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GLOBAL ASYMPTOTIC STABILITY FOR NONLINEAR MULTI-DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. The aim of this paper is to study the existence and the stability of solutions for a system of nonlinear delay partial differential equations of fractional order. The Schauder fixed point theorem for the existence of solutions is used, and it is proved that all solutions are globally asymptotically stable.

რეზიუმე. ნაშრომის მიზანს წარმოადგენს არაწრფივი დაგვიანებული წილადური ხარისხის კერძო წარმოებულებიანი დიფერენციალური განტოლებებისათვის ამონახსნების არსებობისა და მდგრადობის შესწავლა. გამოყენებულია შაუდერის უძრავი წერტილის თეორემა ამონახსნების არსებობისათვის და დამტკიცებულია, რომ ყველა ამონახსნი არის გლობალურად ასიმპტოტურად მდგრადი.

1. Introduction

The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bodes analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [3], Baleanu et al. [10], Diethelm [16], Hilfer [18], Kilbas et al. [19], Lakshmikantham et al. [24], Podlubny [28], Tarasov [30], and the papers by Abbas et al. [1, 2, 4], Agarwal et al. [5, 6, 7, 8], Araya et al. [9], Cuevas et al. [13, 14], Mophu and N'Guérékata [25, 26], N'Guérékata

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[27], Lakshmikantham et al. [20, 21, 22, 23], Vityuk and Golushkov [31]. Recently interesting results of the stability of the solutions of various classes of integral equations of fractional order have obtained by Banaś et al. [11], Darwish et al. [15] and the references therein.

In this paper we establish a sufficient condition for the existence and the stability of solutions of the following system of nonlinear delay differential equations of fractional order of the form

$$^{c}D_{\theta}^{r}u(t,x)=p(t,x)-q(t,x)f\big(t,x,u(t-\tau_1,x-\xi_1),\dots,u(t-\tau_m,x-\xi_m)\big)$$
$$\text{for} \quad (t,x)\in J:=\mathbb{R}_+\times[0,b], \tag{1}$$

$$u(t,x)=\Phi(t,x) \quad \text{for} \quad (t,x)\in\tilde{J}:=[-T,\infty)\times[-\xi,b]\backslash(0,\infty)\times(0,b], \tag{2}$$

$$\begin{cases} u(t,0)=\varphi(t), & t\in[0,\infty),\\ u(0,x)=\psi(x), & x\in[0,b], \end{cases} \tag{3}$$

where $b>0$, $\theta=(0,0)$, $\mathbb{R}_+=[0,\infty)$, $\tau_i,\xi_i\geq 0$, $i=1,\dots,m$, $T=\max_{i=1,\dots,m}\{\tau_i\}$, $\xi=\max_{i=1,\dots,m}\{\xi_i\}$, $^{c}D_{\theta}^{r}$ is the Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $p,q:J\to\mathbb{R}$, $f:J\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, $\varphi:\mathbb{R}_+\to\mathbb{R}$, $\psi:[0,b]\to\mathbb{R}$ are absolutely continuous functions with $\lim_{t\to\infty}\varphi(t)=0$ and $\psi(x)=\varphi(0)$ for each $x\in[0,b]$, and $\Phi:\tilde{J}\to\mathbb{R}$ is continuous with $\varphi(t)=\Phi(t,0)$ for each $t\in\mathbb{R}_+$, and $\psi(x)=\Phi(0,x)$ for each $x\in[0,b]$.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([0,a]\times[0,b])$, $a,b>0$, we denote the space of Lebesgue-integrable functions $u:[0,a]\times[0,b]\to\mathbb{R}$ with the norm

$$\|u\|_1=\int_0^a\int_0^b|u(t,x)|dx\,dt.$$

By $BC:=BC([-T,\infty)\times[-\xi,b])$ we denote the Banach space of all bounded and continuous functions from $[-T,\infty)\times[-\xi,b]$ into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC}=\sup_{(t,x)\in[-T,\infty)\times[-\xi,b]}|u(t,x)|.$$

