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EXTREMAL SOLUTIONS FOR NONLOCAL FRACTIONAL DIFFERENTIAL EQUATIONS

J. WANG, X. DONG AND W. WEI

Abstract. In this paper, we study extremal solutions for nonlocal fractional differential equations. Some definitions such as $L^{\frac{1}{\delta}}$-Lipschitz, $L^{\frac{1}{\beta}}$-Carathéodory and $L^{\frac{1}{\gamma}}$-Chandrabhan, absolutely continuous solution, lower solution and supper solution, maximal solution and minimal solution are introduced. Existence results for extremal solutions are obtained by applying the Dhage hybrid fixed point theorem. At last, an example on biomedical sciences is given to illustrate the usefulness of our main results.

რეზიუმე. ნაშრომში გამოკვლეულია არალოკალური წილადური რიგის დიფერენციალური განტოლებების ექსტრემალური ამონახსნები. შემოღებულია $L^{\frac{1}{\delta}}$-ლიფშიცის, $L^{\frac{1}{\beta}}$-კარათეოდორისა, $L^{\frac{1}{\gamma}}$-ჩანდრაბანის, აბსოლუტურად უწყვეტი, ქვედა და ზედა ამონახსნების ცნებები. ექსტრემალური ამონახსნების არსებობის პრობლემა გადაწყვეტილია დეიჯის ჰიბრიდული უძრავი წერტილის პრინციპის გამოყენებით. და ბოლოს, მიღებული შედეგების გამოყენების თვალსაზრისით, მოყვანილია ერთი მაგალითი ბიოსამედიცინო მეცნიერებებიდან.

1. Introduction

During the past two decades, fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of biomedical sciences, engineering, physics and economics. For more details, one can see the monographs of Diethelm [8], Kilbas et al. [12], Lakshmikantham et al. [13], Miller and Ross [14], Podlubny [18], Tarasov [19]. Very recently, fractional differential equations (inclusions) and optimal controls in Banach spaces are studied by Balachandran et al. [3, 4], Benchohra et al. [5, 6], N'Guérékata [15, 16], Mophou and N'Guérékata [17], Wang et al. [20, 21, 22, 23, 24, 25, 26, 27, 28], Zhou et al. [29, 30, 31, 32] etc.

Throughout this paper, $(X, \|\cdot\|)$ will be a Banach space, and $J = [0, T]$, $T > 0$. Let $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm $\|u\|_C := \sup\{\|u(t)\| : t \in J\}$ for $u \in C(J, X)$.

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We consider the following fractional differential equation with nonlocal conditions

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t)) + g(t, u(t)) + h(t, u(t)), \text{ a.e. } t \in J, \\ u(0) = u_0 + G(u), \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, $f : J \times X \to X$, $g : J\times X \to X$, $h : J\times X \to X$, the nonlocal term $G : C(J, X) \to X$ are given functions satisfying some assumptions that will be specified latter.

Nonlocal conditions were initiated by Byszewski [1]. As remarked by Byszewski and Lakshmikantham [2], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. Although, a pioneering work on the existence results of solutions for nonlocal problems for fractional differential equations has been reported by N'Guérékata [15], N'Guérékata [16] reported that the results in [15] hold only in finite dimensional spaces. Very recently, Dong et al. [9] revisit this interesting problem and establish some new existence principles of solutions by virtue of fractional calculus and fixed point theorems under some suitable conditions, which extend the results in [15] to abstract Banach spaces.

On the existence results of extremal solutions for fractional differential equations involving Riemann-Liouville derivative and Caputo derivative have been reported in [11] and [30]. However, the results obtained in [11] and [30] hold only in finite dimensional spaces. There are few papers deal with the extremal solutions for fractional differential equations in abstract Banach spaces.

In the present paper, we study the existence of extremal solutions to the semilinear fractional differential equation with nonlocal conditions in X. Many definitions such as $L^{\frac{1}{\delta}}$-Lipschitz, $L^{\frac{1}{\beta}}$-Carathéodory and $L^{\frac{1}{\gamma}}$-Chandrabhan, absolutely continuous solution, lower solution and supper solution, maximal solution and minimal solution are introduced, where δ, β, γ are associated with the fractional derivative of order $\alpha \in (0, 1)$. Subsequently, the existence results for extremal solutions are proved by applying the Dhage hybrid fixed point theorem.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, some definitions of solutions such as lower solution, supper solution, maximal solution, minimal solution and a important lemma are given. In Section 4, the existence results for extremal solutions are proved. Finally, we give an example to illustrate the usefulness of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us recall the following known definitions. For more details see [12].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as

$$^{L}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3. The Caputo derivative of order γ for a function $f : [0, \infty) \to R$ can be written as

$$^{c}D^{\gamma} f(t) = \ ^{L}D^{\gamma} \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \ t > 0, \ n-1 < \gamma < n.$$

