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AN APPLICATION OF MEASURE OF NONCOMPACTNESS IN THE STUDY OF INTEGRODIFFERENTIAL EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper, we prove the existence of mild solutions for a class of integrodifferential evolution equations with nonlocal conditions. The technique relies on the techniques of measures of noncompactness and Mönch's type fixed point theorem.

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1. Introduction

It is well known that integrodifferential evolution equations form an important class of systems with distributed parameters to describe phenomena in real worlds. In this paper we discuss the following integrodifferential evolution equations with nonlocal initial conditions

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f\left(t, x(t)\right), \quad t \in J = [0, b], \\ x(0) = g(x) + x_0. \end{cases} \tag{1}$$

in a general Banach space $(X, \|\cdot\|)$, where $x_0 \in X$ and g: $C(J, X) \to X$ are given X-valued functions which constitutes a nonlocal Cauchy problem. It is well known that the study of nonlocal Cauchy problem arises to deal specially with some situations in physics. For the comments and motivations of nonlocal Cauchy problem via integrodifferential equations, we refer the reader to [1], [5], [6], [7], [8], [9], [10], [15], [16], [21], [25], [26], [30] and the references contained therein.

We make the following assumption.

[HA]: A is the infinitesimal generator of a strongly continuous semigroup $\{T\,(t)\,, t \geq 0\}$ on X with domain $D(A)$. Hence $D(A)$ endowed with the

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graph norm $|x| = \|x\| + \|Ax\|$ is a Banach space which will be denoted by $(Y, |\cdot|)$.

Then $\{B(t) \mid t \in J\}$ is a family of unbounded operators in X to be defined later. We now define the resolvent operator for

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x(t)), t \in J. \tag{2}$$

Definition 1. A family $\{R(t) \mid t \geq 0\}$ of continuous linear operators on X is called a resolvent operator for (2) if and only if

(R1) $R(0) = I$, where I is the identity operator on X,

(R2) the map $t \to R(t)x$ is continuous from J to X, for all $x \in X$,

(R3) $R(t)$ is a continuous linear operator on Y, for all $t \in J$, and the map $t \to R(t)y$ belongs to $C(J, X) \cap C^1(J, X)$ and satisfies

$$\frac{dR(t)y}{dt} = AR(t)y + \int_0^t B(t-s)R(s)yds = R(t)Ay + \int_0^t R(t-s)B(s)yds.$$

This definition follows [18], so we know from [18] that the existence of such a resolvent operator is well established. Accordingly, we can define a mild solution of nonlocal Cauchy problem (1).

Definition 2. A function $x \in C(J, X)$ is called a mild solution of the nonlocal Cauchy problem (1) if x satisfies

$$x(t) = R(t)[x_0 + g(x)] + \int_0^t R(t-s)f(s, x(s))ds, \quad t \in J.$$

In order to study the existence of mild solutions for nonlocal Cauchy problem (1), some authors assume that f (and/or g) satisfy Lipschitz conditions, so that the contraction mapping principle can be applied to derive fixed points of certain mappings so as to derive the existence of mild solutions [21], [25], [26], [27], [28]. On the other hand, by using the compactness of $T(\cdot)$ and Schauder's fixed point theorem, Liang et al prove the existence result for mild solutions of nonlocal Cauchy problem (1) without assuming Lipschitz conditions on nonlinear and nonlocal terms [29]. Further, we study a class of nonlinear integrodifferential impulsive (periodic) systems of mixed type and optimal control on Banach space [32], [33]. Existence of PC-mild solutions is proved and existence of optimal pairs of systems governed by nonlinear impulsive integrodifferential equations of mixed type is also presented.

As far as we know, there are very few papers related to the nonlocal Cauchy problem (1) by using the technique measure of noncompactness. The technique of measures of noncompactness which is often used in several branches of nonlinear analysis. Especially, that technique turns out to be very useful tool in the existence for several types of integral equations, details are found in [2], [3], [4], [11], [12], [13], [20], [22], [23], [31].

In this paper, applying the Hausdorff measure of noncompactness and Mönch's type fixed point theorem, we investigate the nonlocal Cauchy problem (1), we are able to prove the existence of solutions for the investigated nonlocal Cauchy problem (1).

