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REFINEMENTS OF HÖLDER AND MINKOWSKI INEQUALITIES WITH WEIGHTS

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Abstract. We establish refinements of the inequality between quasi-arithmetic means with the help of generalized mixed means. This leads to generalizations of the Hölder's and Minkowski's inequalities. To derive the main results we use refinements of the discrete Jensen's inequality for functions of several variables, given in [3].

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1. Introduction and Preliminary Results

Let U be a convex subset of a real linear space, and let $f : U \to \mathbb{R}$ be a convex function. If $x_i \in U$ $(1 \le i \le n)$ and $p_i \ge 0$ $(1 \le i \le n)$ such that $\sum\limits_{i=1}^{n} p_i = 1$, then the discrete Jensen's inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

holds.

Let $I \subset \mathbb{R}$ be an interval, let $h : I \to \mathbb{R}$ be a continuous and strictly monotone function, let $\mathbf{a} = (a_1, \dots, a_n) \in I^n$, and let $\mathbf{p} = (p_1, \dots, p_n)$ be a nonnegative n-tuple such that $\sum\limits_{i=1}^{n} p_i = 1$. The quasi-arithmetic h-mean of $\mathbf{a}$ with weights $\mathbf{p}$ is defined by

$$h_n(\mathbf{a};\mathbf{p}) = h_n(a_i; 1 \le i \le n; \mathbf{p}) = h(\mathbf{a};\mathbf{p};n) := h^{-1}\left(\sum_{i=1}^{n} p_i h(a_i)\right).$$

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If $p_i = \frac{1}{n}$ $(1 \leq i \leq n)$, then $\mathbf{p}$ will be ignored from the previous notations.

First, we extend Beck's results (see [1]). The following hypothesis is assumed:

(A_1) Let $L_t : I_t \to \mathbb{R}$ $(t = 1, \ldots, m)$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_1 \times \cdots \times I_m \to I_N$ be a continuous function. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{x}^{(t)} \in I_t^n$ for each $t = 1, \ldots, m$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^{n} p_i = 1$.

The result is a simple consequence of the discrete Jensen's inequality.

Theorem 1.1. *Assume (A_1). If N is an increasing function, then the inequality*

$$f(L_1(\mathbf{x}^{(1)};\mathbf{p};n), \ldots, L_m(\mathbf{x}^{(m)};\mathbf{p};n)) \geq$$
$$\geq N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)}, \ldots, x_i^{(m)}))\right), \tag{2}$$

holds for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ and $\mathbf{p}$, if and only if the function H defined on $L_1(I_1) \times \cdots \times L_m(I_m)$ by

$$H(t_1, \ldots, t_m) := N\left(f\left(L_1^{-1}(t_1), \ldots, L_m^{-1}(t_m)\right)\right) \tag{3}$$

is concave. The inequality in (2) is reversed for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ and $\mathbf{p}$, if and only if H is convex.

Proof. We replace the convex function f by $-H$ or H, and x_i by $L_t(x_i^{(t)})$ in (1) and then applying the increasing function N^{-1} we get the required results. □

Beck's original result was the special case of Theorem 1.1, where $m = 2$ and $I_1 = [k_1, k_2]$, $I_2 = [l_1, l_2]$ and $I_N = [n_1, n_2]$ (see [2], p. 249).

For simplicity, in the case $m = 2$ we use the following form of (A_1):

(A_2) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^{n} p_i = 1$.

Then (2) has the form

$$f\left(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\right) \geq N_n(f(\mathbf{a},\mathbf{b});\mathbf{p}), \tag{4}$$

where $f(\mathbf{a},\mathbf{b})$ means $(f(a_1, b_1), \ldots, f(a_n, b_n))$.

The following results are important special cases of Theorem 1.1, and generalize the corresponding results of Beck. The next hypothesis will be used:

(A_3) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$ such that either $I_K + I_L \subset I_N$ and $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$ or $I_K,\, I_L \subset]0,\infty[$, $I_K \cdot I_L \subset I_N$ and $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. Assume further that the functions K, L and N are twice continuously differentiable on the interior of their domains, respectively. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$.

The interior of a subset A of $\mathbb{R}$ is denoted by A°.

Corollary 1.2. *Assume* (A_3) *with* $f(x,y) = x + y$ $((x,y) \in I_K \times I_L)$, *and assume that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (4) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ. \tag{5}$$

Corollary 1.3. *Assume* (A_3) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N°, *respectively. Assume further that* K', L', N', A, B *and* C *are all positive. Then* (4) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

To prove these corollaries, similar arguments can be applied as in the analogous results of Beck. We just sketch the proof of Corollary 1.2.

Proof. By Theorem 1.1, it is enough to prove that the function

$$H : K(I_K) \times L(I_L), \quad H(t,s) := N\left(K^{-1}(t) + L^{-1}(s)\right)$$

is concave. Since H is continuous, and twice continuously differentiable on the interior $K(I_K^\circ) \times L(I_L^\circ)$ of its domain, we have to show that

$$D_{11}H(t,s)h_1^2 + 2D_{12}H(t,s)h_1h_2 + D_{22}H(t,s)h_2^2 \leq 0$$

for all $(t,s) \in K(I_K^\circ) \times L(I_L^\circ)$ and $(h_1,h_2) \in \mathbb{R}^2$. By computing the partial derivatives of H of order 2 at the points of $K(I_K^\circ) \times L(I_L^\circ)$, we have that this condition follows from (5). □

In [4], Mitrinović and Pečarić obtained a new inequality like (4), which is based on the following refinement of the discrete Jensen's inequality (see Pečarić and Volenec [6]):

Lemma A. *Let* f *be a real valued convex function defined on a convex set* U *from a real linear space. If* $x_1, \ldots, x_n \in U$, *and*

$$f_{k,n} = f_{k,n}(x_1, \ldots, x_n) :=$$
$$:= \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{1}{k}(x_{i_1} + \cdots + x_{i_k})\right), \quad 1 \leq k \leq n, \tag{6}$$

then

$$f\left(\sum_{i=1}^{n}\frac{1}{n}x_i\right) = f_{n,n} \leq \cdots \leq f_{k,n} \leq \cdots \leq f_{1,n} = \sum_{i=1}^{n}\frac{1}{n}f(x_i). \tag{7}$$