For $u_0\in BC$ and $\eta\in(0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1 ([31])**.** Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1([0,a]\times[0,b])$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} u(s,\tau)\,ds\,d\tau,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt, \quad \nu > 0.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x), \ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(\tau,s)\,ds\,d\tau$$
$$\text{for almost all} \quad (t,x) \in [0,a] \times [0,b],$$

where $\sigma = (1,1)$. For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1([0,a]\times[0,b])$. Note also that when $u \in C([0,a]\times[0,b])$, then $(I_\theta^r u) \in C([0,a]\times[0,b])$, moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0, \quad t \in [0,a], \quad x \in [0,b].$$

Example 2.2. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}$$
$$\text{for almost all} \quad (t,x) \in [0,a] \times [0,b].$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1]\times(0,1]$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2.3 ([31])**.** Let $r \in (0,1]\times(0,1]$ and $u \in L^1([0,a]\times[0,b])$. The Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_\theta^r u(t,x) = (I_\theta^{1-r} D_{tx}^2 u)(t,x)$.

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x) \ \text{ for almost all } \ (t,x) \in [0,a]\times[0,b].$$

Example 2.4. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1,r_2) \in (0,1]\times(0,1]$, then

$${}^cD_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}$$
$$\text{for almost all} \quad (t,x) \in [0,a] \times [0,b].$$

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{4}$$

Inspired by the definition of the attractivity of solutions of integral equations (for instance [2]), we introduce the following concept of attractivity of solutions for equation (4).

Definition 2.5. Solutions of equation (4) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of equations (4) belonging to $B(u_0,\eta) \cap \Omega$, we have that, for each $x \in [0,b]$,

$$\lim_{t\to\infty} (v(t,x) - w(t,x)) = 0. \tag{5}$$

When the limit (5) is uniform with respect to $B(u_0,\eta) \cap \Omega$, solutions of equation (4) are said to be uniformly locally attractive (or equivalently that solutions of (4) are locally asymptotically stable).

Definition 2.6 ([2])**.** The solution $v = v(t)$ of equation (4) is said to be globally attractive if (5) hold for each solution $w = w(t)$ of (4). If condition (5) is satisfied uniformly with respect to the set Ω, solutions of equation (4) are said to be globally asymptotically stable (or uniformly globally attractive).

Lemma 2.7 ([12], p. 62)**.** *Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*

(a) *D is uniformly bounded in BC,*

(b) *The functions belonging to D are almost equicontinuous on $\mathbb{R}_+ \times [0,b]$, i.e. equicontinuous on every compact of $\mathbb{R}_+ \times [0,b]$,*

(c) *The functions from D are equiconvergent, that is, given $\epsilon > 0$, $x \in [0,b]$ there corresponds $T(\epsilon,x) > 0$ such that $|u(t,x) - \lim_{t\to\infty} u(t,x)| < \epsilon$ for any $t \geq T(\epsilon,x)$ and $u \in D$.*

3. Main Results

Let us start by defining what we mean by a solution of of the problem (1)–(3).

Definition 3.1. A function $u \in BC$ is said to be a solution of (1)–(3) if u satisfies equation (1) on J, equation (2) on $\tilde{J}$ and condition (3) is satisfied.

Lemma 3.2 ([1])**.** *Let $f \in L^1([0,a] \times [0,b])$, $a, b > 0$. A function $u \in AC([0,a] \times [0,b])$ is a solution of problem*

$$\begin{cases} ({}^cD_\theta^r u)(t,x) = f(t,x), \ (t,x) \in [0,a] \times [0,b], \\ u(t,0) = \varphi(t), \ t \in [0,a], \ \ u(0,x) = \psi(x), \ x \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(t,x)$ *satisfies*

$$u(t,x)=\mu(t,x)+(I_\theta^r f)(t,x), \quad (t,x)\in[0,a]\times[0,b],$$

where

$$\mu(t,x)=\varphi(t)+\psi(x)-\varphi(0).$$

The following hypotheses will be used in the sequel:

(H_1) The functions Φ, p and q are in BC. Moreover, assume that

$$\lim_{t\to\infty} I_\theta^r p(t,x)=0, \quad x\in[0,b],$$

(H_2) There exist continuous functions $d_i:\mathbb{R}_+\times[0,b]\to\mathbb{R}_+$ such that

$$\left(1+\sum_{i=1}^m |u_i|\right)\big|f(t,x,u_1,u_2,\dots,u_m)\big|\le\sum_{i=1}^m |u_i| d_i(t,x)$$

for $\quad (t,x)\in\mathbb{R}_+\times[0,b] \quad$ and for $\quad u_i\in\mathbb{R}, \quad i=1,\dots,m.$

