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ON ANISOTROPIC WEIGHTED SOBOLEV INEQUALITIES

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Abstract. Two weighted versions of the known anisotropic Sobolev inequality are obtained.

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1. Introduction

Let $I_i = (a_i, b_i)$, $-\infty \le a_i < b_i \le \infty$, $i = 1, 2, \dots, n$ and $\Omega = I_1 \times I_2 \times \dots \times I_n$. As usual, $C_0^\infty(\Omega)$ will denote the space of infinitely differentiable functions with compact support in Ω. The standard Sobolev inequality asserts that if $1 \le p < n$ then for all $u \in C_0^\infty(\Omega)$, there exists a constant $K > 0$ such that

$$\|u\|_q \le K \sum_{i=1}^{n} \|D_i u\|_p, \tag{1.1}$$

where q is the Sobolev conjugate of p, i.e., $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ or $q = \frac{np}{n-p}$ and the norms involved are the norms in $L^p(\Omega)$, i.e., for $u \in L^p(\Omega)$

$$\|u\|_p = \left(\int_\Omega |u|^p \right)^{1/p}.$$

In [3], the space $L^p(\Omega)$ was studied with mixed norm : for $P = (p_1, p_2, \dots, p_n)$, $1 \le p_i \le \infty$, the mixed norm space $L^P(\Omega)$ consists of all measurable functions u defined on Ω for which

$$\|u\|_P = \Big\| \big\| \|u\|_{p_1} \big\|_{p_2} \cdots \Big\|_{p_n} < \infty, \tag{1.2}$$

where first, L^{p_1}-norm is calculated w.r.t. x_1, then on the result L^{p_2}-norm of u is calculated w.r.t. x_2 and so on. In case $p_i = p$, $i = 1, 2, \dots, n$, we shall denote $\|.\|_p$ for $\|.\|_{(p,p,\dots,p)}$. Also, by $\frac{1}{P}$, we shall mean the vector $(\frac{1}{p_1}, \frac{1}{p_2}, \dots, \frac{1}{p_n})$.

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In [2], the anisotropic version of the inequality (1.1) was derived. Precisely, the following was proved:

Theorem A. *For $1 \le i \le n$, let $p_i \ge 1$ and $\sum_{i=1}^{n} \frac{1}{p_i} > 1$. If $\frac{1}{q} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i} - \frac{1}{n}$, then there exists a constant $K > 0$ such that the anisotropic Sobolev inequality*

$$\|u\|_q \le K \sum_{i=1}^{n} \|D_i u\|_{p_i} \tag{1.3}$$

holds for all $u \in C_0^\infty(\Omega)$.

In the present paper, as one of the aims, we shall obtain a weighted version of (1.3). As done for the proof of Theorem A, we shall also use mixed norm Lebesgue spaces as a tool. Let w be a weight function, i.e., a function which is measurable, positive and finite a.e. The weighted Lebesgue space, denoted by $L_w^p(\Omega)$ is the space of all functions $u \in \Omega$ such that

$$\|u\|_{p,w} = \left(\int_\Omega |u|^p w \right)^{1/p} < \infty, \quad 1 \le p < \infty$$

with usual modifications when $p = \infty$.

Next, another anisotropic version of the inequality (1.1) was given, again by Adams, in [1]. In fact, he proved the following:

Theorem B. *Let $1 < n < \infty$, $1 \le p \le q$ and r satisfies*

$$\frac{n}{r} = \frac{1}{p} + \frac{n-1}{q} - 1 > 0.$$

Then there exists a positive constant K such that for all $u \in C_0^\infty(\Omega)$.

$$\|u\|_r \le K \sum_{i=1}^{n} \|D_i u\|_{v_i(p,q)}, \tag{1.4}$$

where $v_i(p,q) = (q, q, \dots, p, \dots, q)$ having p at the i^{th} place.

As another aim of this paper, we shall obtain a weighted version of the inequality (1.4). In [4, 5], various imbeddings and inequalities have been obtained in the framework of anisotropic Sobolev spaces.