2. Preliminaries

In this section, we introduce some basic definitions and lemmas which are used throughout this paper. Denote by $C(J, X)$ the Banach space of continuous functions $x : J \to X$, with the usual supremum norm

$$\|x\|_\infty = \sup_{t\in J} \|x(t)\|.$$

A measurable function $x\colon J \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. Let $L^1(J, X)$ denote the Banach space of functions $x\colon J \to X$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_J \|x(t)\|dt.$$

Let us recall the following definitions.

Definition 3. Let E^+ be the positive cone of an order Banach space $(E, \leq)$. A function Φ defined on the set of all bounded subsets of the Banach space X with values in E^+ is called a measure of noncompactness (MNC) on X if $\Phi(\overline{co}\Omega) = \Phi(\Omega)$ for all bounded subsets $\Omega \subset X$, where $\overline{co}\Omega$ stands for the closed convex hull of Ω.

The MNC Φ is said to be:

(1) *monotone* if for all bounded subsets Ω_1, Ω_2 of X, $\Omega_1 \subseteq \Omega_2$ implies

$$\Phi(\Omega_1) \leq \Phi(\Omega_2).$$

(2) *nonsingular* if $\Phi(a \cup \Omega) = \Phi(\Omega)$ for every $a \in X$ and every nonempty subset $\Omega \subseteq X$;

(3) *regular* if $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X.

One of the most important examples of MNC is the noncompactness measure of Hausdorff χ defined on each bounded subset Ω of X by

$$\chi(\Omega) = \inf\{\varepsilon > 0 \mid \Omega \text{ has a finite } \varepsilon\text{-net in } X\}.$$

It is well known that MNC χ enjoys the above properties (1), (2) and (3) and other properties (see [11] and [24]).

Definition 4. A countable set $\{f_n\}_{n=1}^{+\infty} \subset L^1(J, X)$ is said to be *semi-compact* if the sequence $\{f_n(t)\}_{n=1}^{+\infty}$ is compact in X for a.e. $t \in J$ and if there is a function $\mu \in L^1(J, R^+)$ satisfying

$$\sup_{n\geq 1} \|f_n(t)\| \leq \mu(t) \text{ for a.e. } t \in J.$$

[HB]: $\{B(t) \mid t \in J\}$ is a family of continuous linear operators from $(Y, |\cdot|)$ into $(X, \|\cdot\|)$. Moreover, there is an integrable function $c : J \to R^+$ such that for any $y \in Y$, the map $t \to B(t)y$ belongs to $W^{1,1}(J, X)$ and

$$\left\| \frac{dB(t)y}{dt} \right\| \leq c(t)|y|, \;\; y \in Y, \;\; t \in J.$$

In fact, this assumption is satisfied in the study of heat conduction in materials with memory [18] and viscoelasticity [19], where $B(t) = K(t)A$ for a family of continuous operators $\{K(t) \mid t \in J\}$ on X satisfying some additional conditions. It follows [14] that the corresponding resolvent operator $R(\cdot)$ of (2) exists under the assumptions [HA] and [HB].

Definition 5. We call the operator G: $L^1(J, X) \to C(J, X)$ defined by

$$Gf(t) = \int_0^t R(t-s)f(s)ds, \; t \in J, \tag{3}$$

as the Cauchy operator, where $R(\cdot)$ is the resolvent operator of (2).

Similar to the proof of Theorem 4.2.2 and Theorem 5.1.1 in [24], we also can give the following properties about Cauchy operator G.

Lemma 1. *Let G be the Cauchy operator defined by* (3)*.*

(1) *$\{f_n\}_{n=1}^{+\infty}$ is a sequence of functions in $L^1(J, X)$. Assume that there exist $\mu, \eta \in L^1(J, R^+)$ satisfying*

$$\sup_{n\geq 1} \|f_n(t)\| \leq \mu(t) \text{ and } \chi(\{f_n(t)\}_{n=1}^{+\infty}) \leq \eta(t) \text{ a.e. } t \in J.$$

Then for all $t \in J$, we have

$$\chi(\{Gf_n)(t)\}_{n=1}^{+\infty}) \leq 2M \int_0^t \eta(s)ds,$$

where $M = \sup_{t\in J}\{\|R(t)\|_{\pounds(X)}\}$, $\pounds(X)$ denotes all continuous linear operators on X.

(2) *For every semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^1(J, X)$ the set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J, X)$.*

The following fixed point theorem, a nonlinear alternative of Mönch's type, plays a key role in our existence of mild solutions for nonlocal Cauchy problem (1).