Assume (A_2). We denote by α_i^k $(1 \leq i \leq v)$ and β_i^k $(1 \leq i \leq v)$ the k-tuples of $\mathbf{a}$ and $\mathbf{b}$ respectively, where $v = \begin{pmatrix} n \\ k \end{pmatrix}$. Following [4], we introduce the mixed N-K-L means of $\mathbf{a}$ and $\mathbf{b}$:

$$M(N,K,L;k) := N_v(f(K_k(\alpha_i^k), L_k(\beta_i^k)); 1 \leq i \leq v), \quad 1 < k < n, \tag{8}$$

and

$$M(N,K,L;1) := N_n(f(\mathbf{a},\mathbf{b})),$$
$$M(N,K,L;n) := f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right).$$

The promised theorem from [4] is the next:

Theorem A. *Assume* (A_2). *Let N be an increasing* (*decreasing*) *function, and let*

$$H : K(I_K) \times L(I_L) \to \mathbb{R}, \quad H(s,t) := N\left(f\left(K^{-1}(s), L^{-1}(t)\right)\right)$$

be a convex (*concave*) *function. Then*

$$M(N,K,L;k+1) \leq M(N,K,L;k), \quad k = 1,\ldots,n-1. \tag{9}$$

If N is increasing (*decreasing*) *but H is concave* (*convex*) *then the inequalities in* (9) *are reversed.*

Here we can apply Lemma A to the function H and to the points $(K(a_i), L(b_i))$ $(1 \leq i \leq n)$.

On the analogy of Corollary 1.2 and Corollary 1.3, we have the following consequences of Theorem A.

Corollary A. *Assume* (A_3) *with* $f(x,y) = x + y$ $((x,y) \in I_K \times I_L)$. *Assume further that* K', L', N', K'', L'' *and* N'' *are all positive and* $E(x) + F(y) \leq G(x+y)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, *where* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$. *Then* (9) *with reverse inequality is valid.*

Corollary B. *Assume* (A_3) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N°, *respectively. If* K', L', M', A, B *and* C *are all positive and* $A(x) + B(y) \leq C(xy)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, *then* (9) *with reverse inequality is valid.*

The results given in [4] are without weights. But in this paper, we give results with weights. We improve the results given in [4] by using a new refinement of the discrete Jensen's inequality from [3]. First, we give the

notations introduced in [3]:
Let X be a set, $P(X)$ its power set and $|X|$ denotes the number of elements in X. Let $u \geq 1$ and $v \geq 2$ be fixed integers. Define the functions

$$S_{v,w} : \{1, \ldots, u\}^v \to \{1, \ldots, u\}^{v-1}, \quad 1 \leq w \leq v,$$

$$S_v : \{1, \ldots, u\}^v \to P\left(\{1, \ldots, u\}^{v-1}\right),$$

and

$$T_v : P\left(\{1, \ldots, u\}^v\right) \to P\left(\{1, \ldots, u\}^{v-1}\right)$$

by

$$S_{v,w}(i_1, \ldots, i_v) := (i_1, \ldots, i_{w-1}, i_{w+1}, \ldots, i_v), \quad 1 \leq w \leq v,$$

$$S_v(i_1, \ldots, i_v) := \bigcup_{w=1}^{v} \{S_{v,w}(i_1, \ldots, i_v)\},$$

and

$$T_v(I) := \begin{cases} \bigcup\limits_{(i_1,\ldots,i_v)\in I} S_v(i_1, \ldots, i_v), & I \neq \phi, \\ \phi, & I = \phi. \end{cases}$$

Further, introduce the function

$$\alpha_{v,i} : \{1, \ldots, u\}^v \to \mathbb{N}, \quad 1 \leq i \leq u,$$

via

$$\alpha_{v,i}(i_1, \ldots, i_v) := \text{Number of occurrences of } i \text{ in the sequence } (i_1, \ldots, i_v).$$

For each $I \in P(\{1, \ldots, u\}^v)$, let

$$\alpha_{I,i} := \sum_{(i_1,\ldots,i_v)\in I} \alpha_{v,i}(i_1, \ldots, i_v), \qquad 1 \leq i \leq u.$$

It is easy to observe from the construction of the functions S_v, $S_{v,w}$, T_v and $\alpha_{v,i}$ that they do not depend essentially on u, so we can write for short S_v for S_v^u, and so on.

(H$_1$) The following considerations concern a subset I_k of $\{1, \ldots, n\}^k$ satisfying

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n, \tag{10}$$

where $n \geq 1$ and $k \geq 2$ are fixed integers.

Next, we proceed inductively to define the sets $I_l \subset \{1, \ldots, n\}^l$ $(k-1 \geq l \geq 1)$ by

$$I_{l-1} := T_l(I_l), \quad k \geq l \geq 2.$$

By (10), $I_1 = \{1, \ldots, n\}$ and this implies that $\alpha_{I_1,i} = 1$ for $1 \leq i \leq n$. From (10) again, we have $\alpha_{I_l,i} \geq 1$ $(k-1 \geq l \geq 1, 1 \leq i \leq n)$.

For every $k \geq l \geq 2$ and for any $(j_1, \ldots, j_{l-1}) \in I_{l-1}$ let

$$H_{I_l}(j_1, \ldots, j_{l-1}) :=$$
$$:= \{((i_1, \ldots, i_l), m) \in I_l \times \{1, \ldots, l\} \mid S_{l,m}(i_1, \ldots, i_l) = (j_1, \ldots, j_{l-1})\}.$$

Using these sets we define the functions $t_{I_k,l} : I_l \to \mathbb{N}$ $(k \geq l \geq 1)$ inductively by

$$t_{I_k,k}(i_1,\dots,i_k) := 1, \quad (i_1,\dots,i_k) \in I_k\,; \tag{11}$$

$$t_{I_k,l-1}(j_1,\dots,j_{l-1}) := \sum_{((i_1,\dots,i_l),m)\in H_{I_l}(j_1,\dots,j_{l-1})} t_{I_k,l}\,(i_1,\dots,i_l). \tag{12}$$

We need another hypothesis:

(H$_2$) Let U be a convex set in $\mathbb{R}^m$, $\mathbf{x}_1,\dots,\mathbf{x}_n \in U$, and let $\mathbf{p} := (p_1,\dots,p_n)$ be a positive n-tuples such that $\sum_{i=1}^n p_i = 1$. Further, let $f : U \to \mathbb{R}$ be a convex function.