Moreover, assume that

$$\lim_{t\to\infty} I_\theta^r d_i(t,x)=0, \quad x\in[0,b], \quad i=1,\dots,m.$$

Remark 3.3. Set

$$\Phi^*:=\sup_{(t,x)\in\tilde{J}}\Phi(t,x),\ \varphi^*:=\sup_{t\in\mathbb{R}_+}\varphi(t), \quad p^*:=\sup_{(t,x)\in J} I_\theta^r p(t,x),$$

$$q^*:=\sup_{(t,x)\in J} q(t,x) \quad \text{and} \quad d_i^*:=\sup_{(t,x)\in J} I_\theta^r d_i(t,x), \quad i=1,\dots,m.$$

¿From hypotheses, we infer that Φ^*, φ^*, p^*, q^* and d_i^*, $i=1,\dots,m$ are finite.

Now, we shall prove the following theorem concerning the existence and the stability of a solution of problem (1)–(3).

Theorem 3.4. *Assume that the hypotheses* (H_1) *and* (H_2) *hold, then the problem* (1)–(3) *has at least one solution in the space* BC. *Moreover, solutions of problem* (1)–(3) *are globally asymptotically stable.*

Proof. Let us define the operator N such that, for any $u\in BC$,

$$(Nu)(t,x)=\begin{cases}\Phi(t,x), & (t,x)\in\tilde{J},\\ \varphi(t)+I_\theta^r\Big[p(t,x)- \\ \quad -q(t,x)f\big(t,x,u(t-\tau_1,x-\xi_1),\dots \\ \quad \dots,u(t-\tau_m,x-\xi_m)\big)\Big], & (t,x)\in J.\end{cases} \tag{6}$$

The operator N maps BC into BC. Indeed the map $N(u)$ is continuous on $[-T,\infty)\times[-\xi,b]$ for any $u\in BC$, and for each $(t,x)\in J$ we have

$$\big|(Nu)(t,x)\big|\le|\varphi(t)|+I_\theta^r\Big|p(t,x)-q(t,x)\times$$

$$\times\, g\big(t,x,u(t-\tau_1,x-\xi_1),\dots,u(t-\tau_m,x-\xi_m)\big)\Big| \le$$

$$\le |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_1}\times$$

$$\times\left[|p(t,x)| + |q(t,x)|\left(\sum_{i=1}^m \big|u(\tau-\tau_i,s-\xi_i)\big| d_i(\tau,s)\right)\times\right.$$

$$\left.\times\left(1+\sum_{i=1}^m \big|u(\tau-\tau_i,s-\xi_i)\big|\right)^{-1}\right] ds\, d\tau \le$$

$$\le \varphi^* + p^* + q^*\sum_{i=1}^m d_i^*,$$

and for $(t,x)\in\tilde{J}$, we have

$$\big|(Nu)(t,x)\big| = \big|\Phi(t,x)\big| \le \Phi^*.$$

Thus,

$$\big\|N(u)\big\|_{BC} \le \max\left\{\Phi^*, \varphi^* + p^* + q^*\sum_{i=1}^m d_i^*\right\} := \eta. \tag{7}$$

Hence, $N(u)\in BC$. This proves that the operator N maps BC into itself.

By Lemma 3.2, the problem of finding the solutions of the problem (1)–(3) is reduced to finding the solutions of the operator equation $N(u)=u$. Equation (7) implies that N transforms the ball $B_\eta := B(0,\eta)$ into itself. We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [17]. The proof will be given in several steps.