2. Weighted Permutation Inequality

In this section, we shall prove a weighted permutation inequality which is required in the subsequent results. Let w be a product type weight on Ω, i.e., $w(x) = w_1(x_1)w_2(x_2)\dots w_n(x_n)$. Then the weighted mixed norm space $L_w^P(\Omega)$ consists of all measurable functions u on Ω for which

$$\|u\|_{P,w} = \left\| \left\| \|u w_1^{1/p_1}\|_{p_1} w_2^{1/p_2} \right\|_{p_2} \dots w_n^{1/p_n} \right\|_{p_n} < \infty, \tag{2.1}$$

i.e.,

$$\|u\|_{P,w} = \Big\| \big\| \|u\|_{p_1,w_1} \big\|_{p_2,w_2} \cdots \Big\|_{p_n,w_n} < \infty.$$

The permutation inequality asserts that if the components of P are arranged in non-increasing order then the resulting mixed L^P-norm (1.2) will not decrease. This fact was proved in [2] (see also [1]). We prove the weighted version of the same, i.e., for the inequality (2.1).

Let σ be a permutation of $\{1, 2, \dots, n\}$. For $P = (p_1, p_2, \dots, p_n)$, we write $\sigma P = (p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(n)})$ and similarly for $u = u(x_1, x_2, \dots, x_n)$, we write $\sigma u = u(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$. The weighted permutation inequality is formulated in the following lemma:

Lemma 2.1. *Let w be a product type weight on Ω and σ_1, σ_2 be permutations of $\{1, 2, \dots, n\}$ so that the components of $\sigma_1 P$ are in non-decreasing order while the components of $\sigma_2 P$ are in non-increasing order. Then for a function u defined on Ω, the following holds:*

$$\|\sigma_1 u\|_{\sigma_1 P, \sigma_1 w} \le \|\sigma_2 u\|_{\sigma_2 P, \sigma_2 w}.$$

Proof. We shall prove the result for $n = 2$ since then the assertion would follow by induction. Now $P = (p_1, p_2)$. Without loss of generality, we can assume that $p_1 \le p_2$. Then $\sigma_1 P = (p_1, p_2)$ and $\sigma_2 P = (p_2, p_1)$.

Now by taking $r = p_2/p_1$, we have by Minkowski's integral inequality

$$\begin{aligned}
\|\sigma_1 u\|_{\sigma_1 P, \sigma_1 w} &= \left(\int\limits_{I_2} \left(\int\limits_{I_1} |u(x_1, x_2)|^{p_1} w(x_1) dx_1 \right)^{p_2/p_1} w(x_2) dx_2 \right)^{1/p_2} = \\
&= \left\| \int\limits_{I_1} |u(x_1, x_2)|^{p_1} w(x_1) dx_1 \right\|_{r, w(x_2)}^{1/p_1} \le \\
&\le \left(\int\limits_{I_1} \Big\| |u(x_1, x_2)|^{p_1} w(x_1) \Big\|_{r, w(x_2)} dx_1 \right)^{1/p_1} = \\
&= \left(\int\limits_{I_1} \left(\int\limits_{I_2} |u(x_1, x_2)|^{p_1 r} w^r(x_1) w(x_2) dx_2 \right)^{1/r} dx_1 \right)^{1/p_1} = \\
&= \left(\int\limits_{I_1} \left(\int\limits_{I_2} |u(x_1, x_2)|^{p_2} w(x_2) dx_2 \right)^{p_1/p_2} w(x_1) dx_1 \right)^{1/p_1} = \\
&= \|\sigma_2 u\|_{\sigma_2 P, \sigma_2 w}.
\end{aligned}$$

□

3. Weighted Anisotropic Sobolev Inequality

We say that a weight function w defined on $I = (a, b)$, $-\infty \leq a < b \leq \infty$ belongs to the class $D(I)$, written $w \in D(I)$, if w is differentiable and $\sup \frac{w'(x)}{w(x)} < \infty$.

Remark 3.1. The condition of differentiability on weights seems to be restrictive but many of the useful weights are so, e.g, power weights, exponential weights etc. Further, the class $D(I)$ is non-trivial since $e^x \in D(I)$ and all power weights belong to $D(I)$ whenever $a > 0$.

The following is the first main result of the paper in which a weighted version of the inequality (1.3) is obtained.

Theorem 3.2. *Let $1 \leq i \leq n$, $p_i \geq 1$, $\sum_{i=1}^{n} (1/p_i) > 1$ and $w = w_1 w_2 \dots w_n$ be a product type weight defined on Ω such that for each i, $w_i \in D(I_i)$. If*

$$\frac{1}{q} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{p_i}\right) - \frac{1}{n},$$

then there exists constants C and K such that

$$\|u\|_{q,w} \leq \frac{1}{n}\sum_{i=1}^{n}\left(C\|D_i u\|_{p_i,w} + K\|u\|_{p_i,w}\right)$$

holds for all $u \in C_0^\infty(\Omega)$.

