} I_2 &\leq c \left(\psi(n) \right)^{-1} \left\| S_n(\cdot, W_n(f)) - S_n(\cdot, f) \right\|_{p(\cdot),\omega}, \\ I_3 &\leq c \left(\psi(2n) \right)^{-1} \left\| W_n(\cdot, f) - S_n(\cdot, W_n(f)) \right\|_{p(\cdot),\omega} \leq \\ &\leq c \left(\psi(2n) \right)^{-1} E_n \left(W_n(f) \right)_{p(\cdot),\omega}. \end{aligned}$$

Using inequality (13) of [6] we have that the fraction $\psi(n) / \psi(2n)$ is bounded from above by a constant and hence

$$I_3 \leq c \left(\psi(n) \right)^{-1} E_n \left(W_n(f) \right)_{p(\cdot),\omega}.$$

Now we have

$$\begin{aligned} &\left\| S_n(\cdot, W_n(f)) - S_n(\cdot, f) \right\|_{p(\cdot),\omega} \leq \\ &\leq \left\| S_n(\cdot, W_n(f)) - W_n(\cdot, f) \right\|_{p(\cdot),\omega} + \\ &+ \left\| W_n(\cdot, f) - f(\cdot) \right\|_{p(\cdot),\omega} + \left\| f(\cdot) - S_n(\cdot, f) \right\|_{p(\cdot),\omega} \leq \\ &\leq c E_n \left(W_n(f) \right)_{p(\cdot),\omega} + c E_n (f)_{p(\cdot),\omega} + C E_n (f)_{p(\cdot),\omega}. \end{aligned}$$

Since

$$E_n \left(W_n(f) \right)_{p(\cdot),\omega} \leq c E_n (f)_{p(\cdot),\omega}$$

we get

$$\left\| f_\alpha^\psi(\cdot) - (S_n(\cdot, f))_\alpha^\psi \right\|_{p(\cdot),\omega} \leq$$
$$\leq cE_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega} + c\left(\psi\left(n\right)\right)^{-1} E_n\left(W_n(f)\right)_{p(\cdot),\omega} +$$
$$+ cE_n\left(f\right)_{p(\cdot),\omega} \leq cE_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega} + c\left(\psi\left(n\right)\right)^{-1} E_n\left(f\right)_{p(\cdot),\omega}.$$

Since by Theorem 1.1 in [1]

$$E_n\left(f\right)_{p(\cdot),\omega} \leq c\psi\left(n+1\right) E_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega}$$

and we obtain

$$\left\| f_\alpha^\psi(\cdot) - (S_n(\cdot, f))_\alpha^\psi \right\|_{p(\cdot),\omega} \leq cE_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega}. \qquad \square$$

Now we give an inverse theorem for (α, ψ) differentiable functions in weighted variable exponent spaces. The next theorem was proved in [2] and changing in the above Theorem 1.2 forced us to change the hypotesis. The proof will not change.

Theorem 1.3. *Let* $p \in \mathcal{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ *for some* $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$ *and* $f \in L_\omega^{p(\cdot)}$. *If* $\psi \in \mathfrak{M}_0$, $r \in (0, \infty)$ *and*

$$\sum_{\nu=1}^{\infty} \left(\nu\psi\left(\nu\right)\right)^{-1} E_\nu\left(f\right)_{p(\cdot),\omega} < \infty, \tag{1.1}$$

then there exist constants $c, C > 0$ *dependent only on* ψ, r *and* p *such that*

$$\Omega_r\left(f_\alpha^\psi, \frac{1}{n}\right)_{p(\cdot),\omega} \leq \frac{c}{n^{2r}} \sum_{\nu=1}^{n} \nu^{2r-1} \left(\psi\left(\nu\right)\right)^{-1} E_\nu\left(f\right)_{p(\cdot),\omega} +$$
$$+ C \sum_{\nu=n+1}^{\infty} \left(\nu\psi\left(\nu\right)\right)^{-1} E_\nu\left(f\right)_{p(\cdot),\omega}$$

hold.

Proof of Theorem 1.3. The proof is the same as in the proof of Theorem 1.2 of [2]. So we will outline only. First of all we have

$$\Omega_r\left(T_{2^{m+1}}, \delta\right)_{p(\cdot),\omega} \leq c\delta^{2r} \left\| T_{2^{m+1}}^{(2r)} \right\|_{p(\cdot),\omega}. \tag{1.2}$$