Remark 2.4. (i) If $f(t) \in C^n[0, \infty)$, then

$$^{c}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t), \ t > 0, \ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Assume that $1 \leq p \leq \infty$. For measurable functions $m : J \to R$, define the norm

$$\|m\|_{L^p J} = \begin{cases} \left(\int_J |m(t)|^p dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf_{\mu(\bar{J})=0} \left\{ \sup_{t \in J - \bar{J}} |m(t)| \right\}, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J, R)$ be the Banach space of all Lebesgue measurable functions $m : J \to R$ with $\|m\|_{L^p J} < \infty$.

Lemma 2.5 (Hölder inequality)**.** *Assume that* $p, q \geq 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $\phi \in L^p(J, R), \varphi \in L^q(J, R)$, *then for* $1 \leq q \leq \infty$, $\phi\varphi \in L^1(J, R)$ *and*

$$\|\phi\varphi\|_{L^1 J} \leq \|\phi\|_{L^p J} \|\varphi\|_{L^q J}.$$

Lemma 2.6 (Bochner theorem). *A measurable function $f\colon J \to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Lemma 2.7 (Mazur lemma). *If $\mathcal{K}$ is a compact subset of X, then its convex closure $\overline{conv}\mathcal{K}$ is compact.*

Lemma 2.8 (Ascoli-Arzela theorem). *Let $\mathcal{W} = \{s(t)\}$ is a function family of continuous mappings $s : J \to X$. If $\mathcal{W}$ is uniformly bounded and equicontinuous, and for any $t^* \in J$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}$ $(n = 1, 2, \ldots, t \in J)$ in $\mathcal{W}$.*

Definition 2.9. An operator $S : X \to X$ is called compact if $\overline{S(X)}$ is a compact subset of X. $S : X \to X$ is called totally bounded if S maps the bounded subsets of X into the relatively compact subsets of X. Finally, $S : X \to X$ is called a completely continuous operator, if it is a continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on the bounded subsets of X.

Definition 2.10. A non-empty closed set K in a Banach space X is called a cone if
(i) $K + K \subseteq K$,
(ii) $\lambda K \subseteq K$ for $\lambda \in R, \lambda \geq 0$ and
(iii) $\{-K\} \cap K = \{0\}$, where 0 is the zero element of X.

We introduce an order relation " $\leq$ " in X as follows. Let $z, y \in X$. Then $z \leq y$ if and only if $y - z \in K$.

Definition 2.11. A cone K is called *normal* if the norm $\|\cdot\|_X$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|z\|_X \leq N\|y\|_X$ for all $z, y \in K$ with $z \leq y$.

It is known that if the cone K is normal in X, then every order–bounded set in X is norm–bounded. Similarly, the cone K in X is called regular if every monotone increasing (resp. decreasing) order bounded sequence in X converges in norm. The details of cones and their properties appear in Heikkilä and Lakshmikantham [10].

For any $a, b \in X, a \leq b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{z \in X : a \leq z \leq b\}.$$

Definition 2.12. Let X and Y be two ordered Banach spaces. A mapping $S : X \to Y$ is said to be nondecreasing or monotone increasing if $z \leq y$ implies $Sz \leq Sy$ for all $z, y \in [a, b]$.

We use the following hybrid fixed point theorem of Dhage [7].

Lemma 2.13 (Hybrid fixed point theorem). *Let X be a Banach space and let $A, B, C : X \to X$ be three monotone increasing operators such that*

(i) *A is a contraction with contraction constant $\ell < 1$,*

(ii) *B is completely continuous,*

(iii) *C is totally bounded, and*

(iv) *there exist elements a and b in X such that*

$$a \leq Aa + Ba + Ca \text{ and } b \geq Ab + Bb + Cb \text{ with } a \leq b.$$

Further if the cone K in X is normal, then the operator equation

$$Az + Bz + Cz = z$$

has a least and a greatest solution in the order interval $[a, b]$.

