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THE PROBLEM OF FINDING EQUISTRONG HOLES IN AN ELASTIC SQUARE

L. GOGOLAURI

Abstract. An elastic square weakened by four unknown equal holes whose boundaries are free from external forces and the sides of the square are under the action of absolutely rigid punches of rectilinear base, is considered. Concentrated forces P are applied to the middle points of the punches.

Unknown boundaries of the holes are found under the condition that tangential normal stress takes on them one and the same constant value.

რეზიუმე. განხილულია ოთხი უცნობი ტოლი სიდიდის ხვრელით შესუსტებული დრეკადი კვადრატი, როდესაც ხვრელების საზღვარი თავისუფალია გარე დატვირთვისაგან, ხოლო კვადრატის გვერდებზე მოქმედებს სწორფუძიანი აბსოლუტურად ხისტი შტამპები. შტამპების შუა წერტილებზე მოდებულია P სიდიდის ტოლი შეყურსული ძალები.

მოძებნილია ხვრელების უცნობი საზღვარი იმ პირობით, რომ მასზე ტანგენციალური ნორმალური ძაბვა ღებულობდეს ერთი და იგივე მუდმივ მნიშვნელობას.

We consider the problem of elastic equilibrium of an elastic square weakened by four unknown holes which are intersected by the square diagonals and are symmetric both with respect to these diagonals and to the straight lines connecting middle points of the opposite square sides. The boundaries of the holes are assumed to be free from external loads, the square sides are under the action of absolutely rigid punches of rectilinear base, and concentrated forces P are applied to the middle points of the punches.

Since the problem is axially symmetric (the symmetry axes are the square diagonals and the straight lines connecting middle points of opposite square sides), we consider a curvilinear pentagon $A_1A_2A_3A_4A_5$ (Figure 1).

Introduce the notation: $A_kA_{k+1} = \Gamma_k$ $(k = 1, 2, 3)$, $\Gamma_4 = A_5A_1$, $\Gamma = \bigcup\limits_{k=1}^{4} \Gamma_k$, the arc A_4A_5 we denote by Γ_5 and the domain occupied by the curvilinear pentagon by S.

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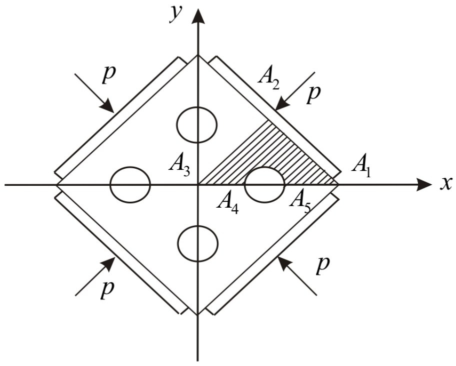

FIGURE 1

The surfaces of the bodies are assumed to be absolutely smooth, and hence the frictional force will be neglected.

The problem is formulated as follows: Find unknown holes and stressed state of the square under the condition that the tangential normal stress σ_s at the hole boundaries takes constant value, i.e., $\sigma_s = k = \text{const}$, on the square sides the tangential stress $\tau_{ns} = 0$, while on the unknown part of the square boundary $\sigma_n = \tau_{ns} = 0$.

Using the Kolosov-Muskhelishvili's [1] formulas, the boundary conditions of the problem can be written in the form

$$\operatorname{Re} e^{-i\alpha(t)}\big(\varkappa\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)}\big) = 2\mu v_n, \qquad t \in \Gamma, \tag{1}$$

$$\operatorname{Re} e^{-i\alpha(t)}\big(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\big) = b(t), \qquad t \in \Gamma, \tag{2}$$

$$\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} = c_0, \qquad t \in \Gamma_5, \tag{3}$$

$$\operatorname{Re}\varphi'(t) = \sigma_n + \sigma_s = \frac{k}{4}\,, \qquad t \in \Gamma_5, \tag{4}$$

where μ and $\varkappa$ are the elastic constants, $\alpha(t)$ is the size of the angle made by the normal and the OX-axis,

$$b(t) = \int\limits_{A_1}^{t} \sigma_n(t_0)\sin(\alpha(t) - \alpha(t_0))ds_0, \quad t \in \Gamma.$$