Indeed using

$$\left(1 - \frac{\sin x}{x}\right) \leq x^2 \text{ for } x \in \mathbb{R}^+$$

and Marcinkiewicz Multiplier theorem for weighted variable exponent Lebesgue spaces we get

$$\Omega_r(T_n,\delta)_{p(\cdot),\omega} = \sup_{0<h_i,t<\delta}\left\|\prod_{i=1}^{[r]}(I-\mathcal{A}_{h_i})\,\sigma_t^{\{r\}}T_n\right\|_{p(\cdot),\omega} =$$

$$= \sup_{0<h_i,t<\delta}\left\|\sum_{k=1}^{n}\Big(1-\frac{\sin kh_1}{kh_1}\Big)\ldots\Big(1-\frac{\sin kh_{[r]}}{kh_{[r]}}\Big)\Big(1-\frac{\sin kt}{kt}\Big)^{\{r\}}A_k(x,T_n)\right\|_{p(\cdot),\omega} \le$$

$$\le c\sup_{0<h_i,t<\delta} h_1^2\ldots h_{[r]}^2 t^{2\{r\}}\left\|\sum_{k=1}^{n}k^{2[r]}k^{2\{r\}}A_k(x,T_n)\right\|_{p(\cdot),\omega} \le$$

$$\le c\delta^{2r}\left\|\sum_{k=1}^{n}k^{2r}A_k(x,T_n)\right\|_{p(\cdot),\omega} =$$

$$= c\delta^{2r}\left\|\sum_{k=1}^{n}k^{2r}\Big[A_k\Big(x+\frac{2r\pi}{2k},T_n\Big)\cos r\pi + A_k\Big(x+\frac{2r\pi}{2k},\widetilde{T_n}\Big)\sin r\pi\Big]\right\|_{p(\cdot),\omega}.$$

Since

$$A_k\left(x,T_n^{(2r)}\right) = k^{2r}A_k\Big(x+\frac{2r\pi}{2k},T_n\Big)$$

we get

$$\Omega_r(T_n,\delta)_{p(\cdot),\omega} \le c\delta^{2r}\left(\left\|T_n^{(2r)}\right\|_{p(\cdot),\omega} + \left\|\left(\widetilde{T_n}\right)^{(2r)}\right\|_{p(\cdot),\omega}\right).$$

Now using the boundedness of conjugate operator $f\to\tilde{f}$ and $\left(\widetilde{T_n}\right)^{(2r)} = \widetilde{T_n^{(2r)}}$ we conclude

$$\Omega_r\Big(T_n,\delta\Big)_{p(\cdot),\omega} \le c\delta^{2r}\left\|T_n^{(2r)}\right\|_{p(\cdot),\omega}.$$

Using last inequality we get by standard computations that

$$\Omega_r\Big(f,\frac{1}{n}\Big)_{p(\cdot),\omega} \le \frac{c}{n^{2r}}\sum_{\nu=1}^{n}\nu^{2r-1}E_{\nu-1}(f)_{p(\cdot),\omega}. \tag{1.3}$$

Hence we have

$$\Omega_r\Big(f_\alpha^\psi,\frac{1}{n}\Big)_{p(\cdot),\omega} \le \frac{c}{n^{2r}}\sum_{\nu=1}^{n}\nu^{2r-1}E_{\nu-1}\left(f_\alpha^\psi\right)_{p(\cdot),\omega}.$$

Using Theorem 1.3 of [1]

$$E_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega} \le c\left((\psi(n))^{-1}E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty}(\nu\psi(\nu))^{-1}E_\nu(f)_{p(\cdot),\omega}\right)$$

and therefore the required result

$$\Omega_r\left(f_\alpha^\psi, \frac{1}{n}\right)_{p(\cdot),\omega} \leq \frac{c}{n^{2r}} \sum_{\nu=1}^{n} \nu^{2r-1} \left(\psi\left(\nu\right)\right)^{-1} E_\nu\left(f\right)_{p(\cdot),\omega} + \\ +C \sum_{\nu=n+1}^{\infty} \left(\nu\psi\left(\nu\right)\right)^{-1} E_\nu\left(f\right)_{p(\cdot),\omega}$$

follows. □

Note that the latter estimate in refined form is given in [2] (see Theorem 1.3).

Namely, there the following statement is proved.

Theorem 1.4. *Let* $p \in \mathcal{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ *for some* $p_0 \in (1, p_*(\boldsymbol{T}))$. *Suppose that* $\alpha \in \mathbb{R}$, $\psi \in \mathfrak{M}_0$, $\gamma := \min\{2, p_*\}$, $r \in (0,\infty)$ *and*

$$\sum_{\nu=1}^{\infty} \left(\nu\left(\psi\left(\nu\right)\right)^\gamma\right)^{-1} \left(E_\nu\left(f\right)_{p(\cdot),\omega}\right)^\gamma < \infty.$$

Then there exist positive constants c *and* C *depending only on* ψ, r *and* p *such that the inequality*

$$\Omega_r\left(f_\alpha^\psi, \frac{1}{n}\right)_{p(\cdot),\omega} \leq \frac{c}{n^{2r}} \left(\sum_{\nu=1}^{n} \nu^{2\gamma r} \left(\nu\left(\psi\left(\nu\right)\right)^\gamma\right)^{-1} \left(E_\nu\left(f\right)_{p(\cdot),\omega}\right)^\gamma\right)^{1/\gamma} + \\ +C\left(\sum_{\nu=n+1}^{\infty} \left(\nu\left(\psi\left(\nu\right)\right)^\gamma\right)^{-1} \left(E_\nu\left(f\right)_{p(\cdot),\omega}\right)^\gamma\right)^{1/\gamma}$$

holds.

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