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ON A STAGE OF A NUMERICAL ALGORITHM FOR A TIMOSHENKO TYPE NONLINEAR EQUATION

G. PAPUKASHVILI, J. PERADZE AND Z. TSIKLAURI

Abstract. An initial boundary value problem for a differential equation describing the beam oscillation is considered.As a result of application of the variational method and a difference scheme, a nonlinear system of equations is obtained, which is solved by iteration. The convergence conditions and the error estimate of the iteration method are obtained.

რეზიუმე. განხილულია საწყის-სასაზღვრო ამოცანა დიფერენციალური განტოლებისათვის, რომელიც აღწერს ძელის რხევას. ვარიაციული მეთოდისა და სხვაობიანი სქემის გამოყენების შედეგად მიიღება არაწრფივ განტოლებათა სისტემა, რომელიც იხსნება იტერაციულად. მიღებულია იტერაციული მეთოდის კრებადობის პირობები და ცდომილების შეფასება.

1. Statement of the problem

Let us consider the initial boundary value problem

$$
\begin{gathered}
u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t)- \\
-\left(\beta+\rho\int_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t)- \\
-\sigma\left(\int_0^L u_x(x,t)u_{xt}(x,t)\,dx\right)u_{xx}(x,t)=0,\quad 0<x<L,\quad 0<t\le T, \qquad (1)
\end{gathered}
$$

$$
\begin{gathered}
u(x,0)=u^0(x),\quad u_t(x,0)=u^1(x), \\
u(0,t)=u(L,t)=0,\quad u_{xx}(0,t)=u_{xx}(L,t)=0,
\end{gathered} \qquad (2)
$$

where α, γ, ρ, σ, β and δ are the given constants, among which the first four are positive numbers, while $u^0(x)\in W_2^2(0,L)$ and $u^1(x)\in L_2(0,L)$ are given functions such that $u_0(0)=u_1(0)=u_0(L)=u_1(L)=0$. In the sequel it is assumed that the inequality $|\delta|<\gamma(\frac{\pi}{L})^4$ is fulfilled when $\delta<0$, and $\alpha(\frac{\pi}{L})^2>|\beta|$ holds when $\beta<0$. It will be assumed that there exists a solution $u(x,t)\in W_2^2((0,L)\times(0,T))$ of problem (1), (2).

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Equation (1) obtained by J. Ball [1] using the Timoshenko theory describes the vibration of the beam. Moreover, in [1], the existence of a global solution for (1) is shown. The problem of construction of an approximate solution for this equation is investigated in [2], [3], [4].

Here we consider a numerical solution algorithm for problem (1), (2).

2. Algorithm

a. *Galerkin method.* A solution of the problem (1), (2) will be sought in the form of a finite sum

$$u_n(x,t)=\sum_{i=1}^{n} u_{ni}(t)\,\sin\frac{i\pi x}{L}\,, \tag{3}$$

where the coefficients $u_{ni}(t)$ are defined by the Galerkin method from the system of ordinary differential equations

$$\begin{aligned} u''_{ni}(t)+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)u'_{ni}(t)+\Bigg[\alpha\left(\frac{i\pi}{L}\right)^4+ \\ +\left(\frac{i\pi}{L}\right)^2\Bigg(\beta+\rho\,\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}^2(t)+ \\ +\sigma\,\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}(t)u'_{nj}(t)\Bigg)\Bigg]u_{ni}(t)=0,\quad i=1,2,\dots,n, \end{aligned} \tag{4}$$

with the initial conditions

$$u_{ni}(0)=a_i^0,\quad u'_{ni}(0)=a_i^1, i=1,2,\dots,n, \tag{5}$$

where

$$a_i^p=\frac{2}{L}\int_0^L u^p(x)\sin\frac{i\pi x}{L}\,dx,\quad p=0,1,\quad i=1,2,\dots,n.$$

The convergence of the Galerkin method for equation (1) and an equation with similar nonlinearity is studied in [6] and [7].

