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ON BOUNDEDNESS OF THE MULTIFUNCTIONAL BERGMAN TYPE OPERATORS IN TUBE DOMAINS OVER SYMMETRIC CONES

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Abstract. We introduce and study new multifunctional Bergman type integral operators in tube domains over symmetric cones. We obtain a new sufficient condition for the continuity of the Bergman-type projection in tube domains over symmetric cones using multy-functional embeddings.

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1. Introduction and Statements of the Results

Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional euclidean space V. Following the notation of [6] we denote the rank of the cone Ω by r and by Δ the determinant function on V. Letting $V = \mathbb{R}^n$, we have as an example of a symmetric cone on $\mathbb{R}^n$ the Lorentz cone Λ_n which is a rank 2 cone defined for $n \geq 3$ by

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, y_1 > 0\}.$$

The determinant function in this case is given by the Lorentz form

$$\Delta(y) = y_1^2 - \cdots - y_n^2.$$

Let us introduce some convenient notation regarding multi-indices.

If $t = (t_1, \ldots, t_r)$, then $t^\star = (t_r, \ldots, t_1)$ and, for $a \in \mathbb{R}$, $t + a = (t_1 + a, \ldots, t_n + a)$. Also, if $t, k \in \mathbb{R}^n$, then $t < k$ means $t_j < k_j$ for all $1 \leq j \leq r$.

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We are going to use the following multi-index,where d can be determined from equation below via r,n,mentioned above.

$$g_0 = \left((j-1)\frac{d}{2}\right)_{1\le j\le r}, \quad \text{where} \quad (r-1)\frac{d}{2} = \frac{n}{r} - 1.$$

For $1 \le p, q < +\infty$ and $\nu \in \mathbb{R}^r$, we denote by $A^{p,q}_\nu(T_\Omega)$ the mixed-norm Bergman space consisting of analytic functions f in T_Ω such that

$$\|f\|_{L^{p,q}_\nu} = \left(\int\limits_\Omega \left(\int\limits_V |F(x+iy)|^p d\,x\right)^{q/p} \Delta_\nu(y)\frac{d\,y}{\Delta(y)^{n/r}}\right)^{1/q} < \infty,$$

where Δ_ν is the generalized power function to be defined in the next section. The space $A^{p,q}_\nu(T_\Omega)$ is nontrivial if and only if $\nu > g_0$, see [5]. When $p = q$ we write $A^{p,q}_\nu(T_\Omega) = A^p_\nu(T_\Omega)$; the classical Bergman space $A^p(\Omega)$ corresponds to $\nu = (n/r, \dots, n/r)$.

The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$ and it is given by the following integral formula

$$P_\nu f(z) = d_\nu \int\limits_{T_\Omega} B_\nu(z,w) f(w) dV_\nu(w), \tag{1}$$

where $B_\nu(z,w) = c_\nu \Delta^{-(\nu+\frac{n}{r})}((z-\overline{w})/i)$ is the Bergman reproducing kernel for A^2_ν, see [6]. Here we used notation $dV_\nu(w) = \Delta^{\nu-\frac{n}{r}}(v)dudv$, where $w = u + iv \in T_\Omega$.

The problem of boundedness of the Bergman projection on tube domains over symmetric cones has been considered by several authors (see [1], [4], [2], [3] and references therein) and still remains open. The best known results have been obtained in [7] in the setting of the light cone. Recently, an equivalent condition for the boundedness of the Bergman projection in terms of Hardy-type inequalities and duality was obtained in [3]. We introduce here the operators T_β, $\beta = (\beta_1, \dots, \beta_m)$ which generalize the Bergman projection and are defined by

$$T_\beta(\overrightarrow{f})(\overrightarrow{z}) = \int\limits_{T_\Omega} \frac{\left(\prod\limits_{j=1}^m f_j(z)\right)\Delta^{\frac{1}{m}\sum\limits_{j=1}^m \beta_j}(\Im z)}{\prod\limits_{j=1}^m \Delta^{\frac{1}{m}(\frac{n}{r}+\beta_j)}(\frac{z_j-\overline{z}}{i})} \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)},$$

where $\overrightarrow{f} = (f_1, \dots, f_n)$, $\overrightarrow{z} = (z_1, \dots, z_n)$, $z_j \in T_\Omega$ and $f_j \in L^1_{\text{loc}}(T_\Omega)$ for $1 \le j \le m$. Combining classical arguments with integrability properties of the Bergman kernel and determinant function we obtain the following sufficient condition for the boundedness of the operator T_β from the product space

$$\prod_{k=1}^{m} L^p_{m\nu_k+(m-1)\frac{n}{r}}(T_\Omega) = L^p_{m\nu_1+(m-1)\frac{n}{r}}(T_\Omega) \times \cdots \times L^p_{m\nu_m+(m-1)\frac{n}{r}}(T_\Omega)$$

to the space $L^p((T_\Omega)^m, \prod\limits_{k=1}^{m} \Delta^{\nu_k - \frac{n}{r}} dV(z_k))$. The idea to consider such multifunctional operator is motivated by [8]. Some results of this paper are analogous to results of [8] proven in the case of the unit ball in $\mathbb{C}^n$. We note here that almost all multifunctional results of this paper are well known in the case $m = 1$. For example, the case $m = 1$ of the following theorem is contained in [4].