3. Some Definitions and an Important Lemma

Let R_+ be the set of nonnegative numbers. We give the following definitions in the sequel.

Definition 3.1 ($L^{\frac{1}{\delta}}$-Lipschitz). A mapping $f : J \times X \to X$ is called $L^{\frac{1}{\delta}}$-Lipschitz if

(i) $f(t, u)$ is Lebesgue measurable with respect to t for any $(t, u) \in J \times X$,

(ii) there exist a constant $\delta \in [0, \alpha)$ and a function $l \in L^{\frac{1}{\delta}}(J, R_+)$ such that

$$\|f(t, u) - f(t, v)\| \leq l(t)\|u - v\|, \quad \text{a.e. } \ t \in J$$

for all $u, v \in X$.

Definition 3.2 ($L^{\frac{1}{\beta}}$-Carathéodory). A mapping $g : J \times X \to X$ is said to be Carathéodory if

(i) $g(t, u)$ is Lebesgue measurable with respect to t for any $(t, u) \in J \times X$,

(ii) $g(t, u)$ is continuous with respect to u for any $u \in X$ and almost all $t \in J$.

Furthermore, a Carathéodory function $g(t, u)$ is called $L^{\frac{1}{\beta}}$-Carathéodory if

(iii) there exist a constant $\beta \in [0, \alpha)$ and a function $m \in L^{\frac{1}{\beta}}(J, R_+)$ such that

$$\|g(t, u)\| \leq m(t), \quad \text{a.e. } \ t \in J$$

for all $u \in X$.

Definition 3.3 ($L^{\frac{1}{\gamma}}$-Chandrabhan). A mapping $h : J \times X \to X$ is said to be Chandrabhan if

(i) $h(t, u)$ is Lebesgue measurable with respect to t for any $(t, u) \in J \times X$,

(ii) $h(t, u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$.

Furthermore, a Chandrabhan function $h(t, u)$ is called $L^{\frac{1}{\gamma}}$-Chandrabhan if

(iii) there exist a constant $\gamma \in [0, \alpha)$ and a function $w \in L^{\frac{1}{\gamma}}(J, R_+)$ such that

$$\|h(t, u)\| \le w(t), \quad \text{a.e.} \ \ t \in J$$

for all $u \in X$.

Definition 3.4. A function $u \in C(J, X)$ is called a solution of system (1) on J if

(i) the function $u(t)$ is absolutely continuous on J,

(ii) $u(0) = u_0 + G(u)$, and

(iii) u satisfies the equation in (1).

We need the following hypotheses in the sequel.

(H_1) $f, g, h : J \times X \to X$, $G : C(J, X) \to X$,

(f_1) f is $L^{\frac{1}{\delta}}$-Lipschitz, and there exists $\eta \in [0, \alpha)$ such that $\|f(t, 0)\| \in L^{\frac{1}{\eta}}(J, R_+)$,

(g_1) g is $L^{\frac{1}{\beta}}$-Carathéodory,

(h_1) h is $L^{\frac{1}{\gamma}}$-Chandrabhan.

For any positive constant r, let $B_r = \{u \in C(J, X) : \|u\|_C \le r\}$. Set

$$\begin{aligned}
q_0 &= \frac{\alpha - 1}{1 - \delta} \in (-1, 0), \quad L = \|l\|_{L^{\frac{1}{\delta}} J},\\
q_1 &= \frac{\alpha - 1}{1 - \eta} \in (-1, 0), \quad F = \|f(t, 0)\|_{L^{\frac{1}{\eta}} J},\\
q_2 &= \frac{\alpha - 1}{1 - \beta} \in (-1, 0), \quad M = \|m\|_{L^{\frac{1}{\beta}} J},\\
q_3 &= \frac{\alpha - 1}{1 - \gamma} \in (-1, 0), \quad W = \|w\|_{L^{\frac{1}{\gamma}} J}.
\end{aligned}$$

By Definition 2.1–2.3, one can obtain the following lemma.

Lemma 3.5. *Assume that the hypotheses (H_1), (f_1), (g_1) and (h_1) hold. A function $u \in C(J, X)$ is a solution of the fractional integral equation*

$$\begin{aligned}
u(t) = u_0 + G(u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} [f(s, u(s)) + \\
= g(s, u(s)) + h(s, u(s))] ds, \qquad (2)
\end{aligned}$$

if and only if u is a solution of the system (1).