Proof. Since $w_i \in D(I_i)$, there exists a constant $K > 0$ such that

$$D_i w_i(x_i) \leq K w_i(x_i). \tag{3.1}$$

For $1 \leq i \leq n$, take $s_i \geq 1$. We have

$$|u(x)|^{s_i} w_i(x_i) = \int_{a_i}^{x_i} D_i\left(|u(x)|^{s_i} w_i(x_i)\right) dx_i$$

so that using (3.1) we get

$$\begin{aligned}
\sup_{x_i} |u(x)|^{s_i} w_i(x_i) &\leq \int_{I_i} D_i\left(|u(x)|^{s_i} w_i(x_i)\right) dx_i = \\
&= \int_{I_i} D_i\left(|u(x)|^{s_i}\right) w_i(x_i) dx_i + \int_{I_i} |u(x)|^{s_i} D_i w_i(x_i) dx_i \leq \\
&\leq s_i \int_{I_i} |u(x)|^{s_i-1} D_i|u(x)| w_i(x_i) dx_i + K \int_{I_i} |u(x)|^{s_i} w_i(x_i) dx_i. \qquad (3.2)
\end{aligned}$$

Following the notation of Adams [1], let us recall that

$$v_i(\alpha,\beta) = (\beta,\beta,\dots,\alpha,\dots,\beta),$$

where α occupies the i^{th} position. Let $C = \max\limits_{1\le i\le n} s_i$. By taking L^1_w norm on both sides of (3.2) w.r.t. other $n-1$ components and using Hölder's inequality we get

$$\begin{aligned}\|\sigma|u|^{s_i}\|_{\sigma v_i(\infty,1),\sigma w} &\le s_i\||u|^{s_i-1}D_iu\|_{1,w} + K\|u^{s_i}\|_{1,w} \le \\ &\le C\||u|^{s_i-1}\|_{p_i',w}\|D_iu\|_{p_i,w} + K\|u^{s_i-1}\|_{p_i',w}\|u\|_{p_i,w} = \\ &= C\|u\|^{s_i-1}_{(s_i-1)p_i',w}\|D_iu\|_{p_i,w} + K\|u\|^{s_i-1}_{(s_i-1)p_i',w}\|u\|_{p_i,w} = \\ &= \|u\|^{s_i-1}_{(s_i-1)p_i',w}\left(C\|D_iu\|_{p_i,w} + K\|u\|_{p_i,w}\right),\end{aligned}$$

where σ is a permutation of $\{1,2,\dots,n\}$ with $\sigma(1)=i$ and $\frac{1}{p_i}+\frac{1}{p_i'}=1$. By the weighted permutation inequality (Lemma 2.1)

$$\begin{aligned}\||u|^{s_i}\|_{v_i(\infty,1),w} &\le \|\sigma|u|^{s_i}\|_{\sigma v_i(\infty,1),\sigma w} \le \\ &\le \|u\|^{s_i-1}_{(s_i-1)p_i',w}\left(C\|D_iu\|_{p_i,w} + K\|u\|_{p_i,w}\right).\end{aligned}$$

Let $s = s_1+s_2+\cdots+s_n$ and denote

$$\frac{1}{T} = \sum_{i=1}^{n}\frac{1}{v_i(\infty,1)} = (n-1,n-1,\dots,n-1)$$

so that $T = (\frac{1}{n-1},\frac{1}{n-1},\dots,\frac{1}{n-1})$. By applying Hölder's inequality we get

$$\begin{aligned}\|u\|^s_{s/(n-1),w} &= \||u|^s\|_{1/(n-1),w} = \\ &= \||u|^{s_1+s_2+\cdots+s_n}\|_{T,w} \le \\ &\le \prod_{i=1}^{n}\||u|^{s_i}\|_{v_i(\infty,1),w} \le \\ &\le \|u\|^{s_i-1}_{(s_i-1)p_i',w}\left(C\|D_iu\|_{p_i,w} + K\|u\|_{p_i,w}\right).\end{aligned} \tag{3.3}$$