For any $k \geq l \geq 1$ set

$$A_{l,l} = A_{l,l}(I_l;\mathbf{x}_1,\dots,\mathbf{x}_n;\mathbf{p}) := \\ := \sum_{(i_1,\dots,i_l)\in I_l}\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_l,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_l,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_l,i_s}}}\right), \tag{13}$$

and associate to each $k-1 \geq l \geq 1$ the number

$$A_{k,l} = A_{k,l}(I_k;\mathbf{x}_1,\dots,\mathbf{x}_n;\mathbf{p}) := \frac{1}{(k-1)\dots l}\times \\ \times \sum_{(i_1,\dots,i_l)\in I_l} t_{I_k,l}(i_1,\dots,i_l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right). \tag{14}$$

The following refinement of the discrete Jensen's inequality is developed in [3]:

Theorem B. *Assume* (H_1) *and* (H_2). *Then*

$$f\left(\sum_{i=1}^{n} p_i\mathbf{x}_i\right) \leq A_{k,k} \leq A_{k,k-1} \leq \dots \leq A_{k,2} \leq A_{k,1} = \sum_{i=1}^{n} p_i f(\mathbf{x}_i), \tag{15}$$

where the numbers $A_{k,l}$ $(k \geq l \geq 1)$ *are defined in* (13) *and* (14). *If* f *is a concave function then the inequalities in* (15) *are reversed.*

The following result is also given in [3].

Theorem C. *Assume* (H_1) *and* (H_2), *and suppose* $|H_{I_l}(j_1,\dots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\dots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$A_{k,l} = A_{l,l} = \\ = \frac{n}{l|I_l|}\sum_{(i_1,\dots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l} p_{i_s}}\right), \qquad k \geq l \geq 1, \tag{16}$$

and thus

$$f\left(\sum_{r=1}^{n} p_r \mathbf{x}_r\right) \le A_{k,k} \le A_{k-1,k-1} \le \cdots \le A_{2,2} \le A_{1,1} = \sum_{r=1}^{n} p_r f(\mathbf{x}_r). \quad (17)$$

If f is a concave function then the inequalities in (17) *are reversed.*

2. Generalizations of Beck's Result

In what follows (A_1) and (H_1) are assumed. The weighted mixed means relative to (13) and (14) are defined in the following ways:

$$\begin{aligned} &M^1_{k,k}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) := \\ &:= N^{-1}\Bigg(\sum_{\mathbf{i}^k\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)\times \\ &\times N\Big(f\Big(L_1(\mathbf{x}^{(1)};\frac{\mathbf{p}}{\alpha_{I_k}};k),\dots,L_m(\mathbf{x}^{(m)};\frac{\mathbf{p}}{\alpha_{I_k}};k)\Big)\Big)\Bigg) \end{aligned} \quad (18)$$

and for $k-1 \ge l \ge 1$

$$\begin{aligned} &M^1_{k,l}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) := \\ &:= N^{-1}\Bigg(\frac{1}{(k-1)\dots l}\sum_{\mathbf{i}^l\in I_l} t_{I_k,l}(\mathbf{i}^l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)\times \\ &\times N\Big(f\Big(L_1(\mathbf{x}^{(1)};\frac{\mathbf{p}}{\alpha_{I_k}};l),\dots,L_m(\mathbf{x}^{(m)};\frac{\mathbf{p}}{\alpha_{I_k}};l)\Big)\Big)\Bigg) \end{aligned} \quad (19)$$

where for $k \ge l \ge 1$

$$L_t(\mathbf{x}^{(t)};\frac{\mathbf{p}}{\alpha_{I_k}};l) := L_t^{-1}\left(\frac{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}L_t(x^{(t)}_{i_s})}{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right), \quad t = 1,\dots,m,$$

respectively, and $\mathbf{i}^l := (i_1,\dots,i_l)$.

Now, we get an interpolation of (2) by the direct application of Theorem B as follows.

Theorem 2.1. *Assume* (A_1) *and* (H_1). *If* N *is an increasing* (*decreasing*) *function, then the inequalities*

$$
\begin{aligned}
f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \leq &\\
\leq M^1_{k,k}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) \leq &\\
\leq M^1_{k,k-1}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) \leq &\\
\leq \cdots \leq &\\
\leq M^1_{k,2}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) \leq &\\
\leq M^1_{k,1}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) = &\\
= N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},\ldots,x_i^{(m)}))\right) &
\end{aligned}
\tag{20}
$$

hold for all possible $\mathbf{x}^{(t)}$ $(t = 1,\ldots,m)$ *and* $\mathbf{p}$, *if and only if the function* H *defined in Theorem* 1.1 *is convex* (*concave*). *If* N *is an increasing* (*decreasing*) *function, then the inequalities in* (20) *are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1,\ldots,m)$ *and* $\mathbf{p}$, *if and only if* H *is concave* (*convex*).

Proof. Suppose N is increasing and the function $H : L_1(I_1)\times\cdots\times L_m(I_m) \to \mathbb{R}$,

$$H(t_1,\ldots,t_m) = N\left(f\left(L_1^{-1}(t_1),\ldots,L_m^{-1}(t_m)\right)\right)$$

is convex. We apply Theorem B to the function H and to the vectors $(L_1(x_i^1),\ldots,L_m(x_i^m))$, $i = 1,\ldots,n$. Then the first term in (15) gives

$$H\left(\sum_{i=1}^{n} p_i(L_1(x_i^1),\ldots,L_m(x_i^m))\right) =$$

$$= N\left(f\left(L_1^{-1}\left(\sum_{i=1}^{n} p_i L_1(x_i^1)\right),\ldots,L_m^{-1}\left(\sum_{i=1}^{n} p_i L_m(x_i^m)\right)\right)\right) =$$

$$= N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right)\right).$$

The last term in (15) will be

$$\sum_{i=1}^{n} p_i H(L_1(x_i^1),\ldots,L_m(x_i^m)) =$$

$$= \sum_{i=1}^{n} p_i N\left(f\left(x_i^1,\ldots,x_i^m\right)\right).$$

$A_{k,k}$ in (15) has the form

$$\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)H\left(\frac{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}(L_1(x_{i_s}^1),\ldots,L_m(x_{i_s}^m))}{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)=$$

$$=\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)N\left(f\left(L_1^{-1}\left(\frac{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}L_1(x_{i_s}^1)}{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right),\ldots,\right.\right.$$

$$\left.\left.\ldots L_m^{-1}\left(\frac{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}L_m(x_{i_s}^m)}{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)\right)\right)=$$

$$=\sum_{\mathbf{i}^k\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)N\left(f\left(L_1(\mathbf{x}^{(1)};\frac{\mathbf{p}}{\alpha_{I_k}};k),\ldots,L_m(\mathbf{x}^{(m)};\frac{\mathbf{p}}{\alpha_{I_k}};k)\right)\right)=$$

$$=M_{k,k}^1(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}).$$

A similar argument shows that for $k-1\geq l\geq 1$ $A_{k,l}$ in (15) can be written as

$$M_{k,l}^1(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}).$$

The inequality (20) follows from these observations and Theorem B since N^{-1} is increasing.