Proof. For any $r > 0$ and $u \in B_r$. According to (g_1) and Definition 3.2 (i)–(ii), $g(t, u)$ is a measurable function on J. Direct calculation gives that $(t - s)^{\alpha - 1} \in L^{\frac{1}{1 - \beta}} J$, for $t \in J$ and $\beta \in [0, \alpha)$. By using Hölder inequality and Definition 3.2 (iii), for $t \in J$, we obtain that

$$\int_0^t \|(t-s)^{\alpha-1} g(s,u(s))\| ds \leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\beta}} ds \right)^{1-\beta} \|m\|_{L^{\frac{1}{\beta}} J} =$$

$$= \left(\int_0^t (t-s)^{q_2} ds \right)^{1-\beta} \|m\|_{L^{\frac{1}{\beta}} J} \leq \frac{M}{(1+q_2)^{1-\beta}} \, T^{(1+q_2)(1-\beta)}, \qquad (3.2)$$

which means that $(t-s)^{\alpha-1} g(s,u(s))$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $u \in B_r$.

According to (f_1), for $t \in J$ and $u \in B_r$, we get that

$$\|f(t,u(t))\| \leq l(t)\|u(t)\| + \|f(t,0)\| \leq l(t)r + \|f(t,0)\|.$$

Using the similar argument and noting that (f_1) and (h_1), we can get that $(t-s)^{\alpha-1} f(s,u(s))$ and $(t-s)^{\alpha-1} h(s,u(s))$ are Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $x \in B_r$.

Thus, we get that $(t-s)^{\alpha-1}[f(s,u(s))+g(s,u(s))+h(s,u(s))]$ are Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $u \in B_r$.

Let $\overline{G}(\tau,s) = (t-\tau)^{-\alpha}|\tau - s|^{\alpha-1} m(s)$. Since $\overline{G}(\tau,s)$ is a nonnegative, measurable function on $D = [0,t] \times [0,t]$ for $t \in J$, we have

$$\int_0^t \left[\int_0^t \overline{G}(\tau,s) ds \right] d\tau = \int_D \overline{G}(\tau,s) ds d\tau = \int_0^t \left[\int_0^t \overline{G}(\tau,s) d\tau \right] ds$$

and

$$\begin{aligned}
\int_D \overline{G}(\tau,s) ds\, d\tau &= \int_0^t \left[\int_0^t \overline{G}(\tau,s) ds \right] d\tau = \\
&= \int_0^t (t-\tau)^{-\alpha} \left[\int_0^t |\tau - s|^{\alpha-1} m(s) ds \right] d\tau = \\
&= \int_0^t (t-\tau)^{-\alpha} \left[\int_0^\tau (\tau - s)^{\alpha-1} m(s) ds \right] d\tau + \\
&+ \int_0^t (t-\tau)^{-\alpha} \left[\int_\tau^t (s-\tau)^{\alpha-1} m(s) ds \right] d\tau \leq \\
&\leq \frac{2M}{(1+q_2)^{1-\beta}} \, T^{(1+q_2)(1-\beta)} \int_0^t (t-\tau)^{-\alpha} d\tau \leq
\end{aligned}$$

$$\leq \frac{2M}{(1-\alpha)(1+q_2)^{1-\beta}} \; T^{(1+q_2)(1-\beta)+1-\alpha}.$$

Therefore, $G_1(\tau,s)=(t-\tau)^{-\alpha}(\tau-s)^{\alpha-1}g(s,u(s))$ is a Lebesgue integrable function on $D=[0,t]\times[0,t]$, then we have

$$\int_0^t d\tau \int_0^\tau G_1(\tau,s)ds = \int_0^t ds \int_s^t G_1(\tau,s)d\tau.$$

Similarly, $G_2(\tau,s)=(t-\tau)^{-\alpha}(\tau-s)^{\alpha-1}h(s,u(s))$ is a Lebesgue integrable function on $D=[0,t]\times[0,t]$, then we have

$$\int_0^t d\tau \int_0^\tau G_2(\tau,s)ds = \int_0^t ds \int_s^t G_2(\tau,s)d\tau.$$

We now prove that

$$\begin{aligned}&{}^LD^\alpha\Big(I^\alpha\big[f(t,u(t))+g(t,u(t))+h(t,u(t))\big]\Big)=\\&=\big[f(t,u(t))+g(t,u(t))+h(t,u(t))\big],\quad \text{for}\quad t\in(0,T].\end{aligned}$$

where D^α is Riemann–Liouville fractional derivative.