Theorem 1. *Let $\nu_k \in \mathbb{R}$, $k = 1, \dots, m$, $m > 1$, $1 \leq p < \infty$ and $\beta = (\beta_1, \dots, \beta_n)$. If the parameters satisfy the following conditions*

$$\frac{1}{m}\sum_{j=1}^{m} \beta_j > \frac{n}{r} - 1, \tag{2}$$

$$1 \leq p < 1 + m\left(\frac{\min_j \nu_j}{\frac{n}{r} - 1} - 1\right), \tag{3}$$

$$\min_j \beta_j > \frac{1}{m}\sum_{j=1}^{m} \beta_j - \frac{n}{rp} + \frac{m}{p}\left(2\frac{n}{r} - 1 + \max_j \nu_j\right), \tag{4}$$

then T_β is bounded from

$$\prod_{k=1}^{m} L^p_{m\nu_k+(m-1)\frac{n}{r}}(T_\Omega) \quad to \quad L^p((T_\Omega)^m, \prod_{k=1}^{m} \Delta^{\nu_k - \frac{n}{r}} dV(z_k)).$$

Among our applications of the above result, we obtain a sufficient condition for the boundedness of the Bergman projection in terms of the reproducing formula, which is new in this setting. More precisely, we prove the following theorem.

Theorem 2. *Let $\nu > \frac{n}{r} - 1$ and $1 < p < \infty$. If for any $f \in L^p_\nu(T_\Omega)$ the following representation formula holds*

$$P_\nu f(z_1) P_\nu f(z_2) = C_\beta \int\limits_{T_\Omega} \frac{f(z) P_\nu f(z) \Delta^{\beta - \frac{n}{r}}(\Im z)}{\Delta^{\frac{1}{2}(\frac{n}{r}+\frac{\beta}{2})}\left(\frac{z_1 - \overline{z}}{i}\right) \Delta^{\frac{1}{2}(\frac{n}{r}+\frac{\beta}{2})}\left(\frac{z_2 - \overline{z}}{i}\right)} dV(z) \tag{5}$$

for some sufficiently large β and all z_1, z_2 in T_Ω, then the Bergman projection P_ν is bounded on $L^p_\nu(T_\Omega)$.

In this theorem the weights ν and β are taken real, but the result generalizes directly to the vector weight case. The condition "β is sufficiently large" is related to the boundedness conditions for the Bergman kernal and determinant function. For example, a necessary condition for the boundedness of the Bergman projection P_β on $L^p_\nu(T_\Omega)$ is that the related Bergman kernel belongs to $L^{p',q'}_\nu(T_\Omega)$, where $1/p+1/p'=1$, $1/q+1/q'=1$, and this can only happen for large values of β for p,q and ν fixed, see [9].

Finally as usual, throughout this paper C or c denote positive constants, not necessarily the same at different occurences; dependence on parameters is indicated by subscripts. As usual given two various quantities A and B, the notation $A \lesssim B$ means that there is an absolute constant C such that $A \lesssim CB$. When both $A \lesssim B$ and $B \lesssim A$ hold we write $A \approx B$.

2. Preliminaries and Auxiliary Results

For reader's convenience, we collect in this section some definitions and results that are used in this paper, they are essentially contained in [6].

2.1. **Symmetric cones and the generalized determinant function.**

Let Ω be an irreducible open cone of rank r in an n-dimensional vector space V endowed with an inner product $(\cdot/\cdot)$ for which Ω is self-dual. Let $G(\Omega)$ be the group of transformations of Ω and G its identity component. It is well known that there is a subgroup H of G acting simply transitively on Ω, i.e. for every $y \in \Omega$ there is a unique $g \in H$ such that $y = g\mathbf{e}$, where $\mathbf{e}$ is a fixed element in Ω.

We recall that Ω induces in V a structure of Euclidean Jordan algebra with identity $\mathbf{e}$ such that

$$\overline{\Omega} = \{x^2 : x \in V\}.$$

We can identify (since Ω is irreducible) the inner product $(\cdot/\cdot)$ with the one given by the trace on V:

$$(x/y) = \mathrm{tr}\,(xy), \quad x, y \in V.$$

Let $\{c_1, \dots, c_r\}$ be a fixed Jordan frame in V and

$$V = \oplus_{1\le i\le j\le r} V_{i,j}$$

be its associated Pierce decomposition of V. We denote by $\Delta_1(x), \dots, \Delta_r(x)$ the principal minors of $x \in V$ with respect to the fixed Jordan frame $\{c_1, \dots, c_r\}$. More precisely, $\Delta_k(x)$ is the determinant of the projection $P_k x$ of x in the Jordan subalgebra $V^{(k)} = \oplus_{1\le i\le j\le k} V_{i,j}$. We have $\Delta = \Delta_r$ and $\Delta_k(x) > 0$, $1 \le k \le r$, when $x \in \Omega$. The generalized power function on Ω is defined as

$$\Delta_s(x) = \Delta_1^{s_1-s_2}(x)\Delta_2^{s_2-s_3}(x)\cdot \Delta_r^{s_r}(x), \;\; x \in \Omega, \; s \in \mathbb{C}^r.$$

Next, we recall the definition of generalized gamma function associated to Ω:

$$\Gamma_\Omega(s) = \int\limits_\Omega e^{-(\mathbf{e}/\xi)} \Delta_s(\xi) \Delta^{-n/r}(\xi) d\xi, \quad s = (s_1, \ldots, s_r) \in \mathbb{C}^r.$$

This integral converges if and only if $\Re s_j > (j-1)\frac{n/r-1}{r-1} = (j-1)\frac{d}{2}$ for all $1 \le j \le r$. In that case we have a formula:

$$\Gamma_\Omega(s) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - (j-1)\frac{d}{2}\right),$$

see Chapter VII of [6] for details. We have the following result on the Laplace transform of the generalized power function (see Proposition VII.1.2 and Proposition VII.1.6 in [6]).