b. *Difference scheme.* Let us introduce the notation

$$y_{ni}(t)=u'_{ni}(t),\quad z_{ni}(t)=\frac{i\pi}{L}\,u_{ni}(t),\quad i=1,2,\dots,n, \tag{6}$$

and rewrite system (4), (5) in the new notation as follows

$$y'_{ni}(t)+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)y_{ni}(t)+\left[\alpha\left(\frac{i\pi}{L}\right)^3+\frac{i\pi}{L}\left(\beta+\rho\,\frac{L}{2}\sum_{j=1}^{n}z_{nj}^2(t)+\right.\right.$$
$$\left.\left.+\sigma\,\frac{L}{2}\sum_{j=1}^{n}\frac{j\pi}{L}\,y_{nj}(t)z_{nj}(t)\right)\right]z_{ni}(t)=0, \tag{7}$$

$$\begin{gathered} z'_{ni}(t)=\frac{i\pi}{L}\,y_{ni}(t), \quad i=1,2,\ldots,n,\\ y_{ni}(0)=a_i^1, \quad z_{ni}(0)=\frac{i\pi}{L}\,a_i^0, \quad i=1,2,\ldots,n. \end{gathered} \tag{8}$$

Problem (7), (8) will be solved using the difference method. On the time interval $[0,T]$ we introduce a net with step $\tau=\frac{T}{M}$ and nodes $t_m=m\tau$, $m=0,1,\ldots,M$.

On the m-th layer, i.e. for $t=t_m$, the approximate values of $y_{ni}(t)$ and $z_{ni}(t)$ are denoted by y_{ni}^m and z_{ni}^m.

We use a Crank-Nicolson type scheme

$$\begin{gathered} \frac{y_{ni}^m-y_{ni}^{m-1}}{\tau}+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)\frac{y_{ni}^m+y_{ni}^{m-1}}{2}+\\ +\left[\alpha\left(\frac{i\pi}{L}\right)^3+\frac{i\pi}{L}\left(\beta+\rho\,\frac{L}{2}\sum_{j=1}^{n}\frac{(z_{nj}^m)^2+(z_{nj}^{m-1})^2}{2}+\right.\right.\\ \left.\left.+\sigma\,\frac{L}{2}\sum_{j=1}^{n}\frac{j\pi}{L}\,\frac{(y_{nj}^m+y_{nj}^{m-1})(z_{nj}^m+z_{nj}^{m-1})}{4}\right)\right]\frac{z_{ni}^m+z_{ni}^{m-1}}{2}=0,\\ \frac{z_{ni}^m-z_{ni}^{m-1}}{\tau}=\frac{i\pi}{L}\,\frac{y_{ni}^m+y_{ni}^{m-1}}{2},\\ m=1,2,\ldots,M, \quad i=1,2,\ldots,n, \end{gathered} \tag{9}$$

with the conditions

$$y_{ni}^0=a_i^1, \quad z_{ni}^0=\frac{i\pi}{L}\,a_i^0, \quad i=1,2,\ldots,n. \tag{10}$$

c. *Iteration method.* System (9), (10) will be solved layer-by-layer. Assuming that the solution has already been obtained on the $(m-1)$-th layer, to find it on the m-th layer we use the Jacobi iteration method. For the sake of simplicity, the error of the final approximation iteration approximation on the $(m-1)$-th layer will be neglected. This means that for fixed m the counting will be carried out by the formulas

$$\frac{y^m_{ni,k+1}-y^{m-1}_{ni}}{\tau}+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)\frac{y^m_{ni,k+1}+y^{m-1}_{ni}}{2}+$$
$$+\Bigg[\alpha\left(\frac{i\pi}{L}\right)^3+\frac{i\pi}{L}\Bigg(\beta+\rho\,\frac{L}{2}\,\frac{(z^m_{ni,k+1})^2+(z^{m-1}_{ni})^2}{2}+$$
$$+\rho\,\frac{L}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(z^m_{nj,k})^2+(z^{m-1}_{nj})^2}{2}+$$
$$+\sigma\,\frac{L}{2}\,\frac{i\pi}{L}\,\frac{(y^m_{ni,k+1}+y^{m-1}_{ni})(z^m_{ni,k+1}+z^{m-1}_{ni})}{4}+$$
$$+\sigma\,\frac{L}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{j\pi}{L}\,\frac{(y^m_{nj,k}+y^{m-1}_{nj})(z^m_{nj,k}+z^{m-1}_{nj})}{4}\Bigg)\Bigg]\frac{z^m_{ni,k+1}+z^{m-1}_{ni}}{2}=0, \quad (11)$$