For the function $b(t)$ we obtain the following relations:

$$b(t) = \int\limits_{A_1}^{t} \sigma_n(t_0)\sin(\alpha(t) - \alpha(t_0))ds_0 = \int\limits_{A_1}^{t} \sigma_n(t_0)\sin 0\, ds_0 = 0, \ \ t \in A_1A_2,$$

$$b(t)=\int_{A_1}^{t}\sigma_n(t_0)\sin(\alpha(t)-\alpha(t_0))ds_0=\int_{A_1}^{A_2}\sigma_n(t_0)\sin\Big(\frac{3\pi}{4}-\frac{\pi}{4}\Big)ds_0+$$

$$+\int_{A_2}^{t}\sigma_n(t_0)\sin\Big(\frac{3\pi}{4}-\frac{3\pi}{4}\Big)ds_0=\int_{A_1}^{A_2}\sigma_n(t_0)\sin\frac{\pi}{2}=-\frac{p}{2}\,,\quad t\in A_2A_3,$$

$$b(t)=\int_{A_1}^{t}\sigma_n(t_0)\sin(\alpha(t)-\alpha(t_0))ds_0=\int_{A_1}^{A_2}\sigma_n(t_0)\sin\Big(\frac{3\pi}{2}-\frac{\pi}{4}\Big)ds_0+$$

$$+\int_{A_2}^{A_3}\sigma_n(t_0)\sin\Big(\frac{3\pi}{2}-\frac{3\pi}{4}\Big)ds_0+\int_{A_3}^{t}\sigma_n(t_0)\sin\Big(\frac{3\pi}{2}-\frac{3\pi}{2}\Big)ds_0=$$

$$=-\frac{p}{2}\Big(-\frac{\sqrt{2}}{2}\Big)-\frac{p\sqrt{2}}{4}=0,\quad t\in A_3A_4,$$

analogously,

$$b(t)=0,\quad \text{for}\quad t\in A_5A_1.$$

Thus we have

$$\begin{aligned}
&b(t)=0, && t\in\Gamma_1,\\
&b(t)=-\frac{p}{2}\,, && t\in\Gamma_2,\\
&b(t)=0, && t\in\Gamma_3,\\
&b(t)=0, && t\in\Gamma_4.
\end{aligned}\tag{5}$$

Obviously, the functions $\alpha(t)$ and $b(t)$ in the problem under consideration are piecewise continuous ones.

Summarizing the boundary conditions (1) and (2), differentiating with respect to the arc abscissa and taking into account that the functions $b(t)$ and $\alpha(t)$ are piecewise constant, we obtain

$$\operatorname{Im}\varphi'(t)=0,\quad t\in\Gamma.$$

The above condition and the condition (4) are the Keldysh-Sedov's problem [2] for the domain S:

$$\begin{cases}\operatorname{Re}\varphi'(t)=\dfrac{k}{4}\,, & t\in\Gamma_5,\\ \operatorname{Im}\varphi'(t)=0, & t\in\Gamma,\end{cases}\tag{6}$$

which in our case has a unique solution

$$\varphi(z)=\frac{k}{4}\,z+\ell,\tag{7}$$

where ℓ is an arbitrary constant.