Step 1: *N is continuous.* Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n\to u$ in B_η. Then, for each $(t,x)\in[-T,\infty)\times[-\xi,b]$, we have

$$\big|(Nu_n)(t,x)-(Nu)(t,x)\big| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}|q(\tau,s)|\times$$

$$\times\Big|f\big(\tau,s,u_n(\tau-\tau_1,s-\xi_1),\dots,u_n(\tau-\tau_m,s-\xi_m)\big)-$$

$$- f\big(\tau,s,u(\tau-\tau_1,s-\xi_1),\dots,u(\tau-\tau_m,s-\xi_m)\big)\Big| ds\, d\tau. \tag{8}$$

Case 1. If $(t,x)\in\tilde{J}\cup([0,a]\times[0,b])$, $a>0$, then, since $u_n\to u$ as $n\to\infty$ and f is continuous, (8) gives

$$\big\|N(u_n)-N(u)\big\|_{BC}\to 0 \quad \text{as} \quad n\to\infty.$$

Case 2. If $(t,x)\in(a,\infty)\times[0,b]$, $a>0$, then from (H_2) and (8), we get

$$\begin{aligned}&\big|(Nu_n)(t,x)-(Nu)(t,x)\big|\le\\&\le\frac{2q^*}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^t\int\limits_0^x(t-\tau)^{r_1-1}(x-s)^{r_2-1}\left(\sum_{i=1}^m\big|u(\tau-\tau_i,s-\xi_i)\big|d_i(\tau,s)\right)\times\\&\times\left(1+\sum_{i=1}^m\big|u_i(\tau-\tau_i,s-\xi_i)\big|\right)^{-1}ds\,d\tau\le\\&\le\sum_{i=1}^m\frac{2q^*}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^t\int\limits_0^x(t-\tau)^{r_1-1}(x-s)^{r_2-1}d_i(\tau,s)ds\,d\tau\le\\&\le 2q^*\sum_{i=1}^m I_\theta^r d_i(t,x).\end{aligned}$$

Then

$$\big|(Nu_n)(t,x)-(Nu)(t,x)\big|\le 2q^*\sum_{i=1}^m I_\theta^r d_i(t,x). \tag{9}$$

Thus (9) gives

$$\big\|N(u_n)-N(u)\big\|_{BC}\to 0\quad\text{as}\quad n\to\infty.$$

Step 2: *$N(B_\eta)$ is uniformly bounded.* This is clear since $N(B_\eta)\subset B_\eta$ and B_η is bounded.

Step 3: *$N(B_\eta)$ is equicontinuous on every compact subset $[-T,a]\times[-\xi,b]$ of $[-T,a]\times[-\xi,\infty)$, $a>0$.* Let (t_1,x_1), $(t_2,x_2)\in[0,a]\times[0,b]$, $t_1<t_2$, $x_1<x_2$ and let $u\in B_\eta$. Thus we have

$$\begin{aligned}&\big|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)\big|\le\big|\varphi(t_2)-\varphi(t_1)\big|+\big|p(t_2,x_2)-p(t_1,x_1)\big|+\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^{t_1}\int\limits_0^{x_1}\Big[(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}-(t_1-\tau)^{r_1-1}(x_1-s)^{r_2-1}\Big]\times\\&\times|q(\tau,s)|\cdot\Big|f\big(\tau,s,I_{0,s}^{r_2}u(\tau,s),u(\tau-\tau_1,s-\xi_1),\dots,u(\tau-\tau_m,s-\xi_m)\big)\Big|ds\,d\tau+\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int\limits_{t_1}^{t_2}\int\limits_{x_1}^{x_2}(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}|q(\tau,s)|\times\\&\times\Big|f\big(\tau,s,I_{0,s}^{r_2}u(\tau,s),u(\tau-\tau_1,s-\xi_1),\dots,u(\tau-\tau_m,s-\xi_m)\big)\Big|ds\,d\tau+\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}|q(\tau,s)|\times$$

$$\times\left|f\big(\tau,s,I_{0,s}^{r_2}u(\tau,s),u(\tau-\tau_1,s-\xi_1),\dots,u(\tau-\tau_m,s-\xi_m)\big)\right|ds\,d\tau+$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}|q(\tau,s)|\times$$

$$\times\left|f\big(\tau,s,I_{0,s}^{r_2}u(\tau,s),u(\tau-\tau_1,s-\xi_1),\dots,u(\tau-\tau_m,s-\xi_m)\big)\right|ds\,d\tau.$$