$$\frac{z^m_{ni,k+1}-z^{m-1}_{ni}}{\tau}=\frac{i\pi}{L}\,\frac{y^m_{ni,k+1}+y^{m-1}_{ni}}{2}\,, \quad (12)$$
$$m=1,2,\dots,M,\quad k=0,1,\dots\,,\quad i=1,2,\dots,n,$$

where $y^m_{ni,k+p}$ and $z^m_{ni,k+p}$ denote the $(k+p)$-th iteration approximation for y^m_{ni} amd z^m_{ni}, $i=1,2,\dots,n$, $p=0,1$, y^{m-1}_{ni} and z^{m-1}_{ni} are the known values, $i=1,2,\dots,n$, and

$$y^0_{ni}=a^1_i,\quad z^0_{ni}=\frac{i\pi}{L}\,a^0_i,\quad i=1,2,\dots,n.$$

On expressing $y^m_{ni,k+1}$ in (12) through y^{m-1}_{ni}, z^{m-1}_{ni} and $z^m_{ni,k+1}$,

$$y^m_{ni,k+1}=-y^{m-1}_{ni}+2\,\frac{L}{i\pi}\,\frac{z^m_{ni,k+1}-z^{m-1}_{ni}}{\tau}\,, \quad (13)$$

and substituting (13) into (11), we come to the expression

$$\frac{1}{\tau}\left(-y^{m-1}_{ni}+2\,\frac{L}{i\pi}\,\frac{z^m_{ni,k+1}-z^{m-1}_{ni}}{\tau}\right)-\frac{y^{m-1}_{ni}}{\tau}+$$
$$+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)\,\frac{L}{i\pi}\,\frac{z^m_{ni,k+1}-z^{m-1}_{ni}}{\tau}+$$
$$+\Bigg\{\alpha\left(\frac{i\pi}{L}\right)^3+\frac{i\pi}{L}\Bigg[\beta+\rho\,\frac{L}{2}\,\frac{(z^m_{ni,k+1})^2+(z^{m-1}_{ni})^2}{2}+$$
$$+\rho\,\frac{L}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(z^m_{nj,k})^2+(z^{m-1}_{nj})^2}{2}+$$

$$+\sigma\,\frac{L}{4}\,\frac{z^m_{ni,k+1}-z^{m-1}_{ni}}{\tau}\,(z^m_{ni,k+1}+z^{m-1}_{ni})+$$
$$+\sigma\,\frac{L}{4}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{z^m_{nj,k}-z^{m-1}_{nj}}{\tau}\,(z^m_{nj,k}+z^{m-1}_{nj})\Bigg]\Bigg\}\frac{z^m_{ni,k+1}+z^{m-1}_{ni}}{2}=0. \tag{14}$$

Hence it follows that for each k the iteration process means the realization of only one formula (14). On obtaining the final iteration approximation $z^m_{ni,k+1}$, we substitute this value into (13) to find an approximation for y^m_{ni}, $i=1,2,\dots,n$.

From expression (14) it follows that we have to solve a cubic equation with respect to $z^m_{ni,k+1}$ at the $(k+1)$-th iteration step for each i.

Applying Cardano's formula we get

$$z^m_{ni,k+1}=-\frac{z^{m-1}_{ni}}{3}+\sum_{p=1}^{2}(-1)^{p+1}\sigma_{i,p}\,, \tag{15}$$
$$k=0,1,\dots,\quad i=1,2,\dots,n,$$

where

$$\sigma_{i,p}=\left[(-1)^p\,\frac{s_i}{2}+\left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{\frac12}\right]^{\frac13}, \tag{16}$$

and

$$r_i=\frac{8}{L(\rho+\frac{\sigma}{\tau})}\left[2\left(\frac{L}{i\pi}\right)^2\frac{1}{\tau^2}+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)\left(\frac{L}{i\pi}\right)^2\frac{1}{\tau}+\frac12\,\alpha\left(\frac{i\pi}{L}\right)^2+\right.$$
$$\left.+\frac12\,\beta\right]+\sum_{\substack{j=1\\ j\neq i}}^{n}(z^m_{nj,k})^2-\frac13(z^{m-1}_{ni})^2+\frac{\rho-\frac{\sigma}{\tau}}{\rho+\frac{\sigma}{\tau}}\sum_{j=1}^{n}(z^{m-1}_{nj})^2, \tag{17}$$