If we take into account the relations (2), (5) and (7), then we will get

$$\operatorname{Re} e^{-i\alpha(t)}\Big(\frac{k}{2}\,t+\ell+\overline{\psi(t)}\Big)=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_3\cup\Gamma_4,\\ -\dfrac{p}{2}\,, & t\in\Gamma_2.\end{cases}\tag{8}$$

The condition (3) with regard for the formula (7) results in

$$\frac{k}{2}\,t+\ell+\overline{\psi(t)}=C_0,\quad t\in\Gamma_5.\tag{9}$$

For any point $t\in\Gamma$, we can write

$$\operatorname{Re}\big(e^{-i\alpha(t)}\cdot t\big)=\operatorname{Re}\big(e^{-i\alpha(t)}\rho\,e^{i\beta}\big)=\operatorname{Re}\big(\rho\,e^{-i(\alpha-\beta)}\big),\tag{10}$$

where ρ is radius, $\alpha=\beta-\frac{\pi}{2}$.

In view of the formula (10), we obtain the following equalities:

$$\begin{aligned}&\operatorname{Re}\big(e^{-\frac{\pi}{4}\,i}\cdot t\big)=\frac{d\sqrt{2}}{2}\,, \quad t\in\Gamma_1,\\ &\operatorname{Re}\big(e^{-\frac{3\pi i}{4}}\cdot t\big)=\frac{d\sqrt{2}}{2}\,, \quad t\in\Gamma_2,\\ &\operatorname{Re}\big(e^{-\frac{3\pi i}{2}}\cdot t\big)=0, \qquad t\in\Gamma_3\cup\Gamma_4,\end{aligned}\tag{11}$$

where d is the square diagonal.

We rewrite the equalities (11) as follows:

$$\operatorname{Re}\big(te^{-i\alpha(t)}\big)=\begin{cases}\dfrac{\sqrt{2}}{2}\,d, & t\in\Gamma_1,\\ \dfrac{\sqrt{2}}{2}\,d, & t\in\Gamma_2,\\ 0, & t\in\Gamma_3\cup\Gamma_4.\end{cases}\tag{12}$$

Writing the conditions for $\psi(t)$, from the formula (8) we have

$$\operatorname{Re} e^{-i\alpha(t)}\bigg(\frac{k}{2}\,t+\overline{\psi(t)}\bigg)=\begin{cases}-\operatorname{Re}\big(\ell\cdot e^{-i\alpha(t)}\big), & t\in\Gamma_1\cup\Gamma_3\cup\Gamma_4,\\ -\operatorname{Re}\big(\ell\cdot e^{-i\alpha(t)}\big)-\dfrac{p}{2}\,, & t\in\Gamma_2,\end{cases}\tag{13}$$

whence

$$\operatorname{Re}\big(e^{-i\alpha(t)}\cdot\overline{\psi(t)}\big)=-\frac{k}{2}\,\operatorname{Re}\big(e^{-i\alpha(t)}\cdot t\big)-\operatorname{Re}\big(\ell\cdot e^{-i\alpha(t)}\big).\tag{14}$$

Taking into account equalities (12) and (14), the formula (13) yields

$$\operatorname{Re}\big(e^{-i\alpha(t)}\cdot\overline{\psi(t)}\big)=\begin{cases}-\dfrac{kd\sqrt{2}}{4}-\operatorname{Re}\big(\ell\cdot e^{-i\alpha(t)}\big), & t\in\Gamma_1,\\ -\dfrac{kd\sqrt{2}}{4}-\operatorname{Re}\big(\ell\cdot e^{-i\alpha(t)}\big)-\dfrac{p}{2}\,, & t\in\Gamma_2,\\ -\operatorname{Re}\big(\ell\cdot e^{-i\alpha(t)}\big), & t\in\Gamma_3\cup\Gamma_4.\end{cases}\tag{15}$$