Choose the number s_i so that

$$(s_1-1)p_1' = (s_2-1)p_2' = \cdots = (s_n-1)p_n' = q.$$

Then

$$q = (s_i-1)p_i' = (s_i-1)\frac{p_i}{p_i-1} \tag{3.4}$$

or

$$s_i = 1 + q\left(1-\frac{1}{p_i}\right).$$

Therefore

$$s = s_1+s_2+\cdots+s_n = n + q\left(n - \sum_{i=1}^{n}\frac{1}{p_i}\right) = (n-1)q$$

since by assumption

$$\frac{1}{q} = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} - 1}{n}.$$

Then (3.3) and (3.4) give

$$\begin{aligned}\|u\|_{q,w}^{s} &\leq \prod_{i=1}^{n} \|u\|_{q,w}^{s_i - 1} \left(C\|D_i u\|_{p_i,w} + K\|u\|_{p_i,w}\right) = \\ &= \|u\|_{q,w}^{s-n} \prod_{i=1}^{n} \left(C\|D_i u\|_{p_i,w} + K\|u\|_{p_i,w}\right).\end{aligned}$$

Hence

$$\|u\|_{q,w}^{n} \leq \prod_{i=1}^{n} \left(C\|D_i u\|_{p_i,w} + K\|u\|_{p_i,w}\right)$$

so that

$$\|u\|_{q,w} \leq \left(\prod_{i=1}^{n} \Big(C\|D_i u\|_{p_i,w} + K\|u\|_{p_i,w}\Big)\right)^{1/n}$$

which, by the arithmetic-geometric mean inequality, gives that

$$\|u\|_{q,w} \leq \frac{1}{n} \sum_{i=1}^{n} \left(C\|D_i u\|_{p_i,w} + K\|u\|_{p_i,w}\right)$$

and the assertion follows. □

Remark 3.3. Theorem 3.3 extends a result of Adams [2] who proves it for the non-weighted case.

4. Another weighted anisotropic Sobolev Inequality

In [1], Adams proves another type of anisotropic Sobolev inequality as given by (1.4). Below we prove its weighted version.

Theorem 4.1. *Let $2 \leq n < \infty$, $1 \leq p \leq q$ and $w = w_1 w_2 \dots w_n$ be a product type weight defined on Ω such that for each i, $w_i \in D(I_i)$. If $r > 0$ satisfies*

$$\frac{n}{r} = \frac{1}{p} + \frac{n-1}{q} - 1$$

then there exists constants s and K such that

$$\|u\|_{r,w} \leq \frac{1}{n} \sum_{i=1}^{n} \left(s\|D_i u\|_{v_i(p,q)} + K\|u\|_{v_i(p,q)}\right)$$

for all $u \in C_0^{\infty}(\Omega)$.

Proof. For $s \geq 1$ we have

$$|u(x)|^s w_i(x_i) = \int\limits_{a_i}^{x_i} D_i(|u(x)|^s w_i(x_i))dx_i =$$

$$= s\int\limits_{a_i}^{x_i} |u(x)|^{s-1}|D_i u(x)|w_i(x_i)dx_i + \int\limits_{a_i}^{x_i} |u(x)|^s D_i w_i(x_i)dx_i$$

By using (3.1) we get

$$\sup_{x_i} |u(x)|^s w_i(x_i) \leq$$

$$\leq s\int\limits_{I_i} |u(x)|^{s-1}|D_i u(x)|w_i(x_i)dx_i + \int\limits_{I_i} |u(x)|^s D_i w_i(x_i)dx_i \leq$$

$$\leq s\int\limits_{I_i} |u(x)|^{s-1}|D_i u(x)|w_i(x_i)dx_i + K\int\limits_{I_i} |u(x)|^s w_i(x_i)dx_i. \quad (4.1)$$

Let $\lambda \geq 1$ be such that

$$\frac{1}{\lambda} = \frac{1}{q} + \frac{1}{p'} = \frac{1}{q} + 1 - \frac{1}{p}.$$