The converse is obtained by Theorem 1.1. □

The following applications of Theorem 2.1 are motivated by the examples given in [3] corresponding to Theorem B.

Example 2.2. Assume (A_1). Consider

$$I_2:=\{(i_1,i_2)\in\{1,\ldots,n\}^2\mid i_1|i_2\},$$

where $i_1|i_2$ means that i_1 divides i_2. Since $i|i$ $(i=1,\ldots,n)$, therefore (H_1) holds and

$$\alpha_{I_2,i}=\left[\frac{n}{i}\right]+d(i),\quad i=1,\ldots,n,$$

where $[\frac{n}{i}]$ is the largest positive integer not greater than $\frac{n}{i}$, and $d(i)$ means the number of positive divisors of i. Then a corresponding weighted mixed mean is

$$M^1_{2,2}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) = \\ = N^{-1}\Bigg(\sum_{(i_1,i_2)\in I_2}\Bigg(\sum_{s=1}^{2}\frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}\Bigg)\times \\ \times N\Big(f\Big(L_1(\mathbf{x}^{(1)};\frac{\mathbf{p}}{\alpha_{I_2}}),\dots,L_m(\mathbf{x}^{(m)};\frac{\mathbf{p}}{\alpha_{I_2}})\Big)\Big)\Bigg), \quad (21)$$

where

$$L_t(\mathbf{x}^{(t)};\frac{\mathbf{p}}{\alpha_{I_2}}) := L_t^{-1}\left(\frac{\sum\limits_{s=1}^{2}\frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}L_t(x^{(t)}_{i_s})}{\sum\limits_{s=1}^{2}\frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}}\right), \quad t=1,\dots,m.$$

If N is increasing and the function H defined in Theorem 1.1 is convex, then Theorem 2.1 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\dots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \le \\ \le M^1_{2,2}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) \le \\ \le N^{-1}\left(\sum_{i=1}^{n}p_iN(f(x_i^{(1)},\dots,x_i^{(m)}))\right). \quad (22)$$

Example 2.3. Assume (A_1). Let $c_i \ge 1$ $(i=1,\dots,n)$ be integers, let $k := \sum\limits_{i=1}^{n} c_i$, and also let $I_k = P_k^{c_1,\dots,c_n}$ consist of all sequences $(i_1,\dots,i_k)$ in which the number of occurrences of $i \in \{1,\dots,n\}$ is c_i $(i=1,\dots,n)$. Then (H_1) is satisfied, and

$$I_{k-1} = \bigcup_{i=1}^{n} P_{k-1}^{c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_n}, \quad \alpha_{I_k,i} = \frac{k!}{c_1!\dots c_n!}c_i, \ i=1,\dots,n,$$

Moreover, $t_{I_k,k-1}(i_1,\dots,i_{k-1}) = k$ for

$$(i_1,\dots,i_{k-1}) \in P_{k-1}^{c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_n}, \quad i=1,\dots,n.$$

Then we can write a corresponding mixed mean as follows:

$$M^1_{k,k-1}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) = \\ = N^{-1}\left(\frac{1}{k-1}\sum_{i=1}^{n}(c_i-p_i)N\left(f\left(L_1(\mathbf{x};\frac{\mathbf{p}}{\mathbf{c}_i}),\dots,L_m(\mathbf{x};\frac{\mathbf{p}}{\mathbf{c}_i})\right)\right)\right), \quad (23)$$

where

$$L_t(\mathbf{x};\frac{\mathbf{p}}{\mathbf{c}_i}) := L_t^{-1}\left(\frac{\sum\limits_{r=1}^{n} p_r L_t(x_r^{(t)}) - \frac{p_i}{c_i} L_t(x_i^{(t)})}{1-\frac{p_i}{c_i}}\right), \quad t=1,\dots,m.$$

If M is increasing and the function H defined in Theorem 1.1 is convex, then Theorem 2.1 gives

$$\begin{aligned} f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\dots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \leq \\ \leq M^1_{k,k-1}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) \leq \\ \leq N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},\dots,x_i^{(m)}))\right). \end{aligned} \tag{24}$$

Now, we assume (A$_1$), (H$_1$) and suppose $|H_{I_l}(j_1,\dots,j_{l-1})| = \beta_{l-1}$ for any $(j_1,\dots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. Then corresponding to the core term of Theorem C, we define for $k \geq l \geq 1$

$$\begin{aligned} M^2_{l,l}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) = N^{-1}\Bigg(\frac{n}{l\,|I_l|}\sum_{\mathbf{i}^l\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right)\times \\ \times N\Big(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p}_{I_l}),\dots,L_m(\mathbf{x}^{(m)};\mathbf{p}_{I_l})\right)\Big)\Bigg), \end{aligned} \tag{25}$$

where

$$L_t(\mathbf{x}^{(t)};\mathbf{p}_{I_l}) := L_t^{-1}\left(\frac{\sum\limits_{s=1}^{l} p_{i_s} L_t(x_{i_s}^{(t)})}{\sum\limits_{s=1}^{l} p_{i_s}}\right), \quad t=1,\dots,m.$$

In this case Theorem C gives another interpolation of (2) as follows:

Theorem 2.4. *Assume* (A$_1$), (H$_1$) *and suppose* $|H_{I_l}(j_1,\dots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\dots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *If* N *is an increasing* (*decreasing*) *function, then the inequalities*

$$\begin{aligned} f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\dots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \leq \\ \leq M^2_{k,k}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) \leq \\ \leq M^2_{k-1,k-1}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) \leq \\ \leq \dots \leq M^2_{2,2}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) \leq \end{aligned}$$

$$\leq M_{1,1}^2(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) =$$

$$= N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)}, \ldots, x_i^{(m)}))\right) \quad (26)$$

hold for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$*, if and only if the function* H *defined in Theorem* 1.1 *is convex* (*concave*)*. If* N *is an increasing* (*decreasing*) *function, then the inequalities in* (26) *are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$*, if and only if* H *is concave* (*convex*)*.*

Proof. The proof is similar to the proof of Theorem 2.1. □

Now, we give some applications of Theorem 2.4 with the help of examples given in [3].