Indeed, we have

$$\begin{aligned}&{}^LD^\alpha\Big(I^\alpha\big[f(t,u(t))+g(t,u(t))+h(t,u(t))\big]\Big)=\\
&=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\alpha}\int_0^\tau(\tau-s)^{q-1}\big[f(s,u(s))+\\
&\qquad+g(s,u(s))+h(s,u(s))\big]dsd\tau=\\
&=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t\int_0^\tau G_1(\tau,s)\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]dsd\tau=\\
&=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t\int_s^t G_1(\tau,s)\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]d\tau ds=\\
&=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds\int_s^t G_1(\tau,s)d\tau=\\
&=\frac{d}{dt}\int_0^t\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds=\\
&=f(t,u(t))+g(t,u(t))+h(t,u(t)).\end{aligned}$$

If u satisfies the relation (2), then we get that $u(t)$ is absolutely continuous on J. In fact, for any disjoint family of open intervals $\{(a_i,b_i)\}_{1\le i\le n}$ on J with $\sum\limits_{i=1}^{n}(b_i-a_i)\to 0$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\|u(b_i)-u(a_i)\| =\\
&=\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\Big\|\int_0^{b_i}(b_i-s)^{\alpha-1}\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds-\\
&-\int_0^{a_i}(a_i-s)^{\alpha-1}\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds\Big\| \le\\
&\le\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\Big\|\int_{a_i}^{b_i}(b_i-s)^{\alpha-1}\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds\Big\|+\\
&+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\Big\|\int_0^{a_i}(b_i-s)^{\alpha-1}\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds-\\
&-\int_0^{a_i}(a_i-s)^{\alpha-1}\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds\Big\| \le\\
&\le\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\int_{a_i}^{b_i}(b_i-s)^{\alpha-1}\big[l(s)r+\|f(s,0)\|+m(s)+w(s)\big]ds+\\
&+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\int_0^{a_i}\big((a_i-s)^{\alpha-1}-(b_i-s)^{\alpha-1}\big)\big[l(s)r+\|f(s,0)\|+m(s)+w(s)\big]ds\le\\
&\le\sum_{i=1}^{n}\frac{r}{\Gamma(\alpha)}\left(\int_{a_i}^{b_i}(b_i-s)^{\frac{\alpha-1}{1-\delta}}ds\right)^{1-\delta}\|l\|_{L^{\frac{1}{\delta}}J}+\\
&+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\left(\int_{a_i}^{b_i}(b_i-s)^{\frac{\alpha-1}{1-\eta}}ds\right)^{1-\eta}\|f(s,0)\|_{L^{\frac{1}{\eta}}J}+\\
&+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\left(\int_{a_i}^{b_i}(b_i-s)^{\frac{\alpha-1}{1-\beta}}ds\right)^{1-\beta}\|m\|_{L^{\frac{1}{\beta}}J}+
\end{aligned}
$$