Theorem 1 (Theorem 2.2, [22]). *Let X be a Banach space, U is an open subset of X and $0 \in U$. Suppose that $F : \overline{U} \to X$ is a continuous map which satisfies Mönch's condition $\{$that is, if $D \subseteq \overline{U}$ is countable and $D \subseteq \overline{co}(\{0\} \cup F(D))$, then $\overline{D}$ is compact$\}$ and assume that*

$$x \neq \lambda F(x) \text{ for } x \in \partial U \text{ and } \lambda \in (0,1) \text{ holds.}$$

Then F has a fixed point in $\overline{U}$.

3. Existence Results

In general, we are used to study the mild solutions in stead of classical solutions for the nonlinear evolution equations since some rigorous conditions must be assumed. In this section, we give the existence of the mild solutions for nonlocal Cauchy problem (1).

We make the following assumptions.

[Hf]: (1) $f : J \times X \to X$ satisfies the Carathéodory condition, i.e., $f(\cdot, x) : J \to X$ is measurable for all $x \in X$ and $f(t, \cdot) : X \to X$ is continuous for a.e. $t \in J$.

(2) There exists a function $m \in L^1(J, R^+)$ and nondecreasing continuous function $\Omega : R^+ \to R^+$ such that

$$\|f(t,x)\| \leq m(t)\Omega(\|x\|), \text{ for all } x \in X \text{ and } t \in J.$$

(3) There exists a function $h \in L^1(J, R^+)$ such that for every bounded $D \subset X$,

$$\chi(f(t,D)) \leq h(t)\chi(D), \text{ for a.e. } t \in J,$$

where χ is the Hausdorff *MNC*.

[Hg1]: $g : C(J,X) \to X$ is a continuous and compact map such that

$$\|g(x)\| \leq c\|x\|_\infty + d$$

for arbitrary $x \in C(J,X)$, some $c, d > 0$.

[Hg2]: $g : C(J,X) \to X$ is Lipschitz continuous with constant k, that is, there exists a constant $k > 0$ such that

$$\|g(x) - g(y)\| \leq k\|x - y\|_\infty, \quad x, y \in C(J,X).$$

Theorem 2. *Assume that the conditions $[HA]$, $[HB]$, $[Hf]$, $[Hg1]$ are satisfied. If the resolvent operator $R(t)$ is operator norm continuous for $t > 0$, then the nonlocal Cauchy problem (1) has at least one mild solution on J provided that there exists a constant $N > 0$ with*

$$\frac{(1 - Mc)N}{M[\|x_0\| + d] + M\Omega(N)\|m\|_{L^1}} > 1, \quad Mc < 1 \tag{4}$$

and

$$2M\|h\|_{L^1} < 1. \tag{5}$$

Proof. We consider the operator $\Gamma : C(J,X) \to C(J,X)$ defined by

$$(\Gamma x)(t) = (\Gamma_1 x)(t) + (\Gamma_2 x)(t), \tag{6}$$

with

$$(\Gamma_1 x)(t) = R(t)[x_0 + g(x)], \tag{7}$$

and

$$(\Gamma_2 x)(t) = \int\limits_0^t R(t-s) f(s, x(s)) ds, \tag{8}$$

for all $t \in J$.

It is not difficult to see the fixed point of Γ is the mild solution of nonlocal Cauchy problem (1). Subsequently, we will prove that Γ has a fixed point by using Theorem 1. Then we proceed in three steps.

Step 1. The operator Γ is continuous on $C(J,X)$.

For this purpose, we assume that $x_n \to x$ in $C(J,X)$. Then by [Hf](1) we have that

$$f(s, x_n(s)) \to f(s, x(s)), \text{ as } n \to \infty, \ s \in J.$$

Since

$$\|f(s, x_n(s)) - f(s, x(s))\| \leq 2\Omega(N) m(s)$$

for some integer N, by [Hf](2), [Hg1] and the dominated convergence theorem we have

$$\|\Gamma x_n - \Gamma x\|_\infty \leq M \|g(x_n) - g(x)\| +$$
$$+ M \int\limits_0^t \|f(s, x_n(s)) - f(s, x(s))\| ds \to 0, \text{ as } n \to \infty,$$

that is, Γ is continuous.

Step 2. The Mönch's condition holds.