Example 2.5. Assume (A_1). If we set

$$I_k := \{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 < \cdots < i_k\}, \quad 1 \leq k \leq n,$$

then $\alpha_{I_n,i} = 1$ $(i = 1, \ldots, n)$ i.e. (H_1) is satisfied for $k = n$. It comes easily that $T_k(I_k) = I_{k-1}$ $(k = 2, \ldots, n)$, $|I_k| = \binom{n}{k}$ $(k = 1, \ldots, n)$, and for each $k = 2, \ldots, n$

$$|H_{I_k}(j_1, \ldots, j_{k-1})| = n - (k-1), \quad (j_1, \ldots, j_{k-1}) \in I_{k-1}.$$

In this case (25) becomes for $n \geq k \geq 1$

$$M_{k,k}^2(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) =$$

$$= N^{-1}\left(\frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left(\sum_{s=1}^{k} p_{i_s}\right) N\left(f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}_{I_k}), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}_{I_k})\right)\right)\right). \quad (27)$$

If N is increasing and the function H defined in Theorem 1.1 is convex, then Theorem 2.4 gives

$$f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\right) \leq$$

$$\leq M_{n,n}^2(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) \leq$$

$$\leq M_{n-1,n-1}^2(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) \leq \leq \cdots \leq$$

$$\leq M_{2,2}^2(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) \leq$$

$$\leq M_{1,1}^2(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) =$$

$$= N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)}, \ldots, x_i^{(m)}))\right). \quad (28)$$

Remark 2.6. If we take $p_1 = \cdots = p_n = \frac{1}{n}$ and $m = 2$ in (27) then we get (8). Hence the interpolation given in (28) is a generalization of (9).

Example 2.7. Assume (A_1). If we set

$$I_k := \{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \leq \cdots \leq i_k\}, \quad k \geq 1,$$

then $\alpha_{I_k,i} \geq 1$ ($i = 1, \ldots, n$) and thus (H_1) is satisfied. It is easy to see that $T_k(I_k) = I_{k-1}$ ($k = 2, \ldots$), $|I_k| = \binom{n+k-1}{k}$ ($k = 1, \ldots$), and for each $l = 2, \ldots, k$

$$|H_{I_l}(j_1, \ldots, j_{l-1})| = n, \quad (j_1, \ldots, j_{l-1}) \in I_{l-1}.$$

Under these settings (25) becomes

$$\begin{aligned} &M^2_{k,k}(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) = \\ &\quad = N^{-1}\Bigg(\frac{1}{\binom{n+k-1}{k}} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \left(\sum_{s=1}^{k} p_{i_s}\right) \times \\ &\qquad \times N\Big(f\Big(L_1(\mathbf{x}^{(1)}; \mathbf{p}_{I_k}), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}_{I_k})\Big)\Big)\Bigg). \end{aligned} \tag{29}$$

If N is increasing and the function H defined in Theorem 1.1 is convex, then Theorem 2.4 gives

$$\begin{aligned} &f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\right) \leq \cdots \leq \\ &\quad \leq M^2_{k,k}(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) \leq \\ &\quad \leq \cdots \leq M^2_{k,1}(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}) = \\ &\quad = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)}, \ldots, x_i^{(m)}))\right). \end{aligned} \tag{30}$$

Example 2.8. Assume (A_1). Let

$$I_k := \{1, \ldots, n\}^k, \quad k \geq 1.$$

Then $\alpha_{I_k,i} \geq 1$ ($i = 1, \ldots, n$), hence (H_1) holds and $T_k(I_k) = I_{k-1}$ ($k = 2, \ldots$), $|I_k| = n^k$ ($k = 1, \ldots$), also for $l = 2, \ldots, k$

$$|H_{I_l}(j_1, \ldots, j_{l-1})| = n^l, \quad (j_1, \ldots, j_{l-1}) \in I_{l-1}.$$

Therefore under these settings, for $k \geq 1$, (25) leads to

$$\begin{aligned} &M^2_{k,k}(L_1, L_2; \mathbf{x}^{(1)}, \mathbf{x}^{(2)}; \mathbf{p}) = \\ &\quad = N^{-1}\Bigg(\frac{1}{kn^{k-1}} \sum_{(i_1, \ldots, i_k) \in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) \times \\ &\qquad \times N\Big(f\Big(L_1(\mathbf{x}^{(1)}; \mathbf{p}_{I_k}), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}_{I_k})\Big)\Big)\Bigg). \end{aligned} \tag{31}$$

If N is increasing and the function H defined in Theorem 1.1 is convex, then for $k \geq 1$ Theorem 2.4 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \leq$$
$$\leq \cdots \leq M^2_{k,k}(L_1,.,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) \leq$$
$$\leq \cdots \leq M^2_{1,1}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) =$$
$$= N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},\ldots,x_i^{(m)}))\right). \quad (32)$$

Example 2.9. Assume (A_1). Let $1 \leq k \leq n$ and let I_k consist of all sequences $(i_1,\ldots,i_k)$ of k distinct numbers from $\{1,\ldots,n\}$. Then $\alpha_{I_n,i} \geq 1$ $(i = 1,\ldots,n)$, and (H_1) is satisfied. It is immediate that $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots)$, $|I_k| = n(n-1)\ldots(n-k+1)$ $(k = 1,\ldots,n)$, and for every $k = 2,\ldots,n$

$$|H_{I_k}(j_1,\ldots,j_{k-1})| = (n-k+1)k, \quad (j_1,\ldots,j_{k-1}) \in I_{k-1}.$$

Therefore under these settings, for $k = 1,\ldots,n$, (25) gives

$$M^2_{k,k}(L_1,.,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) =$$
$$= N^{-1}\Bigg(\frac{n}{kn(n-1)(n-k+1)} \sum_{(i_1,.,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) \times$$
$$\times N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p}_{I_k}),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p}_{I_k})\right)\right)\Bigg). \quad (33)$$

If N is increasing and the function H defined in Theorem 1.1 is convex, then Theorem 2.4 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \leq$$
$$\leq M^2_{n,n}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) \leq$$
$$\leq \cdots \leq M^2_{k,k}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) \leq \cdots \leq$$
$$\leq M^2_{1,1}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) =$$
$$= N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},\ldots,x_i^{(m)}))\right). \quad (34)$$