Using now the relations (9), (12) and (15), the boundary conditions for our problem can be written in the form

$$\frac{k}{2}\,t+\overline{\psi(t)}=c_0-\ell,\quad t\in\Gamma_5, \tag{16}$$

$$\operatorname{Re}\left(t\cdot e^{-i\alpha(t)}\right)=\begin{cases}\dfrac{d\sqrt{2}}{2}, & t\in\Gamma_1,\\ \dfrac{d\sqrt{2}}{2}, & t\in\Gamma_2,\\ 0, & t\in\Gamma_3\cup\Gamma_4,\end{cases} \tag{17}$$

$$\begin{cases}\operatorname{Re}\left(e^{-\frac{\pi i}{4}}\cdot\overline{\psi(t)}\right)=-\dfrac{kd\sqrt{2}}{4}-\ell\dfrac{\sqrt{2}}{2}, & t\in\Gamma_1,\\ \operatorname{Re}\left(e^{-\frac{3\pi i}{4}}\cdot\overline{\psi(t)}\right)=-\dfrac{kd\sqrt{2}}{4}-\ell\cdot\left(-\dfrac{\sqrt{2}}{2}\right)-\dfrac{p}{2}, & t\in\Gamma_2,\\ \operatorname{Re}\left(e^{-\frac{3\pi i}{2}}\cdot\overline{\psi(t)}\right)=0, & t\in\Gamma_3\cup\Gamma_4.\end{cases} \tag{18}$$

Thus we have reduced our problem to the boundary problem (16), (17), (18).

Let the function

$$z=\omega(\zeta) \tag{19}$$

map conformally the domain S onto the semi-circle $|\zeta|<1$, $\operatorname{Im}\zeta>0$, of unit radius. In addition, we may assume that the arc A_4A_5 is mapped onto the diameter $(-1,1)$; the mappings of the points A_k $(k=1,2,3,4,5)$ at the plane ζ we denote, respectively, by a_k $(k=1,2,3,4,5)$. The points $a_4=-1$, $a_5=1$, $a_2=i$ may be assumed to be fixed and the points a_1 and a_3 to be unknown ones (Figure 2).

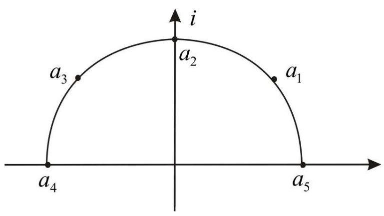

FIGURE 2

By means of the transformation (19) we can rewrite the relations (16), (17) and (18) as follows:

$$\frac{k}{2}\,\omega(\sigma)+\overline{\psi_0(\sigma)}=c_0-\ell,\qquad \sigma\in(-1,1), \tag{20}$$

$$\operatorname{Re}\left(\omega(\sigma)\,e^{-\frac{\pi i}{4}}\right)=\frac{d\sqrt{2}}{2},\qquad \sigma\in\gamma_1, \tag{21}$$

$$\operatorname{Re}\left(\omega(\sigma)\, e^{-\frac{3\pi i}{4}}\right) = \frac{d\sqrt{2}}{2}, \qquad \sigma \in \gamma_2, \tag{22}$$

$$\operatorname{Re}\left(\omega(\sigma)\, e^{-\frac{3\pi i}{2}}\right) = 0, \qquad \sigma \in \gamma_3 \cup \gamma_4, \tag{23}$$

$$\operatorname{Re}\left(e^{-\frac{\pi i}{4}} \cdot \overline{\psi_0(\sigma)}\right) = -\frac{kd\sqrt{2}}{4} - \ell\frac{\sqrt{2}}{2}, \qquad \sigma \in \gamma_1, \tag{24}$$

$$\operatorname{Re}\left(e^{-\frac{3\pi i}{4}} \cdot \overline{\psi_0(\sigma)}\right) = -\frac{kd\sqrt{2}}{4} + \ell\frac{\sqrt{2}}{2} - \frac{p}{2}, \qquad \sigma \in \gamma_2, \tag{25}$$

$$\operatorname{Re}\left(e^{-\frac{3\pi i}{2}} \cdot \overline{\psi_0(\sigma)}\right) = 0, \qquad \sigma \in \gamma_3 \cup \gamma_4, \tag{26}$$

where $\psi_0(\zeta) = \psi(\omega(\zeta))$, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are, respectively, the mappings of the segments $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ onto the semi-circle of unit radius. We denote $\gamma = \bigcup\limits_{k=1}^{4} \gamma_k$ and introduce the function

$$W(\zeta) = \begin{cases} \dfrac{k}{2}\,\omega(\zeta), & \operatorname{Im}\zeta > 0, \\ -\overline{\psi_0(\overline{\zeta})} + c_0 - \ell, & \operatorname{Im}\zeta < 0. \end{cases} \tag{27}$$