$$s_i=\frac{2(z^{m-1}_{ni})^3}{27}-\frac{16y^{m-1}_{ni}}{i\pi(\sigma+\tau\rho)}+\frac{8z^{m-1}_{ni}}{3L(\rho+\frac{\sigma}{\tau})}\left[-8\left(\frac{L}{i\pi}\right)^2\frac{1}{\tau^2}-\right.$$
$$\left.-4\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)\left(\frac{L}{i\pi}\right)^2\frac{1}{\tau}+\alpha\left(\frac{i\pi}{L}\right)^2+\beta\right]+$$
$$+\frac23\,z^{m-1}_{ni}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}(z^m_{nj,k})^2+\frac{\rho-\frac{\sigma}{\tau}}{\rho+\frac{\sigma}{\tau}}\sum_{j=1}^{n}(z^{m-1}_{nj})^2\right). \tag{18}$$

The considered algorithm of solution of problem (1), (2) should be understood as counting by formula (15). Using $z^m_{ni,k}$ and taking (6) and (3) into consideration, we construct the approximate value of the function $u(x,t)$

for $t=t_m$ as the sum

$$u_{n,k}^m(x)=\sum_{i=1}^n \frac{L}{i\pi}\, z_{ni,k}^m \sin\frac{i\pi x}{L}\,. \tag{19}$$

3. Estimate of the iteration method error

Our aim consists in finding convergence conditions and estimating the accuracy of the iteration method (15).

Let us estimate the sums $\sum\limits_{i=1}^n (y_{ni}^m)^2$ and $\sum\limits_{i=1}^n (\frac{i\pi}{L})^{2p}(z_{ni}^m)^2$, $p=0,1$. For this, we multiply the first equation in (9) by $\frac12(y_{ni}^m+y_{ni}^{m-1})$, sum the obtained relation over $i=1,2,\dots,n$ and take into consideration the second equality in (9). We obtain

$$\begin{aligned}
&\frac{1}{2\tau}\sum_{i=1}^n \big((y_{ni}^m)^2-(y_{ni}^{m-1})^2\big)+\frac14\sum_{i=1}^n\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)(y_{ni}^m+y_{ni}^{m-1})^2+\\
&+\alpha\,\frac{1}{2\tau}\sum_{i=1}^n\left(\frac{i\pi}{L}\right)^2\big((z_{ni}^m)^2-(z_{ni}^{m-1})^2\big)+\beta\,\frac{1}{2\tau}\sum_{i=1}^n\big((z_{ni}^m)^2-(z_{ni}^{m-1})^2\big)+\\
&\quad+\rho L\,\frac{1}{8\tau}\sum_{i=1}^n\big((z_{ni}^m)^2+(z_{ni}^{m-1})^2\big)\sum_{i=1}^n\big((z_{ni}^m)^2-(z_{ni}^{m-1})^2\big)+\\
&\qquad+\sigma\,\frac{L}{32}\left(\sum_{i=1}^n\frac{i\pi}{L}\left(y_{ni}^m+y_{ni}^{m-1}\right)\left(z_{ni}^m+z_{ni}^{m-1}\right)\right)^2=0,
\end{aligned}$$

whence we have

$$\begin{aligned}
&\sum_{i=1}^n (y_{ni}^m)^2+\alpha\sum_{i=1}^n\left(\frac{i\pi}{L}\right)^2(z_{ni}^m)^2+\beta\sum_{i=1}^n(z_{ni}^m)^2+\rho\,\frac{L}{4}\left(\sum_{i=1}^n(z_{ni}^m)^2\right)^2\le\\
&\qquad\le\sum_{i=1}^n(y_{ni}^{m-1})^2+\alpha\sum_{i=1}^n\left(\frac{i\pi}{L}\right)^2(z_{ni}^{m-1})^2+\\
&\qquad\qquad+\beta\sum_{i=1}^n(z_{ni}^{m-1})^2+\rho\,\frac{L}{4}\left(\sum_{i=1}^n(z_{ni}^{m-1})^2\right)^2.
\end{aligned}$$