$$+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\Big(\int_{a_i}^{b_i}(b_i-s)^{\frac{\alpha-1}{1-\gamma}}ds\Big)^{1-\gamma}\|w\|_{L^{\frac{1}{\gamma}}J}+$$
$$+\sum_{i=1}^{n}\frac{r}{\Gamma(\alpha)}\Big(\int_{0}^{a_i}(a_i-s)^{\frac{\alpha-1}{1-\delta}}-(b_i-s)^{\frac{\alpha-1}{1-\delta}}ds\Big)^{1-\delta}\|l\|_{L^{\frac{1}{\delta}}J}+$$
$$+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\Big(\int_{0}^{a_i}(a_i-s)^{\frac{\alpha-1}{1-\eta}}-(b_i-s)^{\frac{\alpha-1}{1-\eta}}ds\Big)^{1-\eta}\|f(s,0)\|_{L^{\frac{1}{\eta}}J}+$$
$$+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\Big(\int_{0}^{a_i}(a_i-s)^{\frac{\alpha-1}{1-\beta}}-(b_i-s)^{\frac{\alpha-1}{1-\beta}}ds\Big)^{1-\beta}\|m\|_{L^{\frac{1}{\beta}}J}+$$
$$+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\Big(\int_{0}^{a_i}(a_i-s)^{\frac{\alpha-1}{1-\gamma}}-(b_i-s)^{\frac{\alpha-1}{1-\gamma}}ds\Big)^{1-\gamma}\|w\|_{L^{\frac{1}{\gamma}}J}\leq$$
$$\leq\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}}r\|l\|_{L^{\frac{1}{\delta}}J}+\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}}\|f(s,0)\|_{L^{\frac{1}{\eta}}J}+$$
$$+\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}\|m\|_{L^{\frac{1}{\beta}}J}+\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_3)(1-\gamma)}}{\Gamma(\alpha)(1+q_3)^{1-\gamma}}\|w\|_{L^{\frac{1}{\gamma}}J}+$$
$$+\sum_{i=1}^{n}\frac{(a_i^{1+q_0}-b_i^{1+q_0}+(b_i-a_i)^{1+q_0})^{1-\delta}}{\Gamma(\alpha)(1+q_0)^{1-\delta}}\ r\|l\|_{L^{\frac{1}{\delta}}J}+$$
$$+\sum_{i=1}^{n}\frac{(a_i^{1+q_1}-b_i^{1+q_1}+(b_i-a_i)^{1+q_1})^{1-\eta}}{\Gamma(\alpha)(1+q_1)^{1-\eta}}\ \|f(s,0)\|_{L^{\frac{1}{\eta}}J}+$$
$$+\sum_{i=1}^{n}\frac{(a_i^{1+q_2}-b_i^{1+q_2}+(b_i-a_i)^{1+q_2})^{1-\beta}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}\ \|m\|_{L^{\frac{1}{\beta}}J}+$$
$$+\sum_{i=1}^{n}\frac{(a_i^{1+q_3}-b_i^{1+q_3}+(b_i-a_i)^{1+q_3})^{1-\gamma}}{\Gamma(\alpha)(1+q_3)^{1-\gamma}}\ \|w\|_{L^{\frac{1}{\gamma}}J}\leq$$
$$\leq 2\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}}\ rL\ +\ 2\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}}\ F+$$
$$+2\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}\ M\ +\ 2\sum_{i=1}^{n}\frac{(b_i-a_i)^{(1+q_3)(1-\gamma)}}{\Gamma(\alpha)(1+q_3)^{1-\gamma}}\ W\longrightarrow 0.$$

Therefore, $u(t)$ is absolutely continuous on J which implies that $u(t)$ is differentiable for almost all $t\in J$.

According to the argument above and Remark 2.4, for almost all $t \in (0,T]$, we have

$$
\begin{aligned}
&{}^cD^\alpha u(t) = \\
&=^c D^\alpha\left[u_0+G(u)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds\right]= \\
&=^c D^\alpha\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[f(s,u(s))+g(s,u(s))+h(s,u(s))\big]ds\right] = \\
&=^c D^\alpha\Big(I^\alpha\big[f(t,u(t))+g(t,u(t))+h(t,u(t))\big]\Big) = \\
&=^L D^\alpha\Big(I^\alpha\big[f(t,u(t))+g(t,u(t))+h(t,u(t))\big]\Big)- \\
&\quad -\Big(I^\alpha\big[f(t,u(t))+g(t,u(t))+h(t,u(t))\big]\Big)_{t=0}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Since $(t-s)^{\alpha-1}[f(s,u(s))+g(s,u(s))+h(s,u(s))]$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$, we known that $\big(I^\alpha[f(s,u(s))+g(s,u(s))+h(s,u(s))]\big)_{t=0} = 0$ which implies that

$$
{}^cD^\alpha u(t) = f(t,u(t))+g(t,u(t))+h(t,u(t)), \text{ a.e. } t \in J.
$$

Moreover, $u(0) = u_0 + G(u)$. Thus, $u \in C(J,X)$ is a solution of system (1). On the other hand, if $u \in C(J,X)$ is a solution of system (1), then u satisfies the integral equation (2). □

4. Existence of Extremal Solutions

Define the order relation " $\leq$ " by the cone K in $C(J,X)$, given by

$$
K = \big\{z \in C(J,X) \mid z(t) \geq 0 \ \text{ for all } \ t \in J\big\}.
$$

Clearly, the cone K is normal in $C(J,X)$.