Taking into account the relation (20), for the function $W(\zeta)$ we get

$$W^+(\xi) - W^-(\xi) = \frac{k}{2}\,\omega(\xi) + \overline{\psi_0(\xi)} - c_0 + \ell = 0,$$

that is,

$$W^+(\xi) = W^-(\xi), \quad \xi \in (-1, 1).$$

Thus the function $W(\zeta)$ defind by the formula (27) is analytic in the circle $|\zeta| < 1$.

Using the formula (27), from the relations (21), (22), (23), (24), (25) and (26) we obtain

$$\operatorname{Re}\left(e^{-i\alpha(\sigma)}\, W(\sigma)\right) = f(\sigma), \quad \sigma \in \gamma, \tag{28}$$

$$\operatorname{Re}\left(e^{-i\alpha(\sigma)}\, W(\sigma)\right) = f^*(\sigma), \quad \sigma \in \gamma^*, \tag{29}$$

where the function $f(\sigma)$ is the right-hand side of the relations (21), (22) and (23), while the function $f^*(\sigma)$ is that of the relations (24), (25) and (26); γ^* is the mirror mapping of γ with respect to the OX-axis.

Using the formulas (28) and (29), we find the relation

$$W(\sigma) + e^{2i\alpha(\sigma)}\overline{W(\sigma)} = \begin{cases} e^{i\alpha(\sigma)} \cdot f(\sigma), & \sigma \in \gamma, \\ e^{i\alpha(\sigma)} \cdot f^*(\sigma), & \sigma \in \gamma^*, \end{cases} \tag{30}$$

The above relation is the Riemann-Hilbert problem for the circle of unit radius whose solution is represented in the form [3]

$$W(\zeta) = \frac{X(\zeta)}{4\pi i} \int\limits_{\gamma \cup \gamma^*} \frac{\zeta + \sigma}{\sigma - \zeta}\, \frac{f_1(\sigma)\, e^{i\alpha(\sigma)} d\sigma}{X(\sigma)\, \sigma}, \tag{31}$$

where

$$f_1(\sigma) = \begin{cases} f(\sigma), & \text{for} \quad \sigma \in \gamma, \\ f^*(\sigma), & \text{for} \quad \sigma \in \gamma^*, \end{cases} \tag{32}$$

$$X(\zeta) = \frac{1}{4\pi i} \int \frac{\zeta + \sigma}{\sigma - \zeta} \cdot \frac{2\alpha(\sigma)\, i}{\sigma}\, d\sigma, \quad |\zeta| < 1. \tag{33}$$

Taking into account the relations (21)–(26), after appropriate calculations, for the function $X(\zeta)$ from the formula (33) we obtain

$$X(\zeta) = \sqrt[4]{\frac{\zeta - a_2}{\zeta - a_1} \cdot \left(\frac{\zeta - a_3}{\zeta - a_2}\right)^3 \cdot \left(\frac{\zeta - \overline{a}_3}{\zeta - a_3}\right)^2 \left(\frac{\zeta - \overline{a}_2}{\zeta - \overline{a}_3}\right)^3 \frac{\zeta - \overline{a}_1}{\zeta - \overline{a}_2} \cdot \left(\frac{\zeta - a_1}{\zeta - \overline{a}_1}\right)^2} \times$$

$$\times\, e^{-\frac{1}{2\pi i} \int_{\gamma \cup \gamma^*} \frac{\alpha(\sigma) d\sigma}{\sigma}},$$

$$X(\sigma) = \big|X(\sigma)\big|\, e^{i\alpha(\sigma)}.$$

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Author's address:

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186
Georgia