By taking L^λ_w norm of both sides of (4.1) w.r.t. other $(n-1)$ variables, we get

$$\|\sigma|u|^s\|_{\sigma v_i(\infty,\lambda),\sigma w} \leq s\|\sigma|u|^{s-1}D_i u\|_{\sigma v_i(1,\lambda),\sigma w} + K\|\sigma|u|^s\|_{\sigma v_i(1,\lambda),\sigma w}, \quad (4.2)$$

where σ be any permutation of $(1,2,\ldots,n)$ with $\sigma(1) = i$. By using (4.2) and an application of Hölder's and weighted permutation inequality (Lemma 2.1), we have

$$\begin{aligned}
\|u\|^s_{v_i(\infty,s\lambda),w} &= \||u|^s\|_{v_i(\infty,\lambda),w} \\
&\leq \|\sigma|u|^s\|_{\sigma v_i(\infty,\lambda),\sigma w} \\
&\leq s\|\sigma|u|^{s-1}D_i u\|_{\sigma v_i(1,\lambda),\sigma w} + \\
&\quad + K\|\sigma|u|^s\|_{\sigma v_i(1,\lambda),\sigma w} \leq \\
&\leq s\|\sigma|u|^{s-1}\|_{\sigma v_i(p',p'),\sigma w}\|\sigma D_i u\|_{\sigma v_i(p,q),\sigma w} + \\
&\quad + K\|\sigma|u|^{s-1}\|_{\sigma v_i(p',p'),\sigma w}\|\sigma|u|\|_{\sigma v_i(p,q),\sigma w} \leq \\
&\leq s\|u\|^{s-1}_{(s-1)p',w}\|D_i u\|_{v_i(p,q),w} + K\|u\|^{s-1}_{(s-1)p',w}\|u\|_{v_i(p,q),w} = \\
&= \|u\|^{s-1}_{(s-1)p',w}\left(s\|D_i u\|_{v_i(p,q),w} + K\|u\|_{v_i(p,q),w}\right)
\end{aligned}$$

or

$$\|u\|_{v_i(\infty,s\lambda),w} \leq \|u\|^{1-1/s}_{(s-1)p',w}\left(s\|D_i u\|_{v_i(p,q),w} + K\|u\|_{v_i(p,q),w}\right)^{1/s}.$$

Denote

$$\frac{1}{T} = \sum_{i=1}^{n} \frac{1}{v_i(\infty, s\lambda)} = \left(\frac{n-1}{s\lambda}, \frac{n-1}{s\lambda}, \dots, \frac{n-1}{s\lambda}\right)$$

so that $T = \left(\frac{s\lambda}{n-1}, \frac{s\lambda}{n-1}, \dots, \frac{s\lambda}{n-1}\right)$. By using Hölder's inequality, we have

$$\begin{aligned}
\|u\|^n_{\frac{ns\lambda}{n-1},w} &= \left\| \prod_{i=1}^{n} u \right\|_{T,w} \le \\
&\le \prod_{i=1}^{n} \|u\|_{v_i(\infty,s\lambda),w} \le \\
&\le \|u\|^{n-n/s}_{(s-1)p',w} \prod_{i=1}^{n} \left(s\|D_i u\|_{v_i(p,q),w} + K\|u\|_{v_i(p,q),w}\right)^{1/s}. \qquad (4.3)
\end{aligned}$$

Choose the number s so that $\frac{ns\lambda}{n-1} = (s-1)p'$ which on using the definition of λ gives

$$s = \frac{(n-1)(q+p')}{p'n - q' - p'}.$$

Denote $r = \frac{ns\lambda}{n-1}$ $(= (s-1)p')$. Then

$$\frac{n}{r} = \frac{1}{p} + \frac{n-1}{q} - 1 > 0.$$

Finally, the estimate (4.3) reduces to

$$\begin{aligned}
\|u\|_{r,w} &\le \left(\prod_{i=1}^{n} \left(s\|D_i u\|_{v_i(p,q),w} + K\|u\|_{v_i(p,q),w}\right) \right)^{1/n} \le \\
&\le \frac{1}{n} \sum_{i=1}^{n} \left(s\|D_i u\|_{v_i(p,q),w} + K\|u\|_{v_i(p,q),w}\right)
\end{aligned}$$

and we are done. □

Remark 4.2. As mentioned in Remark 3.3, Theorem 4.1 also extends a result of Adams [1] to the weighted case.

Remark 4.3. In the final step of the proofs of Theorems 3.2 and 4.1, the arithmetic-geometric mean inequality has been used. In fact, if we consider the more general power means $P_a = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^a\right)^{1/a}$, then it is known that P_a is increasing in a, which for $a = 1$ becomes arithmetic mean and for $a = 0$ becomes geometric mean $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$. It is of interest if the monotonicity of P_a can be used in more generality, e.g., the case when $a < 0$ is of interest.

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