Lemma 1. *Let $s = (s_1, \ldots, s_r) \in \mathbb{C}^n$ with $\Re s_j > (j-1)\frac{d}{2}$, $j = 1, \ldots, r$. Then, for all $y \in \Omega$ we have*

$$\int\limits_\Omega e^{-i(y/\xi)} \Delta_s(\xi) \Delta^{-n/r}(\xi) d\xi = \Gamma_\Omega(s) \Delta_s(y^{-1}) = \Gamma_\Omega(s) [\Delta^*_{s^\star}(y)]^{-1}.$$

Here, $y = h\mathbf{e}$ if and only if $y^{-1} = h^{*-1}\mathbf{e}$ with $h \in H$ and Δ^*_j, $j = 1, \ldots, r$ are the principal minors with respect to the rotated Jordan frame $\{c_1, \ldots, c_r\}$.

2.2. **Bergman spaces and integrability of the Bergman kernel function.** In this section we formulate and prove main results of this note.

Let us recall some estimates for the functions in the Bergman space or the projections of the functions in $L^{p,q}_\nu(T_\Omega)$. We begin with a pointwise estimate of elements in $A^{p,q}_\nu(T_\Omega)$. The following lemma follows from the invariance of the Bergman spaces with respect to the transformation group $G(\Omega)$ (see [5]).

Lemma 2. *Let $1 \le p, q < \infty$ and $\nu \in \mathbb{R}^r$, $\nu > g_0$. Then*

$$|f(z)| \lesssim \Delta_{-\frac{\nu}{q} - \frac{n}{rp}}(\Im z) \|f\|_{A^{p,q}_\nu}, \quad z \in T_\Omega. \tag{6}$$

We also need a pointwise estimate for the Bergman projection of functions in $L^{p,q}(T_\Omega)$, defined by integral formula (1), when this projection makes sense. Let us first recall the following integrability properties for the determinant function.

Lemma 3. *Let $\alpha \in \mathbb{C}^r$ and $y \in \Omega$.*

1) *The integral*

$$J_\alpha(y) = \int\limits_{\mathbb{R}^n} \left| \Delta_{-\alpha}\left(\frac{x+iy}{i}\right) \right| dx$$

converges if and only if $\Re\alpha > g_0^* + \frac{n}{r}$. In that case $J_\alpha(y) = C_\alpha |\Delta_{-\alpha+n/r}(y)|$.

2) *For any multi-indices s and β and $t \in \Omega$ the function $y \mapsto \Delta_\beta(y+t)\Delta_s(y)$ belongs to $L^1(\Omega, \frac{dy}{\Delta^{n/r}(y)})$ if and only if $\Re s > g_0$ and $\Re(s+\beta) < g_0^*$. In that case we have*

$$\int_\Omega \Delta_\beta(y)\Delta_s(y)\frac{dy}{\Delta^{n/r}(y)} = C_{\beta,s}\Delta_{s+\beta}(y).$$

We refer to Corollary 2.18 and Corollary 2.19 of [5] for the proof of the above lemma. Let τ denotes the set of all triples (p,q,ν) such that $1 \le p, q < \infty$, $\nu > g_0$ and the function $B_\nu(\cdot, i\mathbf{e})$ belongs to $L^{p',q'}_\nu(T_\Omega)$. We have the following pointwise estimate.

Lemma 4. *Suppose $(p,q,\nu) \in \tau$. Then*

$$|P_\nu f(z)| \le \Delta_{-\frac{\nu}{q}-\frac{n}{rp}}(\Im z)\|f\|_{L^{p,q}_\nu}. \tag{7}$$

Proof. This is an easy consequence of the above lemma and Hölder's inequality. □

3. Bergman-type Operators and Multifunctional Embeddings

We denote by $\Box = \Delta(\frac{1}{i}\frac{\partial}{\partial x})$ the partial differential operator of order r on $\mathbb{R}^n$ defined by

$$\Box[e^{i(x|\xi)}] = \Delta(\xi)e^{i(x|\xi)}, \qquad x, \xi \in \mathbb{R}^n. \tag{8}$$

3.1. Multifunctional Bergman-type operators.

Now we investigate boundedness of T_β from $\prod_{k=1}^m L^p_{m\nu_k+(m-1)\frac{n}{r}}(T_\Omega)$ to $L^p((T_\Omega)^m, \prod_{k=1}^m \Delta^{\nu_k-\frac{n}{r}}dV(z_k))$. We apply the obtained result to multifunctional embeddings for functions in the Bergman spaces $A^p_\nu(T_\Omega)$ where $\nu > \frac{n}{r}-1$ and $1 \le p < \infty$. We begin with the following result, which is known in the case $m=1$, see [4].