Thus

$$\big|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)\big|\le\big|\varphi(t_2)-\varphi(t_1)\big|+\big|p(t_2,x_2)-p(t_1,x_1)\big|+$$

$$+\frac{q^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}\big[(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}-(t_1-\tau)^{r_1-1}(x_1-s)^{r_2-1}\big]\times$$

$$\times\sum_{i=0}^{m}d_i(\tau,s)ds\,d\tau+\frac{q^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}\sum_{i=0}^{m}d_i(\tau,s)ds\,d\tau+$$

$$+\frac{q^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}\sum_{i=0}^{m}d_i(\tau,s)ds\,d\tau+$$

$$+\frac{q^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1}\sum_{i=0}^{m}d_i(\tau,s)ds\,d\tau.$$

From continuity of $\varphi,p,d_i,\ i=0,\dots,m$ and as $t_1\to t_2$ and $x_1\to x_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1<t_2<0$, $x_1<x_2<0$ and $t_1\le 0\le t_2$, $x_1\le 0\le x_2$ is obvious.

Step 4: *$N(B_\eta)$ is equiconvergent.* Let $(t,x)\in\mathbb{R}_+\times[0,b]$ and $u\in B_\eta$, then we have

$$\big|(Nu)(t,x)\big|\le|\varphi(t)|+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t}\int_0^{x}(t-\tau)^{r_1-1}(x-s)^{r_2-1}\Big[p(\tau,s)-q(\tau,s)\times\right.$$

$$\left.\times f(\tau,s,u(\tau-\tau_1,s-\xi_1),\dots,u(\tau-\tau_m,s-\xi_m))\Big]ds\,d\tau\right|\le$$

$$\le|\varphi(t)|+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t}\int_0^{x}(t-\tau)^{r_1-1}(x-s)^{r_2-1}|p(\tau,s)|ds\,d\tau+$$

$$+\frac{q^*}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^t\int\limits_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}\sum_{i=0}^m d_i(\tau,s)ds\,d\tau \leq$$

$$\leq |\varphi(t)| + I_\theta^r p(t,x) + q^*\sum_{i=0}^m I_\theta^r d_i(t,x).$$

Thus, for each $x\in[0,b]$, we get

$$\big|(Nu)(t,x)\big| \to 0 \quad \text{as} \quad t\to+\infty.$$

Hence,

$$\big|(Nu)(t,x)-(Nu)(+\infty,x)\big| \to 0 \quad \text{as} \quad t\to+\infty.$$

As a consequence of Steps 1 to 4 together with the Lemma 2.7, we can conclude that $N : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem [17], we deduce that N has a fixed point u which is a solution of the problem (1)–(3).

Step 5: *The global asymptotic stability of solutions.* Now we investigate the global asymptotic stability of solutions of problem (1)–(3). Let us assume that u and v are solutions of problem (1)–(3). Then for each $(t,x)\in J$

$$\big|u(t,x)-v(t,x)\big| = \big|(Nu)(t,x)-(Nv)(t,x)\big| \leq$$

$$\leq \frac{q^*}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^t\int\limits_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}\times$$

$$\times\Big|f\big(\tau,s,u(\tau-\tau_1,s-\xi_1),\dots,u(\tau-\tau_m,s-\xi_m)\big)-$$

$$-f\big(\tau,s,v(\tau-\tau_1,s-\xi_1),\dots,v(\tau-\tau_m,s-\xi_m)\big)\Big|ds\,d\tau \leq$$

$$\leq \frac{2q^*}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^t\int\limits_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}\sum_{i=1}^m d_i(\tau,s)ds\,d\tau \leq$$

$$\leq 2q^*\sum_{i=1}^m I_\theta^r d_i(t,x). \tag{10}$$

By using (10) and the fact that $I_\theta^r d_i(t,x)\to 0$ as $t\to\infty$, $i=0,\dots,m$ we deduce that

$$\lim_{t\to\infty}\big|u(t,x)-v(t,x)\big| = 0.$$

Consequently, all solutions of problem (1)–(3) are globally asymptotically stable. □