Definition 4.1. A function $a \in C(J,X)$ is called a lower solution of system (1) on J if the function $a(t)$ is absolutely continuous on J, and

$$
\begin{cases}
{}^cD^\alpha a(t) \leq f(t,a(t))+g(t,a(t))+h(t,a(t)), & \text{a.e. } t \in J, \\
a(0) \leq u_0 + G(a).
\end{cases}
$$

Definition 4.2. A function $b \in C(J,X)$ is called a upper solution of system (1) on J if the function $b(t)$ is absolutely continuous on J, and

$$
\begin{cases}
{}^cD^\alpha b(t) \geq f(t,b(t))+g(t,b(t))+h(t,b(t)), & \text{a.e. } t \in J, \\
b(0) \geq u_0 + G(b).
\end{cases}
$$

Definition 4.3. A function $u \in C(J,X)$ is a solution of system (1) on J if it is a lower as well as a upper solution of system (1) on J.

Definition 4.4. A solution u_{max} of system (1) is said to be maximal if for any other solution u to system (1), one has $u(t) \leq u_{max}(t)$ for all $t \in J$.

Definition 4.5. A solution u_{min} of system (1) is said to be minimal if if for any other solution u to system (1), one has $u_{min}(t) \leq u(t)$ for all $t \in J$.

In addition to the hypotheses in Section 3, we introduce the following hypotheses.

(H_2) system (1) has a lower solution a and an upper solution b with $a \leq b$.

(f_2) $f(t,u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$.

(g_2) $g(t,u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$.

(g_3) for every $t \in J$, the set $\mathbb{S}_g = \{(t-s)^{\alpha-1} g(s,u(s)) : u \in C(J,X), s \in [0,t]\}$ is relatively compact.

(h_2) for every $t \in J$, the set $\mathbb{S}_h = \{(t-s)^{\alpha-1} h(s,u(s)) : u \in C(J,X), s \in [0,t]\}$ is relatively compact.

(G_1) for arbitrary $u \in C(J,X)$, there exists a $l_G \in (0,1)$ such that $\|G(u)\| \leq l_G \|u\|_C$.

(G_2) for arbitrary $u,v \in C(J,X)$ there exists a $l'_G \in (0,1)$ such that $\|G(u)-G(v)\| \leq l'_G \|u-v\|_C$.

(G_3) $G(u)$ is nondecreasing with respect to u for any $u \in C(J,X)$.

Theorem 4.6. *Assume that the hypotheses (H_1)–(H_2), (f_1)–(f_2), (g_1)–(g_3), (h_1)–(h_2), and (G_1)–(G_3) hold. Then system (1) has a minimal and a maximal solution in the order interval $[a,b]$ provided that*

$$l_G + \frac{LT^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}} < 1, \tag{3}$$

and

$$\Omega_{\alpha,\delta,q_0} = l'_G + \frac{LT^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}} < 1. \tag{4}$$

Proof. By Lemma 3.5, system (1) is equivalent to the fractional integral equation (2). Consider the order interval $[a,b]$ in $C(J,X)$ which is well defined in view of hypothesis (H_2).

Define three operators A, B and C on $C(J,X)$ as follows

$$\begin{cases}(Au)(t) = G(u) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds, \\ (Bu)(t) = u_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} g(s,u(s))ds, \qquad \text{for} \quad t \in J, \\ (Cu)(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} h(s,u(s))ds,\end{cases}$$

where $u \in X$. Clearly the operators A, B, C are well defined on $[a,b]$ in view of hypotheses (f_1), (g_1) and (h_1).

Then fractional integral equation (2) is equivalent to the operator equation

$$Au(t) + Bu(t) + Cu(t) = u(t), \ t \in J.$$

We shall show that A, B and C satisfy the conditions of Lemma 2.13 on $[a,b]$.

Due to condition (3), we can choose

$$r \geq \frac{1}{1 - l_G - \dfrac{LT^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}}} \times$$
$$\times \left(\|u_0\| + \frac{FT^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}} + \frac{MT^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}} + \frac{WT^{(1+q_3)(1-\gamma)}}{\Gamma(\alpha)(1+q_3)^{1-\gamma}} \right),$$

and define

$$B_r = \{u \in C(J,X) : \|u\|_C \leq r\}.$$

The proof is divided into several steps.

Step 1. $Au + Bu + Cu \in B_r$ for every $u \in B_r$.