Theorem 3. *Let $\nu = (\nu_1, \dots, \nu_m) \in \mathbb{R}^m$, $m > 1$ and $1 \le p < \infty$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$. If the parameters satisfy the following conditions*

$$\frac{1}{m}\sum_{j=1}^m \beta_j > \frac{n}{r} - 1, \tag{9}$$

$$1 \le p < 1 + m\left(\frac{\min_j \nu_j}{\frac{n}{r}-1} - 1\right), \tag{10}$$

and

$$\min_j \beta_j > \frac{1}{m}\sum_{j=1}^m \beta_j - \frac{n}{rp} + \frac{m}{p}\left(2\frac{n}{r} - 1 + \max_j \nu_j\right), \tag{11}$$

then T_β is bounded from

$$\prod_{k=1}^m L^p_{m\nu_k+(m-1)\frac{n}{r}}(T_\Omega) \quad to \quad L^p\left((T_\Omega)^m, \prod_{k=1}^m \Delta^{\nu_k-\frac{n}{r}}dV(z_k)\right).$$

The idea of proof is taken from [8] where similar arguments can be found in higher dimensional case.

Proof. Using Hölder inequality we obtain

$$\begin{aligned}|T_\beta(\overrightarrow{f}(z_1,\dots,z_m)|^p &= \left| \int\limits_{T_\Omega} \frac{\left(\prod\limits_{j=1}^m f_j(z)\right)\Delta^{\frac{1}{m}\sum\limits_{j=1}^m \beta_j}(\Im z)}{\prod\limits_{j=1}^m \Delta^{\frac{1}{m}(\frac{n}{r}+\beta_j)}(\frac{z_j-\overline{z}}{i})} \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)}\right|^p \le \\ &\le I\times J,\end{aligned}$$

where

$$I=\int\limits_{T_\Omega} \frac{\left(\prod\limits_{j=1}^m |f_j(z)|^p\right)\Delta^{\frac{1}{m}\sum\limits_{j=1}^m \beta_j}(\Im z)}{\prod\limits_{j=1}^m |\Delta(\frac{z_j-\overline{z}}{i})|^{p\alpha_j}} \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)},$$

$$J^{p'/p}=\int\limits_{T_\Omega} \frac{\Delta^{\frac{1}{m}\sum\limits_{j=1}^m \beta_j}(\Im z)}{\prod\limits_{j=1}^m |\Delta(\frac{z_j-\overline{z}}{i})|^{p'\gamma_j}} \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)},$$

and

$$\alpha_j+\gamma_j=\frac{1}{m}\left(\frac{n}{r}+\beta_j\right). \tag{12}$$

Let us choose γ_j such that $\gamma_j > \frac{1}{mp'}\left(\frac{1}{m}\sum\limits_{j=1}^m \beta_j + 2\frac{n}{r}-1\right)$. Then we estimate the integral J using Hölder's inequality and Lemma 4:

$$\begin{aligned}J^{p'/p} &= \int\limits_{T_\Omega}\prod_{j=1}^m \left|\Delta\left(\frac{z_j-\overline{z}}{i}\right)\right|^{-p'\gamma_j} \Delta^{\frac{1}{m}\sum\limits_{j=1}^m \beta_j-\frac{n}{r}}(\Im z)dV(z) \le \\ &\le C\prod_{j=1}^m\left(\int\limits_{T_\Omega}\left|\Delta\left(\frac{z_j-\overline{z}}{i}\right)\right|^{-mp'\gamma_j}\Delta^{\frac{1}{m}\sum\limits_{j=1}^m \beta_j-\frac{n}{r}}(\Im z)dV(z)\right)^{1/m} = \\ &= C\prod_{j=1}^m \Delta^{-p'\gamma_j+\frac{1}{m^2}\sum\limits_{j=1}^m \beta_j+\frac{n}{rm}}(\Im z_j).\end{aligned}$$

Hence we obtained:

$$J\le C\prod_{j=1}^m \Delta^{-p\gamma_j+\frac{p}{m^2p'}\sum\limits_{j=1}^m \beta_j+\frac{p}{p'}\frac{n}{rm}}(\Im z_j). \tag{13}$$

Using the estimate (13) and Lemma 4 we finally obtain

$$\int\limits_{T_\Omega} \cdots \int\limits_{T_\Omega} \prod_{k=1}^{m} |T_\beta(\overrightarrow{f})(z_1,\dots,z_m)|^p \Delta^{\nu_k-\frac{n}{r}}(\Im z_k) dV(z_1)\cdots dV(z_m) \quad \le$$

$$\le C \int\limits_{T_\Omega} \left(\prod_{j=1}^{m} |f_j(z)|^p \right) g(z) \Delta^{\frac{1}{m}\sum\limits_{j=1}^{m}\beta_j}(\Im z) \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)}$$

where

$$g(z) = \int\limits_{T_\Omega} \cdots \int\limits_{T_\Omega} \prod_{k=1}^{m} \left(\left| \Delta\left(\frac{z_k - \overline{z}}{i} \right) \right|^{-p\alpha_k} \Delta^{\nu_k - \frac{n}{r} - p\gamma_j + \frac{p}{m^2 p'} \sum\limits_{j=1}^{k} \beta_j + \frac{np}{rmp'}}(\Im z_k) \right)$$
$$dV(z_1)\cdots dV(z_m).$$