3. Generalizations of the Consequences of Beck's Result

Assume (A_2) and (H_1). Then, for $m = 2$, the reverse of (20) can be written as

$$f\left(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\right) \geq M^1_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq$$

$$\geq \cdots \geq M^1_{k,2}(K, L; \mathbf{a}, \mathbf{b}; \mathbf{p}) \geq$$

$$\geq M^1_{k,1}(K, L; \mathbf{a}, \mathbf{b}; \mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(a_i, b_i))\right). \quad (35)$$

Analogous to the results of Corollary A and Corollary B (see [4] and also [5], p. 195), we have immediately from Theorem 2.1 and Corollaries 1.2, 1.3 that

Corollary 3.1. *Assume* (A_3) *with* $f(x,y) = x + y$ $((x,y) \in I_K \times I_L)$, *assume* (H_1), *and assume that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (35) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I^\circ_K \times I^\circ_L.$$

In this case

$$M^1_{k,k}(K, L; \mathbf{a}, \mathbf{b}; \mathbf{p}) =$$

$$= N^{-1}\left(\sum_{\mathbf{i}^k \in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k, i_s}}\right) N\left(K(\mathbf{a}; \frac{\mathbf{p}}{\alpha_{I_k}}; k) + L(\mathbf{b}; \frac{\mathbf{p}}{\alpha_{I_k}}; k)\right)\right), \quad (36)$$

and for $k - 1 \geq l \geq 1$

$$M_{k,l}(K, L; \mathbf{a}, \mathbf{b}; \mathbf{p}) =$$

$$= N^{-1}\left(\frac{1}{(k-1)\dots l} \sum_{\mathbf{i}^l \in I_l} t_{I_k, l}(\mathbf{i}^l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k, i_s}}\right) \times\right.$$

$$\left. \times N\left(K(\mathbf{a}; \frac{\mathbf{p}}{\alpha_{I_l}}; l) + L(\mathbf{b}; \frac{\mathbf{p}}{\alpha_{I_l}}; l)\right)\right), \quad (37)$$

respectively, where $\mathbf{i}^l := (i_1, \dots, i_l)$.

Corollary 3.2. *Assume* (A_3) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$ *and assume* (H_1). *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x) + xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x) + xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x) + xN''(x)}$ *are defined on* I°_K, I°_L *and* I°_N *respectively. Assume further that* K', L', M', A, B *and* C *are all positive. Then* (35) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I^\circ_K \times I^\circ_L.$$

In this case

$$M^1_{k,k}(K, L; \mathbf{a}, \mathbf{b}; \mathbf{p}) =$$

$$= N^{-1}\left(\sum_{\mathbf{i}^k \in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k, i_s}}\right) N\left(K(\mathbf{a}; \frac{\mathbf{p}}{\alpha_{I_k}}; k) L(\mathbf{b}; \frac{\mathbf{p}}{\alpha_{I_k}}; k)\right)\right), \quad (38)$$

and for $k-1 \geq l \geq 1$

$$M_{k,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) =$$
$$= N^{-1}\Bigg(\frac{1}{(k-1)\dots l}\sum_{\mathbf{i}^l \in I_l} t_{I_k,l}(\mathbf{i}^l)\Bigg(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\Bigg)\times$$
$$\times N\Big(K(\mathbf{a};\frac{\mathbf{p}}{\alpha_{I_l}};l)L(\mathbf{b};\frac{\mathbf{p}}{\alpha_{I_l}};l)\Big)\Bigg), \quad (39)$$

respectively, where $\mathbf{i}^l := (i_1, \dots, i_l)$.

We also give some special cases of the Corollaries 3.1 and 3.2 as illustrations.

Remark 3.3. Under the settings of Example 2.2, if $f(x_1,x_2) = x_1 + x_2$, then (36) becomes

$$M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) =$$
$$= N^{-1}\Bigg(\sum_{(i_1,i_2)\in I_2}\Bigg(\sum_{s=1}^{2}\frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}\Bigg)N\Big(K(\mathbf{a};\frac{\mathbf{p}}{\alpha_{I_2}})+L(\mathbf{b};\frac{\mathbf{p}}{\alpha_{I_2}})\Big)\Bigg). \quad (40)$$

Under the conditions of Corollary 3.1

$$K_n(\mathbf{a};\mathbf{p}) + L_n(\mathbf{a};\mathbf{p}) \geq M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i+b_i)\right). \quad (41)$$

Similarly, if $f(x_1,x_2) = x_1x_2$, then from (38) we have

$$M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) =$$
$$= N^{-1}\Bigg(\sum_{(i_1,i_2)\in I_2}\Bigg(\sum_{s=1}^{2}\frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}\Bigg)N\Big(K(\mathbf{a};\frac{\mathbf{p}}{\alpha_{I_2}})L(\mathbf{b};\frac{\mathbf{p}}{\alpha_{I_2}})\Big)\Bigg). \quad (42)$$

Under the conditions of Corollary 3.2

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{a};\mathbf{p}) \geq M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_ib_i)\right). \quad (43)$$

Remark 3.4. Under the settings of Example 2.3, if $f(x_1,x_2) = x_1 + x_2$ then (37) becomes

$$M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) =$$
$$= N^{-1}\Bigg(\frac{1}{k-1}\sum_{i=1}^{n}(c_i - p_i)N\Big(K(\mathbf{a};\frac{\mathbf{p}}{\mathbf{c}_i})+L(\mathbf{b};\frac{\mathbf{p}}{\mathbf{c}_i})\Big)\Bigg), \quad (44)$$

Under the conditions of Corollary 3.1

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{a};\mathbf{p})\geq M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i+b_i)\right). \quad (45)$$

Similarly if $f(x_1,x_2)=x_1x_2$ then from (39) we have

$$\begin{aligned} &M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ &= N^{-1}\left(\tfrac{1}{k-1}\sum_{i=1}^{n}(c_i-p_i)N\left(K(\mathbf{a};\tfrac{\mathbf{p}}{\mathbf{c}_i})L(\mathbf{b};\tfrac{\mathbf{p}}{\mathbf{c}_i})\right)\right), \end{aligned} \quad (46)$$

Under the conditions of Corollary 3.2

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{a};\mathbf{p})\geq M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_ib_i)\right). \quad (47)$$