Suppose that $D \subseteq B_r$ is countable and $D \subseteq \overline{co}(\{0\} \cup \Gamma(D))$, we show that $\chi(D) = 0$, where B_r is the open ball of the radius r centered at the zero in $C(J,X)$ and χ is the Hausdorff *MNC*.

Without loss of generality, we may suppose that $D = \{x_n\}_{n=1}^{+\infty}$. We can easily verify that $\{\Gamma x_n\}_{n=1}^{+\infty}$ is equicontinuous. In fact, $\{\Gamma_1 x_n\}_{n=1}^{+\infty}$ can be shown to be equicontinuous by using the resolvent operator $R(t)$ is operator norm continuous for $t > 0$ and [Hg1]. Next, we prove that $\{\Gamma_2 x_n\}_{n=1}^{+\infty}$ is also equicontinuous. Let $0 \leq t_1 < t_2 \leq b$, and obtain

$$\begin{aligned}
&\|(\Gamma_2 x_n)(t_1) - (\Gamma_2 x_n)(t_2)\| \leq \\
\leq\ & \left\| \int_0^{t_2} R(t_2 - s) f(s, x_n(s)) ds - \int_0^{t_1} R(t_1 - s) f(s, x_n(s)) ds \right\| \leq \\
\leq\ & \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|_{L(X)} \|f(s, x_n(s))\| ds + \\
& + M \int_{t_1}^{t_2} \|f(s, x_n(s)) ds\| .
\end{aligned}$$

If $t_1 = 0$, then the right-hand side can be made small when t_2 is small independently of $x_n \in D$. If $t_1 > 0$, then the right-hand side can be estimated

$$\begin{aligned}
& \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|_{\pounds(X)} \|f(s, x_n(s))\| ds + \\
& + M \int_{t_1}^{t_2} \|f(s, x_n(s)) ds\| \leq \\
\leq\ & \int_0^{t_1-\gamma} \|R(t_2 - s) - R(t_1 - s)\|_{\pounds(X)} \|f(s, x_n(s))\| ds + \\
& + \int_{t_1-\gamma}^{t_1} \|R(t_2 - s) - R(t_1 - s)\|_{\pounds(X)} \|f(s, x_n(s))\| ds + \\
& + M \int_{t_1}^{t_2} \|f(s, x_n(s)) ds\| \leq \\
\leq\ & \int_0^{t_1-\gamma} \|R(t_2 - s) - R(t_1 - s)\|_{\pounds(X)} \|f(s, x_n(s))\| ds + \\
& + 2M \max_{s \in J, x_n \in D} \{\|f(s, x_n(s))\|\} + \\
& + M \int_{t_1}^{t_2} \|f(s, x_n(s)) ds\|
\end{aligned}$$

when $0 < \gamma < t_1$ is a small number.

It comes from $R(t)$ is operator norm continuous uniformly for $t > 0$ that $R(t)$ is operator norm continuous uniformly for $t \in [\gamma, b]$. Therefore,

$$\|R(t_2 - s) - R(t_1 - s)\|_{\pounds(X)}$$

and

$$\int_0^{t_1-\gamma} \|R(t_2 - s) - R(t_1 - s)\|_{\pounds(X)} \|f(s, x_n(s))\| ds$$

can be made small when $t_2 - t_1$ is small independently of $x_n \in D$. Accordingly, we see that the $\{\Gamma_2 x_n\}_{n=1}^{+\infty}$ is also equicontinuous.

As a result, $\{\Gamma x_n\}_{n=1}^{+\infty}$ is equicontinuous. So, $D \subseteq \overline{co}(\{0\} \cup \Gamma(D))$ is also equicontinuous.

Now, from [Hg1], [Hf](3), (1) of Lemma 1 and properties of *MNC* χ, it follows that

$$\begin{aligned}
\chi(\{\Gamma x_n\}_{n=1}^{+\infty}) \leq & \sup_{t\in J}(\chi(\{R(t)[x_0 + g(x_n)]\}_{n=1}^{+\infty}))+ \\
& +\chi\left(\left\{\int_0^t R(t-s) f(s, x_n(s)) ds\right\}_{n=1}^{+\infty}\right) \leq \\
& \leq \sup_{t\in J}(\chi(\{R(t) g(x_n)\}_{n=1}^{+\infty}))+ \\
& +\chi\left(\left\{\int_0^t R(t-s) f(s, x_n(s)) ds\right\}_{n=1}^{+\infty}\right) \leq \\
& \leq 2M \int_0^b h(s) \sup_{t\in J} \chi(\{x_n(s)\}_{n=1}^{+\infty}) ds = \\
& = 2M \|h\|_{L^1} \chi(\{x_n\}_{n=1}^{+\infty}).
\end{aligned}$$

Thus, we get that

$$\chi(D) \leq \chi(\overline{co}(\{0\} \cup \Gamma(D)) = \chi(\Gamma(D)) \leq 2M\|h\|_{L^1}\chi(D),$$

which implies that $\chi(D) = 0$, since the conditions (5) holds.