Note that (11) implies $p\alpha_k > \nu_k - p\gamma_k + \frac{p}{m^2 p'} \sum\limits_{k=1}^{m} \beta_k + \frac{pn}{rmp'} + 2\frac{n}{r} - 1$. Thus, if we finally choose α_j and γ_j such that (12) holds and, for every $j = 1,\dots,m$, we have

$$\frac{1}{mp'} \left(\frac{1}{m} \sum_{j=1}^{m} \beta_j + 2\frac{n}{r} - 1 \right) < \gamma_j <$$

$$< \min \left\{ \frac{1}{m}(\frac{n}{r} + \beta_j), \frac{\min_j \nu_j - \frac{n}{r} + 1}{p} + \frac{\frac{1}{m}\sum\limits_{j=1}^{m} \beta_j + \frac{n}{r}}{mp'} \right\},$$

then an application of Lemma 4 gives estimate

$$g(z) \le C \Delta^{\sum\limits_{k=1}^{m} \nu_k + m\frac{n}{r} - p\sum\limits_{k=1}^{m}(\alpha_k+\beta_k) + \frac{p}{mp'}\sum\limits_{k=1}^{m}\beta_k + \frac{pn}{rp'}}(\Im z).$$

Finally, using Hölder's inequality we obtain

$$\int\limits_{T_\Omega} \cdots \int\limits_{T_\Omega} \prod_{k=1}^{m} |T_\beta(\overrightarrow{f})(z_1,\dots,z_m)|^p \Delta^{\nu_k-\frac{n}{r}}(\Im z_k) dV(z_1)\cdots dV(z_m) \le$$

$$\le C \int\limits_{T_\Omega} \left(\prod_{j=1}^{m} |f_j(z)|^p \right) \Delta^{\sum\limits_{k=1}^{m} \nu_k + (m-1)\frac{n}{r}}(\Im z) \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)} \le$$

$$\le C \left(\prod_{j=1}^{m} \int\limits_{T_\Omega} |f_j(z)|^{mp} \Delta^{m\nu_j + (m-1)\frac{n}{r}}(\Im z) \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)} \right)^{1/m} < \infty. \qquad \square$$

An analogue of the following lemma in the setting of the unit ball in $\mathbb{C}^n$ is contained in [8]. Note also it is easy to see that the case $m = 1$ is obvious.

Lemma 5. *Let $\nu_k > \frac{n}{r} - 1$, $k = 1, \dots, m$ and $1 \le p < \infty$. Then there is a constant $C > 0$ such that*

$$\int_{T_\Omega} \prod_{k=1}^{m} |f_k(z)|^p \Delta^{(m-1)\frac{n}{r} + \sum_{k=1}^{m} \nu_k - \frac{n}{r}}(\Im z)\, dV(z) \le C \prod_{k=1}^{m} \|f_k\|^p_{A^p_{\nu_k}}. \tag{14}$$

Proof. By embedding from [5] we have $A^{p/m}_{\frac{1}{m}\sum_{k=1}^{m}\nu_k}(T_\Omega) \hookrightarrow A^p_{(m-1)\frac{n}{r} + \sum_{k=1}^{m}\nu_k}(T_\Omega)$. Thus, to prove the lemma, we only need to check that for $f_j \in A^p_{\nu_j}(T_\Omega)$, $j = 1, \dots, m$, the product $f_1 \cdots f_m$ is in $A^{p/m}_{\frac{1}{m}\sum_{k=1}^{m}\nu_k}(T_\Omega)$ with the appropriate norm estimate. An application of Hölder's inequality

$$\int_{T_\Omega} \prod_{k=1}^{m} |f_k(z)|^p \Delta^{\frac{1}{m}\sum_{k=1}^{m}\nu_k - \frac{n}{r}}(\Im z)\, dV(z) \le$$

$$\le \prod_{k=1}^{m} \left(\int_{T_\Omega} |f_k(z)|^p \Delta^{\nu_k - \frac{n}{r}}(\Im z)\, dV(z) \right)^{1/m}$$

finishes the proof since the last expression is equal to $\prod_{k=1}^{m} \|f_k\|^{p/m}_{A^p_{\nu_k}}$. $\square$

A complete analogue of the following multifunctional result in the setting of the unit ball in $\mathbb{C}^n$ can be found in [8].

Theorem 4. *Let $\nu_k > \frac{n}{r}$ for $1 \le k \le m$, $m > 1$. Let $1 \le p < \infty$ and suppose that β_j are sufficiently large so that for any sequence $(z_j)_{j=1}^m$ in T_Ω the following representation holds for $f_1, \dots, f_m \in \mathcal{H}(T_\Omega)$*

$$f_1(z_1) \cdots f_m(z_m) = C_{m,\beta} \int_{T_\Omega} \frac{\prod_{j=1}^{m} f_j(z) \Delta^{\frac{1}{m}\sum_{j=1}^{m}}(\Im z)}{\prod_{j=1}^{m} \Delta^{\frac{1}{m}(\frac{n}{r} + \beta_j)}\left(\frac{z_j - \bar{z}}{i}\right)} \frac{dV(z)}{\Delta^n r(\Im z)}. \tag{15}$$

Assuming none of the functions f_k is identically zero, the following statements are equivalent.