Next, assume (A$_2$), (H$_1$) and suppose $|H_{I_l}(j_1,\ldots,j_{l-1})|=\beta_{l-1}$ for any $(j_1,\ldots,j_{l-1})\in I_{l-1}$ $(k\geq l\geq 2)$. For $m=2$, the reverse of (26) becomes

$$\begin{aligned} f\left(K_n(\mathbf{a};\mathbf{p}),L_n(\mathbf{b};\mathbf{p})\right) &\geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{k-1,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \\ &\geq \cdots \geq M^2_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ &= N^{-1}\left(\sum_{i=1}^{n} p_i N(f(a_i,b_i))\right), \end{aligned} \quad (48)$$

where

$$\begin{aligned} &M^2_{l,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ &= N^{-1}\left(\frac{n}{l\,|I_l|}\sum_{\mathbf{i}^l\in I_l}^{n}\left(\sum_{s=1}^{l}p_{i_s}\right)N\Big(f(K(\mathbf{a};\mathbf{p}_{I_l}),L(\mathbf{b};\mathbf{p}_{I_l}))\Big)\right) \end{aligned} \quad (49)$$

for $k\geq l\geq 1$.
Now using Theorem 2.4 (for $m=2$) and Corollaries 1.2, 1.3, we get generalizations of Beck's results in [1] (see also Mitrinović and Pečarić in [4] (see also [5], page 195).

Corollary 3.5. *Assume* (A_3) *with* $f(x,y)=x+y$ $((x,y)\in I_K\times I_L)$, *assume* (H_1), *and suppose* $|H_{I_l}(j_1,\ldots,j_{l-1})|=\beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1})\in I_{l-1}$ $(k\geq l\geq 2)$. *Assume further that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E:=\frac{K'}{K''}$, $F:=\frac{L'}{L''}$, $G:=\frac{N'}{N''}$, (48) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x)+F(y)\leq G(x+y), \quad (x,y)\in I^\circ_K\times I^\circ_L.$$

In this case for $k \geq l \geq 1$

$$M^2_{l,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ = N^{-1}\left(\frac{n}{l\,|I_l|}\sum_{\mathbf{i}^l\in I_l}^{n}\left(\sum_{s=1}^{l}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_l})+L(\mathbf{b};\mathbf{p}_{I_l})\right)\right), \quad (50)$$

where $\mathbf{i}^l := (i_1,\dots,i_l)$.

Corollary 3.6. *Assume* (A_3) *with* $f(x,y)=xy$ $((x,y)\in I_K\times I_L)$, *assume* (H_1), *and suppose* $|H_{I_I}(j_1,\dots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\dots,j_{l-1})\in I_{l-1}$ $(k \geq l \geq 2)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N° *respectively. Assume further that* K', L', M', A, B *and* C *are all positive. Then* (48) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x)+B(y)\leq C(xy), \quad (x,y)\in I_K^\circ\times I_L^\circ.$$

In this case for $k \geq l \geq 1$

$$M^2_{l,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ = N^{-1}\left(\frac{n}{l\,|I_l|}\sum_{\mathbf{i}^l\in I_l}^{n}\left(\sum_{s=1}^{l}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_l})+L(\mathbf{b};\mathbf{p}_{I_l})\right)\right), \quad (51)$$

where $\mathbf{i}^l := (i_1,\dots,i_l)$.

The special cases correspond to Examples 2.5, 2.7, 2.8 and 2.9 are as follows:

Remark 3.7. Under the settings of Example 2.5, for $n \geq k \geq 1$, (50) becomes

$$N^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ = N^{-1}\left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(\sum_{s=1}^{k}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k})+L(\mathbf{b};\mathbf{p}_{I_k})\right)\right). \quad (52)$$

Under the conditions of Corollary 3.5

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p}) \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{k-1,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \\ \geq \cdots \geq M^2_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ = N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right). \quad (53)$$

Similarly if $f(x_1,x_2)=x_1x_2$ then for $n\geq k\geq 1$, (51) cen be written as

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=$$
$$=N^{-1}\left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(\sum_{s=1}^{k}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right). \quad (54)$$

Under the conditions of Corollary 3.6

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p})\geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\geq M^2_{k-1,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\geq$$
$$\geq\cdots\geq M^2_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=$$
$$=N^{-1}\left(\sum_{i=1}^{n}p_iN(a_ib_i)\right). \quad (55)$$

Remark 3.8. Under the settings of Example 2.7, for $k\geq 1$, (50) becomes

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=$$
$$=N^{-1}\left(\frac{1}{\binom{n+k-1}{k}}\sum_{1\leq i_1\leq\cdots\leq i_k\leq n}\left(\sum_{s=1}^{k}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k})+L(\mathbf{b};\mathbf{p}_{I_k})\right)\right), \quad (56)$$

Under the conditions of Corollary 3.5

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p})\geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\geq M^2_{k-1,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\geq$$
$$\geq\cdots\geq M^2_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=$$
$$=N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right). \quad (57)$$

Similarly if $f(x_1,x_2)=x_1x_2$ then for $k\geq 1$, (51) can be written as

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=$$
$$=N^{-1}\left(\frac{1}{\binom{n+k-1}{k}}\sum_{1\leq i_1\leq\cdots\leq i_k\leq n}\left(\sum_{s=1}^{k}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right), \quad (58)$$

Under the conditions of Corollary 3.6

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p})\geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\geq$$
$$\geq\cdots\geq M^2_{k,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=N^{-1}\left(\sum_{i=1}^{n}p_iN(a_ib_i)\right). \quad (59)$$

Remark 3.9. Under the settings of Example 2.8, for $k \geq 1$, (50) becomes

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ = N^{-1}\left(\frac{1}{kn^{k-1}}\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k})+L(\mathbf{b};\mathbf{p}_{I_k})\right)\right). \quad (60)$$

Under the conditions of Corollary 3.5

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p}) \geq \cdots \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \\ \geq \cdots \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right). \quad (61)$$

Similarly if $f(x_1,x_2) = x_1x_2$ then for $k \geq 1$, (51) can be written as

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ = N^{-1}\left(\frac{1}{kn^{k-1}}\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right). \quad (62)$$

Under the conditions of Corollary 3.6 gives

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \geq \cdots \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \\ \geq \cdots \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n}p_iN(a_ib_i)\right). \quad (63)$$

Remark 3.10. Under the settings of Example 2.9, if $f(x_1,x_2) = x_1+x_2$ then for $1 \leq k \leq n$, (50) becomes