Step 3. Now let $\lambda \in (0, 1)$ and $x = \lambda \Gamma(x)$. Then

$$x(t) = \lambda R(t)[x_0 + g(x)] + \lambda \int_0^t R(t-s) f(s, x(s)) ds, \text{ for } t \in J.$$

and one has

$$\|x(t)\| \leq Mc\|x\|_\infty + M[\|x_0\| + d] + M\Omega(\|x(s)\|) \int_0^t m(s) ds.$$

Consequently,

$$\|x\|_\infty \leq Mc\|x\|_\infty + M[\|x_0\| + d] + M\Omega(\|x\|_\infty)\|m\|_{L^1}.$$

It comes from $Mc < 1$ that

$$\frac{(1 - Mc)\|x\|_\infty}{M[\|x_0\| + d] + M\Omega(\|x\|_\infty)\|m\|_{L^1}} \leq 1.$$

Then by (4) there exists a $N > 0$ such that $\|x\|_\infty \neq N$. Set $U = \{x \in C(J, X) \mid \|x\|_\infty < N\}$. From the choice of U there is no $x \in \partial U$ such that $x = \Gamma(x)$ for some $\lambda \in (0, 1)$. Thus we get a fixed point of Γ in $\overline{U}$ due to the Theorem 1, which is a mild solution to nonlocal Cauchy problem (1). □

To prove the next result, we need the following simple fact about Hausdorff *MNC* χ.

Lemma 2 (Lemma 3.1, [17])**.** *If $D \subseteq C(J, X)$ be bounded, then we have*

$$\sup_{t\in J} \chi(D(t)) \leq \chi(D).$$

Theorem 3. *Assume that the conditions $[HA]$, $[HB]$, $[Hf]$, $[Hg2]$ are satisfied. If the resolvent operator $R(t)$ is operator norm continuous for $t > 0$, then the nonlocal Cauchy problem* (1) *has at least one mild solution on J provided that there exists a constant $N > 0$ with*

$$\frac{(1 - Mk)N}{M[\|x_0\| + \|g(0)\|] + M\Omega(N)\|m\|_{L^1}} > 1, \tag{9}$$

and

$$Mk + 2M\|h\|_{L^1} < 1. \tag{10}$$

Proof. Taking into account of Step 1 of Theorem 2, we can know that operator Γ defined by (6) is continuous on $C(J, X)$. Here, we need check that Γ satisfies the Mönch's condition. For this purpose, let $D \subseteq B_r$ be countable and $D \subseteq \overline{co}(\{0\} \cup \Gamma(D))$, we need show that $\chi(D) = 0$.

Without loss of generality, we may suppose that $D = \{x_n\}_{n=1}^{+\infty}$. By Theorem 2, we know that $\{\Gamma_2 x_n\}_{n=1}^{+\infty}$ is equicontinuous. Moreover, $\Gamma_1 : D \to C(J, X)$ is Lispschitz continuous with constant Mk due to the condition [Hg2]. In fact, for $x, y \in D$, we have

$$\|\Gamma_1 x - \Gamma_2 y\|_\infty = \sup_{t\in J} \|R(t)[x_0 + g(x)] - R(t)[x_0 + g(y)]\| \leq$$
$$\leq M\|g(x) - g(y)\| \leq Mk\|x - y\|_\infty.$$