From this inequality and relations (10) and (5) follow the estimates

$$\sum_{i=1}^n (y_{ni}^m)^2\le\theta_0,\quad \sum_{i=1}^n\left(\frac{i\pi}{L}\right)^{2p}(z_{ni}^m)^2\le\left(\frac{1}{\alpha}\right)^p\theta_{1-p},\quad p=0,1, \tag{20}$$

where

$$\theta_0 = \frac{2}{L}\int_0^L [(u^1(x))^2 + \alpha(u^{0\prime\prime}(x))^2 + \beta(u^{0\prime}(x))^2]\,dx + \\ + \frac{\rho}{L}\biggl(\int_0^L (u^{0\prime}(x))^2 dx\biggr)^2,$$

$$\theta_1 = \frac{1}{2\theta_2}(-\theta_3 + (\theta_3^2 + 4\theta_0\theta_2)^{\frac{1}{2}}), \quad \theta_2 = \rho\,\frac{L}{4}\,, \quad \theta_3 = \beta + \alpha\left(\frac{\pi}{L}\right)^2.$$

We will need these estimates later.

Note that under the conditions imposed on the coefficients of equation (1) and functions $u^0(x)$ and $u^1(x)$ the inequality $\theta_0 \geq 0$ holds.

Under the iteration method error we understand the difference between (19) and the sum

$$u_n^m(x) = \sum_{i=1}^n \frac{L}{i\pi}\, z_{ni}^m \sin\frac{i\pi x}{L}\,,$$

which would give an approximate value of the function $u(x,t)$ for $t = t_m$ if the difference system (9), (10) were solved exactly. So, we mean here the relation

$$u_{n,k}^m(x) - u_n^m(x) = \sum_{i=1}^n \frac{L}{i\pi}\,(z_{ni,k}^m - z_{ni}^m)\sin\frac{i\pi x}{L}\,. \tag{21}$$

To estimate (21), we represent system (15) as

$$z_{ni,k+1}^m = \varphi_i\left(z_{n1,k}^m, z_{n2,k}^m, \dots, z_{nn,k}^m\right) \tag{22}$$

and consider the Jacobi matrix

$$J = \left(\frac{\partial\varphi_i}{\partial z_{nj,k}^m}\right)_{i,j=1}^n. \tag{23}$$

By virtue of (15)–(18) and (22) the diagonal elements of the matrix J are equal to zero, while for the nondiagonal elements we have

$$\frac{\partial\varphi_i}{\partial z_{nj,k}^m} = -\frac{z_{nj,k}^m}{9}\sum_{p=1}^2 \frac{1}{\sigma_{i,p}^2}\biggl[2z_{ni}^{m-1} + \\ + (-1)^p\Bigl(s_i z_{ni}^{m-1} + \frac{1}{3}\,r_i^2\Bigr)\Bigl(\frac{s_i^2}{4} + \frac{r_i^3}{27}\Bigr)^{-\frac{1}{2}}\biggr]. \tag{24}$$

By (16)

$$\sigma_{i,1}\sigma_{i,2} = \frac{r_i}{3}\,, \quad \sigma_{i,2}^3 - \sigma_{i,1}^3 = s_i, \quad \left(\frac{s_i^2}{4} + \frac{r_i^3}{27}\right)^{\frac{1}{2}} = \frac{\sigma_{i,1}^3 + \sigma_{i,2}^3}{2}\,. \tag{25}$$

Formulas (24) are obtained under the condition that $\sigma_{i,p} \neq 0$, $p = 1, 2$, for the fulfilment of which it suffices to assume that $|r_i| > 0$. As will be shown below, this condition will be observed.