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ = N^{-1}\left(\frac{n}{kn(n-1)\ldots(n-k+1)}\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}p_{i_s}\right)\times \right. \\ \left. \times N\left(K(\mathbf{a};\mathbf{p}_{I_k})+L(\mathbf{b};\mathbf{p}_{I_k})\right)\right). \quad (64)$$

Under the conditions of Corollary 3.5

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p}) \geq M^2_{n,n}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \cdots \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \\ \geq \cdots \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right). \quad (65)$$

Similarly if $f(x_1,x_2)=x_1x_2$ then for $1\le k\le n$, (51) can be written as

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=$$
$$=N^{-1}\left(\frac{n}{kn(n-1)\dots(n-k+1)}\sum_{(i_1,\dots,i_k)\in I_k}\left(\sum_{s=1}^k p_{i_s}\right)\times\right.$$
$$\left.\times N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right).\quad(66)$$

Under the conditions of Corollary 3.6

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p})\ge M^2_{n,n}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\ge\dots\ge M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})\ge$$
$$\ge\dots\ge M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})=N^{-1}\left(\sum_{i=1}^n p_iN(a_ib_i)\right).\quad(67)$$

4. Generalization of Minkowski's Inequality

(A_4) Let I be an interval in $\mathbb{R}$, and let $M:I\to\mathbb{R}$ be a continuous and strictly monotone function. Let $\mathbf{x}_i\in I^m$ $(i=1,\dots,n)$, let $\mathbf{p}=(p_1,\dots,p_n)$ be a positive n-tuple such that $\sum\limits_{i=1}^n p_i=1$, and let $\mathbf{w}=(w_1,\dots,w_m)$ be a nonnegative m-tuple such that $\sum\limits_{i=1}^m w_i=1$.

We give a generalization of the Minkowski's inequality by using Theorem B.

Theorem 4.1. *Assume (A_4) and (H_1), and assume that the quasi-arithmetic mean function*

$$\mathbf{x}\to M_m(\mathbf{x};\mathbf{w}),\quad \mathbf{x}\in I^m$$

is convex. Then

$$M_m\left(\sum_{r=1}^n p_r\mathbf{x}_r;\mathbf{w}\right)\le A_{k,k}\le A_{k,k-1}\le\dots\le$$
$$\le A_{k,2}\le A_{k,1}=\sum_{r=1}^n p_rM_m(\mathbf{x}_r;\mathbf{w}),\quad(68)$$

where

$$A_{k,k}:=\sum_{(i_1,\dots,i_l)\in I_k}\left(\sum_{s=1}^k\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)M_m\left(\frac{\sum\limits_{s=1}^k\frac{p_{i_s}}{\alpha_{I_k,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^k\frac{p_{i_s}}{\alpha_{I_k,i_s}}};\mathbf{w}\right),\quad(69)$$

and

$$A_{k,l} := \frac{1}{(k-1)\dots l} \times$$

$$\times \sum_{(i_1,\dots,i_l)\in I_l} t_{I_k,l}(i_1,\dots,i_l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) M_m\left(\frac{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}};\mathbf{w}\right), \quad (70)$$

for $k-1 \geq l \geq 1$.

Proof. This is obtained by applying Theorem B to the function $M_m(\cdot;\mathbf{w})$ and to the vectors $\mathbf{x}_i$ $(i = 1,\dots,n)$. It is enough to show that $A_{k,l}$ in (15) has the form (69) and (70) depending on l, but this is easy to check. □

Similarly, by using Theorem C we get

Theorem 4.2. *Assume* (A_4), (H_1), *and suppose* $|H_{I_l}(j_1,\dots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\dots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$M_m\left(\sum_{r=1}^{n} p_r\mathbf{x}_r;\mathbf{w}\right) \leq A_{k,k} \leq$$

$$\leq A_{k-1,k-1} \leq \cdots \leq A_{2,2} \leq A_{1,1} = \sum_{r=1}^{n} p_r M_m(\mathbf{x}_r;\mathbf{w}), \quad (71)$$

where

$$A_{l,l} := \frac{n}{l\,|I_l|}\sum_{(i_1,\dots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) M_m\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l} p_{i_s}};\mathbf{w}\right), \quad k \geq l \geq 1. \quad (72)$$

The following special case a necessary and sufficient condition for the quasi-arithmetic mean function to be convex is given in ([5], p. 197):

Theorem D. *If* $M : [m_1, m_2] \to R$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then the quasi-arithmetic mean function* $M_m(\cdot; w)$ *is convex if and only if* M'/M'' *is a concave function.*

(A_5) Let $M :]0,\infty[\to]0,\infty[$ be a continuous and strictly monotone function such that $\lim\limits_{x\to 0} M(x) = \infty$ or $\lim\limits_{x\to\infty} M(x) = \infty$. Let $\mathbf{x} = (x_1,\dots,x_m)$ and $\mathbf{w} = (w_1,\dots,w_m)$ be positive m-tuples such that $w_i \geq 1$ $(i = 1,\dots,m)$. Let $\mathbf{p} = (p_1,\dots,p_n)$ be a positive n-tuple such that $\sum\limits_{i=1}^{n} p_i = 1$.

Then we define

$$\widetilde{M}_m(\mathbf{x};\mathbf{w}) = M^{-1}\left(\sum_{i=1}^{m} w_i M(x_i)\right). \quad (73)$$

The following result is also given in ([5], p. 197):

Theorem E. *If* $M :]0,\infty[\to]0,\infty[$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then* $\widetilde{M}_m(\cdot; w)$ *is a convex function if* M/M' *is a convex function.*

By using (73) we have

Theorem 4.3. *Assume* (A_5) *and* (H_1). *If the function*

$$\mathbf{x} \to \widetilde{M}_m(\mathbf{x}; \mathbf{w}), \quad \mathbf{x} \in]0,\infty[^m$$

is convex, then Theorem 4.1 *and Theorem* 4.2 (*in this case we suppose* $|H_{I_I}(j_1, \ldots, j_{l-1})| = \beta_{l-1}$ *for any* $(j_1, \ldots, j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$) *remain valid for* $\widetilde{M}_m(\mathbf{x}; \mathbf{w})$ *instead of* $M_m(\mathbf{x}; \mathbf{w})$.

Remark 4.4. All special cases (as given in Section 2) can be considered for Theorem 4.1, Theorem 4.2 and Theorem 4.3.

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