So, from (1) of Lemma 1 and Lemma 2, it follows that

$$\chi(\{\Gamma x_n\}_{n=1}^{+\infty}) \leq \sup_{t\in J}(\chi(\{R(t)g(x_n)\}_{n=1}^{+\infty}))+$$

$$+\chi\left(\left\{\int_0^t R(t-s)f(s,x_n(s))ds\right\}_{n=1}^{+\infty}\right) \leq$$

$$\leq Mk\chi(\{x_n\}_{n=1}^{+\infty}) + 2M\int_0^b h(s)\sup_{t\in J}\chi(\{x_n(s)\}_{n=1}^{+\infty})ds =$$

$$= M(k+2\|h\|_{L^1})\chi(\{x_n\}_{n=1}^{+\infty}).$$

Thus, we obtain

$$\chi(D) \leq \chi(\overline{co}(\{0\}\cup\Gamma(D)) = \chi(\Gamma(D)) \leq M(k+2\|h\|_{L^1})\chi(D),$$

which implies that $\chi(D)=0$, since the condition (10) holds.

Now, let $\lambda \in (0,1)$ and $x = \lambda\Gamma(x)$. Then

$$x(t) = \lambda R(t)[x_0+g(x)] + \lambda\int_0^t R(t-s)f(s,x(s))ds, \text{ for } t\in J.$$

and one has

$$\|x\|_\infty \leq Mk\|x\|_\infty + M[\|x_0\| + \|g(0)\|] + M\Omega(\|x\|_\infty)\int_0^t m(s)ds.$$

Consequently, it comes from condition (10) that $Mk<1$, hence,

$$\frac{(1-Mk)\|x\|_\infty}{M[\|x_0\|+\|g(0)\|] + M\Omega(\|x\|_\infty)\|m\|_{L^1}} \leq 1.$$

With analogous arguments as in the proof of Theorem 2, we can get an open ball U by the condition (9), and there is no $x\in\partial U$ such that $x=\lambda\Gamma(x)$ for some $\lambda\in(0,1)$. Thus, we get a fixed point of Γ in $\overline{U}$ due to Theorem 1, which is a mild solution to nonlocal Cauchy problem (1). □

At last, we prove the result of Theorem 2 without assuming the operator norm continuity of $R(t)$ for $t>0$ and condition (5). We need introduce Φ as the following measure of noncompactness in $C(J,X)$ defined by

$$\Phi(\Omega) = \max_{E\in\Delta(\Omega)}\{\alpha(E), mod_C(E)\}$$

for all bounded subsets Ω of $C(J,X)$, where $\Delta(\Omega)$ stands for the set of countable subsets of $\Omega\subset C(J,X)$, α is the real *MNC* defined as

$$\alpha(E) = \sup_{t\in J} e^{-Lt}\chi(E(t)) \text{ with } E(t) = \{x(t)\mid x\in E\}, t\in J,$$

and $mod_C(E)$ is the modulus of equicontinuity of the set of functions E given by the formula

$$mod_C(E) = \lim_{\delta\to 0} \sup_{x\in E} \max_{|t_2-t_1|\le\delta} \|x(t_2) - x(t_1)\|,$$

$L > 0$ is a constant that we need appropriately choose.

It was proved in [24] that Φ is well defined and is a monotone, nonsingular, regular *MNC*.

Now, we are ready to prove the result of Theorem 2 without assuming the operator norm continuity of $R(t)$ for $t > 0$ and condition (5).

Theorem 4. *Assume that the conditions* $[HA]$, $[HB]$, $[Hf]$, $[Hg1]$ *are satisfied. Then the nonlocal Cauchy problem* (1) *has at least one mild solution on* J *provided the condition* (4) *holds.*

Proof. On account of Theorem 2, we only prove that the function Γ given by (6) satisfies the Mönch's condition. For this purpose, let $D \subseteq B_r$ be countable and $D \subseteq \overline{co}(\{0\} \cup \Gamma(D))$, we will show that D is relatively compact.

From the regularity of Φ, it is enough to prove that $\Phi(D) = (0, 0)$. Since $\Phi(\Gamma(D))$ is a maximum, let $\{y_n\}_{n=1}^{+\infty} \subseteq \Gamma(D)$ be the denumerable set which achieves its maximum.