1) *There is a constant $C > 0$ such that*

$$\int_{T_\Omega} \prod_{k=1}^{m} |f_k(z)|^p \Delta^{(m-1)\frac{n}{r} + \sum_{k=1}^{m}\nu_k}(\Im z) \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)} \le C < \infty. \tag{16}$$

2) *$f_k \in A^p_{\nu_k}(T_\Omega)$ for all $k = 1, \dots, m$.*

Proof. We have already seen that 2) $\Rightarrow$ 1) independently of the representation formula (15). Let us prove implication 1) $\Rightarrow$ 2) assuming (15). Since the functions f_j are not identically zero, condition

$$\int\limits_{T_\Omega}\cdots\int\limits_{T_\Omega}\prod_{k=1}^{m}(|f_k(z_k)||^p\Delta^{\nu_k-\frac{n}{r}}(\Im z_k)dV(z_1)\cdots dV(z_m)<\infty$$

implies $f_k\in A^p_{\nu_k}(T_\Omega)$ for all $k=1,\dots,m$. Now, using the representation (15) we obtain

$$\begin{aligned}K&=\int\limits_{T_\Omega}\cdots\int\limits_{T_\Omega}\prod_{k=1}^{m}(|f_k(z_k)|^p\Delta^{\nu_k-\frac{n}{r}}(\Im z_k))dV(z_1)\cdots dV(z_m)=\\&=\int\limits_{T_\Omega}\cdots\int\limits_{T_\Omega}|T_\beta(\overrightarrow{f}(z_1,\dots,z_m)|^p\left(\prod_{k=1}^{m}\Delta^{\nu_k-\frac{n}{r}}(\Im z_k)\right)dV(z_1)\cdots dV(z_m),\end{aligned}$$

where $\overrightarrow{f}=(f_1,\dots,f_m)$. The proof of Theorem 5 gives

$$K\le C\int\limits_{T_\Omega}\cdots\int\limits_{T_\Omega}\prod_{k=1}^{m}(|f_k(z_k)||^p\Delta^{\nu_k-\frac{n}{r}}(\Im z_k)dV(z_1)\cdots dV(z_m)<\infty.\qquad\square$$

We write $(\nu,p)\in\sigma$ if $\nu\in\mathbb{R}$, $1\le p<\infty$, $\nu>\frac{n}{r}-1$ and $\Delta^{-(\nu+\frac{n}{r})}(\frac{z-i\mathbf{e}}{i})\in L^{p'}_\nu(T_\Omega)$. Let us define, for $f_k\in L^p_{\nu_k}$, the following operators:

$$S_{\beta,k}(\overrightarrow{f})(\overrightarrow{z})=\int\limits_{T_\Omega}\frac{f_k(z)\prod\limits_{j\ne k}P_{\nu_j}f_j(z)\Delta^{\frac{1}{m}\sum\limits_{j=1}^{m}\beta_j}(\Im z)}{\prod\limits_{j=1}^{m}\Delta^{\frac{1}{m}(\frac{n}{r}+\beta_j)}(\frac{z_j-\overline{z}}{i})}\frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)},\tag{17}$$

and

$$S_\beta=\sum_{k=1}^{m}S_{\beta,k}.\tag{18}$$

Theorem 5. *Suppose $(\nu_k,p)\in\sigma$ for $k=1,\dots,m$. If the parameters satisfy conditions (2), (3), and (4), then the operators S_{β_k} and S_β are bounded from $\prod\limits_{j=1}^{m}L^p_{\nu_j}(T_\Omega)$ to $L^p((T_\Omega)^m,\prod\limits_{j=1}^{m}\Delta^{\nu_j-\frac{n}{r}}(\Im z_j)dV(z_j))$.*

Proof. Clearly we only need to prove the result for S_{β_k} for fixed k. An inspection of the proof of Theorem 5 and Lemma 5 give

$$\int\limits_{T_\Omega}\cdot\int\limits_{T_\Omega}|S_{\beta,k}(\overrightarrow{f}(\overrightarrow{z}|^p\Delta^{\nu_j-\frac{n}{r}}(\Im z_j)dV(z_1)\cdots dV(z_m)\le$$

$$\le C \int_{T_\Omega} |f_k(z)|^p \Big(\prod_{j\neq k}^m |P_{\nu_j} f_j(z)|^p \Big) \Delta^{\sum\limits_{k=1}^m \nu_k + (m-1)\frac{n}{r}}(\Im z) \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)} \le$$

$$\le C \prod_{j\neq k} \|f_j\|_{L^p_{\nu_j}} \int_{T_\Omega} |f_k(z)|^p \Delta^{\nu_k - \frac{n}{r}}(\Im z) dV(z) \le C \prod_{j\neq k} \|f_j\|_{L^p_{\nu_j}},$$

and the proof is complete. □

As a consequence we have the following result.