From (24) and (25) follows

$$\frac{\partial\varphi_i}{\partial z^m_{nj,k}}=-\frac{4}{9}\,z^m_{nj,k}z^{m-1}_{ni}\biggl(\sigma^2_{i,1}-\frac{r_i}{3}+\sigma^2_{i,2}\biggr)^{-1}+\\+\frac{2}{3}\,z^m_{nj,k}s_i\biggl(\sigma^4_{i,1}+\frac{r_i^2}{9}+\sigma^4_{i,2}\biggr)^{-1},\quad i\neq j. \quad (26)$$

Now to the obvious inequality $\sigma^{2p}_{i,1}+\sigma^{2p}_{i,2}\geq 2(\sigma_{i,1}\sigma_{i,2})^p$, $p=1,2$, we apply the first relation in (25). We have

$$\sigma^{2p}_{i,1}+\sigma^{2p}_{i,2}\geq 2\biggl(\frac{r_i}{3}\biggr)^p.$$

By virtue of this inequality, from (26) follows

$$\biggl|\frac{\partial\varphi_i}{\partial z^m_{nj,k}}\biggr|\leq\biggl(\frac{4}{3|r_i|}\,|z^{m-1}_{ni}|+\frac{2}{r_i^2}\,|s_i|\biggr)|z^m_{nj,k}|. \quad (27)$$

Let us estimate $|r_i|$ from below. From (17) and (20) we conclude that

$$|r_i|\geq\mu_i+\sum_{\substack{j=1\\ j\neq i}}^n(z^m_{nj,k})^2-\mu\theta_1,$$

where

$$\mu_i=\frac{8}{L(\rho+\frac{\sigma}{\tau})}\biggl[2\biggl(\frac{L}{i\pi}\biggr)^2\frac{1}{\tau^2}+\biggl(\delta+\gamma\biggl(\frac{i\pi}{L}\biggr)^4\biggr)\biggl(\frac{L}{i\pi}\biggr)^2\frac{1}{\tau}+\\+\frac{1}{2}\,\alpha\biggl(\frac{i\pi}{L}\biggr)^2+\frac{1}{2}\,\beta\biggr], \quad (28)$$

$$\mu=\max\biggl(0,\frac{1}{3}-\frac{\rho-\frac{\sigma}{\tau}}{\rho+\frac{\sigma}{\tau}}\biggr).$$

Let us choose an arbitrary number ε from the interval $(0,1)$ and require that the inequality

$$|r_i|\geq(1-\varepsilon)\biggl(\mu_i+\sum_{\substack{j=1\\ j\neq i}}^n(z^m_{nj,k})^2+\frac{5}{9}\,\theta_1\biggr) \quad (29)$$

be fulfilled. For this it suffices to assume that the step τ is so small that the inequality $\mu_i>\frac{1}{\varepsilon}\theta_1(\mu+\frac{5}{9}(1-\varepsilon))$ is fulfilled. On replacing in the latter inequality μ_i by $\frac{8}{L(\sigma+\tau\rho)}[\omega+2(\frac{L}{n\pi})^2\frac{1}{\tau}+\frac{1}{2}\,\tau(\alpha(\frac{\pi}{L})^2+\beta)]$, where $\omega=2\sqrt{\delta\gamma}$ for $\delta>0$ and $\omega=(\delta+\gamma(\frac{\pi}{L})^4)(\frac{L}{n\pi})^2$ for $\delta<0$, we obtain a simpler but more rigid condition of the fulfillment of relation (29).

Further, (18) and (20) imply

$$|s_i| \le \frac{16|y_{ni}^{m-1}|}{i\pi(\sigma+\tau\rho)} + \frac{8|z_{ni}^{m-1}|}{3L(\rho+\frac{\sigma}{\tau})}\left[8\left(\frac{L}{i\pi}\right)^2\frac{1}{\tau^2}+\right.$$
$$+4\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)\left(\frac{L}{i\pi}\right)^2\frac{1}{\tau}+\alpha\left(\frac{i\pi}{L}\right)^2+\beta\Big]+$$
$$+\left(\frac{2}{3}\sum_{\substack{j=1\\ j\ne i}}^{n}(z_{nj,k}^m)^2+\frac{20}{27}\,\theta_1\right)|z_{ni}^{m-1}|. \tag{30}$$

Using (27)–(30), we obtain

$$\left|\frac{\partial\varphi_i}{\partial z_{nj,k}^m}\right| \le \frac{1}{1-\varepsilon}\,\tau(\sigma+\tau\rho)\left(\frac{i\pi}{L}\right)^2 L\left\{\frac{1}{12}\,|z_{ni}^{m-1}|+\right.$$
$$\left.+\frac{1}{8(1-\varepsilon)}\left[\tau\,\frac{i\pi}{L}\,|y_{ni}^{m-1}|+\frac{4}{3}\,|z_{ni}^{m-1}|\right]\right\}|z_{nj,k}^m|. \tag{31}$$