Of course, there exists a set $\{x_n\}_{n=1}^{+\infty} \subseteq D$ such that

$$\begin{aligned} y_n(t) &= (\Gamma x_n)(t) = (\Gamma_1 x_n)(t) + (\Gamma_2 x_n)(t) = \\ &= R(t)[x_0 + g(x_n)] + \int_0^t R(t-s) f(s, x_n(s))ds, \text{ for } t \in J, n \ge 1. \end{aligned}$$

Now, we need to estimate $\alpha(\{y_n\}_{n=1}^{+\infty})$. By virtue of [Hf](3), we have

$$\begin{aligned} &\chi(\{f(s, x_n(s))\}_{n=1}^{+\infty}) \le \\ &\le e^{Ls} h(s) \sup_{\theta\in J} e^{-L\theta} \chi(\{x_n(\theta)\}_{n=1}^{+\infty}) = e^{Ls} h(s) \alpha(\{x_n\}_{n=1}^{+\infty}). \end{aligned}$$

Further, we obtain

$$\chi(\{(\Gamma_2 x_n)(t))\}_{n=1}^{+\infty}) \le 2M\alpha(\{x_n\}_{n=1}^{+\infty}) \int_0^t e^{Ls} h(s)ds.$$

By elementary computation, we have

$$\begin{aligned} \alpha(\{y_n\}_{n=1}^{+\infty}) &\le \sup_{t\in J} e^{-Lt} 2M\alpha(\{x_n\}_{n=1}^{+\infty}) \int_0^t e^{Ls} h(s)ds = \\ &= \alpha(\{x_n\}_{n=1}^{+\infty}) \left[2M \sup_{t\in J} \int_0^t e^{-L(t-s)} h(s)ds \right]. \end{aligned}$$

It is not difficult to choose a suitable constant $L > 0$ such that

$$q = 2M \sup_{t \in J} \int_0^t e^{-L(t-s)} h(s) ds < 1.$$

Therefore,

$$\alpha(\{x_n\}_{n=1}^{+\infty}) \leq \alpha(D) \leq \alpha(\overline{co}(\{0\} \cup \Gamma(D)) \leq \alpha(\{y_n\}_{n=1}^{+\infty}) \leq q\alpha(\{x_n\}_{n=1}^{+\infty}),$$

which implies that

$$\alpha(\{x_n\}_{n=1}^{+\infty}) = \alpha(D) = \alpha(\{y_n\}_{n=1}^{+\infty}) = 0.$$

By the definition of α, we can obtain

$$\chi(\{x_n(t)\}_{n=1}^{+\infty}) = \chi(\{y_n(t)\}_{n=1}^{+\infty}) = 0, \text{ for every } t \in J.$$

Thus, by [Hf](3) again,

$$\chi(\{f(t, x_n(t))\}_{n=1}^{+\infty}) \leq h(t)\chi(\{x_n(t)\}_{n=1}^{+\infty}) = 0,$$

which implies that $\{f(t, x_n(t))\}_{n=1}^{+\infty}$ is relatively compact for a.e. $t \in J$. So we obtain that $\{f(\cdot, x_n(\cdot))\}_{n=1}^{+\infty} \subset L^1(J, X)$ is semicompact. Moreover, it comes from the fact $\{x_n\}_{n=1}^{+\infty} \subseteq D \subseteq B_r$ and [Hf](2) that

$$\|f(t, x_n(t))\| \leq m(t)\Omega(\|x_n(t)\|) \leq m(t)\Omega(r) \text{ for a.e. } t \in J \text{ and very } n \geq 1.$$

Further, using (2) of Lemma 1, $\Gamma_2(\{x_n\}_{n=1}^{+\infty}) = G(\{f(\cdot, x_n(\cdot))\}_{n=1}^{+\infty})$ is relatively compact set. On the other hand, the set $\Gamma_1(\{x_n\}_{n=1}^{+\infty})$ is also relatively compact due to the strong continuity of $R(\cdot)$ and the compactness of g.

As a result, $\{y_n\}_{n=1}^{+\infty} \subset C(J, X)$ is relatively compact. Since Φ is a monotone, nonsingular, regular *MNC*, we obtain

$$\Phi(D) \leq \Phi(\overline{co}(\{0\} \cup \Gamma(D)) \leq \Phi(\Gamma(D)) = \Phi(\{y_n\}_{n=1}^{+\infty}) = (0, 0).$$

Therefore, D is relatively compact. □

Remark 1. In [29], the authors discuss (1) when $\{T(t), t \geq 0\}$ is a compact semigroup. From Theorem 4, we can see that the compactness of semigroup $\{T(t), t \geq 0\}$ can be replaced by the condition [Hf](3). In fact, if f is Lipschitz continuous or compact, then the condition [Hf](3) is satisfied. Thus, our results can be considered as a contribution to this emerging field.

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