4. An Example

As an application and to illustrate our results, we consider the following system of delay differential equations of fractional order

$$^{c}D_{\theta}^{r}u(t,x)=p(t,x)-q(t,x)f\Big(t,x,u\Big(t-1,x-\frac{1}{4}\Big),u\Big(t-\frac{2}{3},x-\frac{1}{5}\Big)\Big)$$

$$\text{for}\quad (t,x)\in J:=[0,\infty)\times[0,1], \qquad (11)$$

$$u(t,x)=e^{-t}\ \text{ for }\ (t,x)\in\tilde{J}:=[-1,\infty)\times\Big[-\frac{1}{4},1\Big]\backslash(0,\infty)\times(0,1], \qquad (12)$$

$$\begin{cases} u(t,0)=e^{-t}, & t\in[0,\infty),\\ u(0,x)=1, & x\in[0,1], \end{cases} \qquad (13)$$

where $r=(r_1,r_2)=(\frac{1}{4},\frac{1}{2})$, $q(t,x)=\frac{1}{1+t^2+x^2}$, $(t,x)\in(0,\infty)\times[0,1]$,

$$\begin{cases} p(t,x)=xt^{\frac{-3}{4}}\sin t, & (t,x)\in(0,\infty)\times[0,1],\\ p(0,x)=0, & x\in[0,1], \end{cases}$$

and

$$\begin{cases} f(t,x,u,v)=\dfrac{xt^{\frac{-3}{4}}\left(|u|\sin t+|v|e^{-\frac{1}{t}}\right)}{2+|u|+|v|}, & (t,x)\in(0,\infty)\times[0,1] \text{ and } u,v\in\mathbb{R},\\ f(0,x,u,v)=0, & x\in[0,1] \text{ and } u,v\in\mathbb{R}. \end{cases}$$

We have for each $(t,x)\in[0,\infty)\times[0,1]$, $\Phi(t,x)=e^{-t}$ and $\varphi(t)=e^{-t}\to 0$ as $t\to\infty$, $\Phi^*=\varphi^*=1$, $p^*=1$, then the assumption (H_1) is satisfies. Let us notice that the function f satisfies assumption (H_2), where

$$\begin{cases} d_1(t,x)=xt^{\frac{-3}{4}}|\sin t|, & (t,x)\in(0,\infty)\times[0,1],\\ d_1(0,x)=0, & x\in[0,1], \end{cases}$$

and

$$\begin{cases} d_2(t,x)=xt^{\frac{-3}{4}}e^{-\frac{1}{t}}, & (t,x)\in(0,\infty)\times[0,1],\\ d_2(0,x)=0, & x\in[0,1]. \end{cases}$$

Also, for each $x\in[0,1]$, we get

$$I_{\theta}^{r}p(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-\tau)^{r_1-1}(x-s)^{r_2-1}p(\tau,s)ds\,d\tau=$$

$$=\frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}\int_0^t\int_0^x(t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}}s\tau^{\frac{-3}{4}}|\sin\tau|ds\,d\tau\le$$

$$\leq \frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}\int_0^t\int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}}s\tau^{\frac{-3}{4}}ds\,d\tau =$$

$$= \frac{\Gamma(\frac{1}{4})\Gamma(2)}{\Gamma(\frac{5}{4})\Gamma(\frac{5}{2})}t^{\frac{-1}{2}}x^{\frac{3}{2}} \to 0 \text{ as } t\to\infty,$$

$I_\theta^r d_1(t,x) = I_\theta^r p(t,x)$ and

$$I_\theta^r d_2(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}d_2(\tau,s)ds\,d\tau =$$

$$= \frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}\int_0^t\int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}}s\tau^{\frac{-3}{4}}e^{-\frac{1}{\tau}}ds\,d\tau \leq$$

$$\leq \frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}\int_0^t\int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}}s\tau^{\frac{-3}{4}}ds\,d\tau =$$

$$= \frac{\Gamma(\frac{1}{4})\Gamma(2)}{\Gamma(\frac{5}{4})\Gamma(\frac{5}{2})}t^{\frac{-1}{2}}x^{\frac{3}{2}} \to 0 \text{ as } t\to\infty.$$

Thus

$$\lim_{t\to\infty} I_\theta^r p(t,x) = \lim_{t\to\infty} I_\theta^r d_i(t,x) = 0, \quad x\in[0,1], \quad i=0,1,2.$$

Hence by Theorem 3.4, the problem (11)–(13) has a solution defined on $[-1,\infty)\times[-\frac{1}{4},1]$ and all solutions are globally asymptotically stable on $[-1,\infty)\times[-\frac{1}{4},1]$.

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