Theorem 6. *Suppose $(\nu_k, p) \in \sigma$ for $k = 1, \ldots, m$. Suppose also that, for β_j large enough, the following representation*

$$\prod_{k=1}^m P_{\nu_k} f_k(z_k) = C_{m,\beta} \int_{T_\Omega} \frac{f_k(z) \prod\limits_{j\neq k}^m P_{\nu_j} f_j(z) \Delta^{\frac{1}{m}\sum\limits_{j=1}^m \beta_j}(\Im z)}{\prod\limits_{j=1}^m \Delta^{\frac{1}{m}(\frac{n}{r}+\beta_j)}(\frac{z_j - \overline{z}}{i})} \frac{dV(z)}{\Delta^{\frac{n}{r}}(\Im z)} \tag{19}$$

holds for any sequence $(z_j)_{j=1}^m$ in T_Ω and any $f_k \in L^p_{\nu_k}(T_\Omega)$, $1 \le k \le m$. Then $P_{\nu_k} f_k \in L^p_{\nu_k}(T_\Omega)$, $1 \le k \le m$.

We also have the following corollary which gives a sufficient condition for boundedness of the Bergman projection.

Corollary 1. *Let $(\nu, p) \in \sigma$. If the following representation*

$$P_\nu f(z_1) P_\nu f(z_2) =$$
$$= C_\beta \int_{T_\Omega} \frac{f(z) P_\nu f(z)}{\Delta^{\frac{1}{2}(\frac{n}{r}+\frac{\beta}{2})}(\frac{z_1-\overline{z}}{i}) \Delta^{\frac{1}{2}(\frac{n}{r}+\frac{\beta}{2})}(\frac{z_2-\overline{z}}{i})} \Delta^{\beta - \frac{n}{r}}(\Im z) dV(z) \tag{20}$$

holds for all $z_1, z_2 \in T_\Omega$ and $f \in L^p_\nu(T_\Omega)$, where β is large enough, then P_ν is bounded on $L^p_\nu(T_\Omega)$.

Proof. Using Lemma 5 we clearly have

$$\int_{T_\Omega} |f(z)|^p |P_\nu f(z)|^p \Delta^{2\nu}(\Im z) dV(z) \le$$
$$\le C \|f\|^p_{L^p_\nu} \int_{T_\Omega} |f(z)|^p \Delta^{\nu - \frac{n}{r}}(\Im z) dV(z) =$$
$$= C \|f\|^{2p}_{L^p_\nu}.$$

Now, following the proof of Theorem 7 we obtain

$$
\begin{aligned}
&P_\nu f\|_{L^p_\nu}^{2p} = \\
&= \int\limits_{T_\Omega}\int\limits_{T_\Omega} |P_\nu f(z_1)|^p |P_\nu f(z_2)|^p \Delta^{\nu-\frac{n}{r}}(\Im z_1)\Delta^{\nu-\frac{n}{r}}(\Im z_2) dV(z_1)dV(z_2) \leq \\
&\leq C \int\limits_{T_\Omega} |f(z)|^p |P_\nu f(z)|^p \Delta^{2\nu}(\Im z) dV(z) \leq \\
&\leq C\|f\|_{L^p_\nu}^{2p}. \qquad \square
\end{aligned}
$$

3.2. **Multifunctional inequalities involving Bergman projection or the box operator.** Next we derive multifunctional inequalities involving the Bergman projection or the box operator. As a preparation, we first prove the following proposition.

Proposition 1. *Let $(\nu, p) \in \sigma$. If P_ν is bounded on $L^p_\nu(T_\Omega)$, then P_ν is bounded from $L^p_\nu(T_\Omega)$ to $L^{kp}_{k\nu+(k-2)\frac{n}{r}}(T_\Omega)$ for any $k \in \mathbb{N}$.*

Proof. Suppose P_ν is bounded on $L^p_\nu(T_\Omega)$. Then using Lemma 5 we obtain, for any $f \in L^p_\nu(T_\Omega)$:

$$
\begin{aligned}
&\int\limits_{T_\Omega} |P_\nu f(z)|^{kp} \Delta^{k\nu+(k-2)\frac{n}{r}}(\Im z) dV(z) = \\
&= \int\limits_{T_\Omega} (|P_\nu f(z)|^p \Delta^{\nu+\frac{n}{r}}(\Im z))^{k-1} |P_\nu f(z)|^p \Delta^{\nu-\frac{n}{r}}(\Im z) dV(z) \leq \\
&\leq C\|f\|_{L^p_\nu}^{(k-1)p} \int\limits_{T_\Omega} |P_\nu f(z)|^p \Delta^{\nu-\frac{n}{r}}(\Im z) dV(z) \leq \\
&\leq C\|f\|_{L^p_\nu}^{kp}. \qquad \square
\end{aligned}
$$

Proposition 2. *Let $(\nu_k, p) \in \sigma$ for $1 \leq k \leq m$. Suppose P_{ν_k} is bounded on $L^p_{\nu_k}(T_\Omega)$ for all $k = 1, \ldots, m$. Then for any $l \in \mathbb{N}$ we have*

$$
\int\limits_{T_\Omega} \prod_{k=1}^m [|P_{\nu_k}|f_k(z)|^{lp} \Delta^{l\nu_k + l\frac{n}{r}}(\Im z)] \frac{dV(z)}{\Delta^{2\frac{n}{r}}(\Im z)} \leq C \prod_{k=1}^m \|f_k\|_{L^p_{\nu_k}}^{lp}.
$$