We need the vector norm equal to $\|v\| = \sum\limits_{i=1}^{n}|v_i|$ and the corresponding norm for the matrix $\|K\| = \max\limits_{1\le j\le n}\sum_{i=1}^n |k_{ij}|$, where $v=(v_i)_{i=1}^n$ and $K=(k_{ij})_{i,j=1}^n$. By (23), (31) and (20) we get

$$\|J\| \le (a\tau^3+b\tau^2+c\tau)\max_{1\le j\le n}|z_{nj,k}^m|, \tag{32}$$

where the following notation is used

$$a=\frac{\rho L}{8(1-\varepsilon)^2}\left(\frac{\pi}{L}\right)^3\left(\sum_{i=1}^n i^6\right)^{\frac12}\sqrt{\theta_0},\quad b=a\,\frac{\sigma}{\rho}+c\,\frac{\rho}{\sigma}\,,$$
$$c=\frac{1}{6(1-\varepsilon)}\,\sigma L\left(\frac{1}{2}+\frac{1}{1-\varepsilon}\right)\frac{\pi}{L}\left(\sum_{i=1}^n i^2\right)^{\frac12}\sqrt{\frac{\theta_0}{\alpha}}\,.$$

By virtue of Banach's construction principle [5], it can be assumed that the condition $\|J\|\le q$ is fulfilled for $0<q<1$ and $z_{n,k}^m=(z_{ni,k}^m)_{i=1}^n$, $k=0,1,\dots$, belongs to the domain

$$\left\{w\in R^n:\ \|w-z_{n,0}^m\|\le\frac{1}{1-q}\,\|z_{n,1}^m-z_{n,0}^m\|\right\}. \tag{33}$$

According to (32), for this it suffices that the restriction

$$a\tau^3+b\tau^2+c\tau\le q\left(\|z_{n,0}^m\|+\frac{1}{1-q}\,\|z_{n,1}^m-z_{n,0}^m\|\right)^{-1} \tag{34}$$

be fulfilled for the step τ.

If this restriction is fulfilled, then system (9), (10) has a unique solution y_{ni}^m, z_{ni}^m, $i = 1, 2, \dots, n$, in (33), the iteration process (15) converges, $\lim\limits_{k\to\infty} z_{ni,k}^m = z_{ni}^m$, $i = 1, 2, \dots, n$, and the convergence rate is determined by the vector inequality

$$\|z_{n,k}^m - z_n^m\| \le \frac{q^k}{1-q}\,\|z_{n,1}^m - z_{n,0}^m\|,$$

where $z_n^m = (z_{ni}^m)_{i=1}^n$.

Applying this relation to (21), we come to a conclusion that if condition (34) is fulfilled, then the estimate

$$\left\|\frac{d^p}{dx^p}\left(u_{n,k}^m(x) - u_n^m(x)\right)\right\|_{L^2(0,L)} \le \left(\frac{L}{\pi}\right)^{1-p}\sqrt{\frac{L}{2}}\,\frac{q^k}{1-q}\,\|z_{n,1}^m - z_{n,0}^m\|,$$
$$p = 0, 1, \quad m = 1, 2, \dots, M, \quad k = 1, 2, \dots\ ,$$

holds for the $L^2(0,L)$-norm of the iteration method error.

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Authors' addresses:

G. Papukashvili
No. 199 Public School-Pansion (Komarovi)
49, Vaja-Pshavela Ave., Tbilisi 0186
Georgia
E-mail: papukashvili@yahoo.com

J. Peradze
Faculty of Exact and Natural Sciences
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186
Georgia
E-mail: j_peradze@yahoo.com

Department of Mathematics
Georgian Technical University
77, M. Kostava Str., Tbilisi 0175
Georgia

Z. Tsiklauri
Department of Mathematics
Georgian Technical University
77, M. Kostava Str., Tbilisi 0175
Georgia
E-mail: zviad_tsiklauri@yahoo.com