Proof. Using the above proposition, Hölder's inequality and Lemma 5 we obtain

$$\int\limits_{T_\Omega} \prod_{k=1}^{m} [|P_{\nu_k}|f_k(z)|^{kp} \Delta^{l\nu_k + l\frac{n}{r}}(\Im z)] \frac{dV(z)}{\Delta^{2\frac{n}{r}}(\Im z)} \leq$$

$$\leq C \prod_{k=1}^{m} \|f_k\|^{(l-1)p} L^p_{\nu_k} \int\limits_{T_\Omega} \prod_{k=1}^{m} [|P_{\nu_k}|f_k(z)|^p \Delta^{\nu_k + \frac{n}{r}}(\Im z)] \frac{dV(z)}{\Delta^{2\frac{n}{r}}(\Im z)} \leq$$

$$\leq C \prod_{k=1}^{m} \|f_k\|^{(l-1)p}_{L^p_{\nu_k}} \prod_{k=1}^{m} \Big(\int\limits_{T_\Omega} |P_{\nu_k} f_k(z)|^{mp} \Delta^{m\nu_k + (m-2)\frac{n}{r}}(\Im z) dV(z) \Big)^{1/m} \leq$$

$$\leq C \prod_{k=1}^{m} \|f_k\|^{lp}_{L^p_{\nu_k}} \qquad \square$$

It is well-known that the operator $\Box$ satisfies the following boundedness estimate

$$\|\Box f\|_{A^p_{\nu+p}} \leq C \|f\|_{A^p_\nu}, \tag{21}$$

see [4]. It follows, using Hölder's inequality, that for $1 \leq p < \infty$ and $q < p$

$$\int\limits_{T_\Omega} |\Box f(z)|^q |f(z)|^{p-q} \Delta^{\nu + q - \frac{n}{r}}(\Im z) dV(z) \leq C \|f\|^p_{A^p_\nu}. \tag{22}$$

Our goal is to obtain a multifunctional version of the above estimate. To this end, we introduce the following operator, which we still denote by $\Box$, defined for pointwise products of holomorphic functions:

$$\Box(f_1 \cdots f_m) = \sum_{j=1}^{m} f_1 \cdots f_{j-1} (\Box f_j) f_{j+1} \cdots f_m.$$

We note that the $\Box$ inside the sum is the usual $\Box$ as defined at the beginning of this section. The next theorem generalizes (22), this idea probably for the first time appeared in [8].

Theorem 7. *Let $\nu > \frac{n}{r} - 1$, $1 \leq q \leq p < \infty$. Then there exists $C > 0$ such that*

$$\int\limits_{T_\Omega} |\Box(f_1 \cdots f_m)|^q \prod_{j=1}^{m} |f_j(z)|^{p-q} \Delta^{m(\nu + \frac{n}{r}) + q}(\Im z) \frac{dV(z)}{\Delta^{2\frac{n}{r}}(\Im z)} \leq$$

$$\leq C m^q \prod_{j=1}^{m} \|f_j\|^p_{A^p_\nu}. \tag{23}$$

Proof. Using Minkowski's inequality, the pointwise estimate for functions in $A^p_\nu(T_\Omega)$ and the estimate (22) we obtain

$$\int\limits_{T_\Omega} |\Box(f_1\cdots f_m)|^q \prod_{j=1}^m |f_j(z)|^{p-q} \Delta^{m(\nu+\frac{n}{r})+q}(\Im z)\frac{dV(z)}{\Delta^{2\frac{n}{r}}(\Im z)} \leq$$

$$\leq C\Bigg(\sum_{j=1}^m \Bigg(\int\limits_{T_\Omega} \prod_{k\neq j}^m |f_k(z)|^q |\Box f_j(z)|^q \times$$

$$\times \prod_{k=1}^m |f_k(z)|^{p-q}\Delta^{m(\nu+\frac{n}{r})+q}(\Im z)\frac{dV(z)}{\Delta^{2\frac{n}{r}}(\Im z)}\Bigg)^{1/q}\Bigg)^q \leq$$

$$\leq C\Bigg(\sum_{j=1}^m \Bigg(\int\limits_{T_\Omega} \Bigg(\prod_{k\neq j}^m |f_k(z)|^p \Delta^{\nu+\frac{n}{r}}(\Im z)\Bigg)\times$$

$$\times |\Box f_j(z)|^q |f_j(z)|^{p-q}\Delta^{\nu-\frac{n}{r}+q}(\Im z)dV(z)\Bigg)^{1/q}\Bigg)^q \leq$$

$$\leq C\Bigg(\sum_{j=1}^m \Bigg(\prod_{k\neq j}^m \|f_k\|^{p/q}_{A^p_\nu}\Bigg)\times$$

$$\times\Bigg(\int\limits_{T_\Omega} |\Box f_j(z)|^q |f_j(z)|^{p-q}\Delta^{\nu-\frac{n}{r}+q}(\Im z)dV(z)\Bigg)^{1/q}\Bigg)^q \leq$$

$$\leq Cm^q \prod_{k=1}^m \|f_k\|^p_{A^p_\mu}. \qquad \Box$$

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