From (f_1), (g_1), (h_1), (G_1) we have

$$\|Au + Bu + Cu\| \leq$$
$$\leq \|u_0\| + \|G(u)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \|f(s,u(s)) + g(s,u(s)) + h(s,u(s))\| ds \leq$$
$$\leq \|u_0\| + l_G\|u\|_C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \big[l(s)\|u\| + \|f(s,0)\| + m(s) + w(s)\big] ds \leq$$
$$\leq \|u_0\| + l_G r + \frac{Lr}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\delta}} ds\right)^{1-\delta} + \frac{F}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\eta}} ds\right)^{1-\eta} +$$

$$+\frac{M}{\Gamma(\alpha)}\bigg(\int\limits_0^t (t-s)^{\frac{\alpha-1}{1-\beta}}ds\bigg)^{1-\beta}+\frac{W}{\Gamma(\alpha)}\bigg(\int\limits_0^t (t-s)^{\frac{\alpha-1}{1-\gamma}}ds\bigg)^{1-\gamma}\le$$

$$\le \|u_0\|+l_G r+\frac{rLT^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{(1-\delta)}}+\frac{FT^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}}+$$

$$+\frac{MT^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}+\frac{WT^{(1+q_3)(1-\gamma)}}{\Gamma(\alpha)(1+q_3)^{1-\gamma}}\le r.$$

Thus, $Au+Bu+Cu\in B_r$.

From Lemma 3.5, we get that system (1) is equivalent to the operator equation $(Au)(t)+(Bu)(t)+(Cu)(t)=u(t)$ for $t\in J$. Now we show that the operator equation $Au+Bu+Cu=u$ has a least and a greatest solution in $[a,b]$.

Step 2. A is a contraction in B_r.

For any $u,v\in B_r$ and $t\in J$, according to (k_1), (f_1) and (G_2) we have

$$\big\|(Au)(t)-(Av)(t)\big\|\le \|G(u)-G(v)\|+$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\|f(s,u(s))-f(s,v(s))\|ds\le$$

$$\le l'_G\|u-v\|_C+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}l(s)\|u(s)-v(s)\|ds\le$$

$$\le\bigg[l'_G+\frac{1}{\Gamma(\alpha)}\bigg(\int\limits_0^t (t-s)^{\frac{\alpha-1}{1-\delta}}ds\bigg)^{1-\delta}\|l\|_{L^{\frac{1}{\delta}}[0,t]}\bigg]\|u-v\|_C\le$$

$$\le\bigg[l'_G+\frac{LT^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}}\bigg]\|u-v\|_C,$$

which implies that

$$\|Ax-Ay\|_C\le\Omega_{\alpha,\delta,q_0}\|u-v\|_C.$$

Therefore, A is a contraction in B_r according to (4).

Step 3. B is a completely continuous operator and C is a totally bounded operator.

For any $u\in B_r$, Let $\{u_n\}$ be a sequence of B_r such that $u_n\to u$ in B_r. Then, $g(s,u_n(s))\to g(s,u(s))$ as $n\to\infty$ due to the hypotheses (g_1). Moreover, for all $t\in J$, we have

$$\big\|g(s,u_n(s))-g(s,u(s))\big\|\le 2m(s).$$

Note that the functions $s\to(t-s)^{\alpha-1}2m(s)$ is integrable on J, and $\|u_n(s)-u(s)\|\to 0$, $\|g(s,u_n(s))-g(s,u(s))\|\to 0$ a.e. $s\in J$ as $n\to\infty$. By means of the Lebesgue Dominated Convergence Theorem,

$$\big\|(Bu_n)(t)-(Bu)(t)\big\|\leq\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\big\|g(s,u_n(s))-g(s,u(s))\big\|ds\to 0.$$

Therefore, $Bu_n \to Bu$ as $n \to \infty$ which implies that B is continuous.

Since B is a continuous operator, we only need to check that $\{Bu,\ u \in B_r\}$ is relatively compact. For any $u \in B_r$ and $t \in J$, we have

$$\begin{aligned}
\|(Bu)(t)\| &\leq \|u_0\| + \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\|g(s,u(s))\|ds \leq \\
&\leq \|u_0\| + \frac{1}{\Gamma(\alpha)}\bigg(\int\limits_0^t (t-s)^{\frac{\alpha-1}{1-\beta}}ds\bigg)^{1-\beta}\|m\|_{L^{\frac{1}{\beta}}[0,t]} \leq \\
&\leq \|u_0\| + \frac{M}{\Gamma(\alpha)}\bigg(\int\limits_0^t (t-s)^{q_2}ds\bigg)^{1-\beta} \leq \\
&\leq \|u_0\| + \frac{MT^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}.
\end{aligned}$$

Thus $\{Bu,\ u \in B_r\}$ is uniformly bounded.

In the following, we will show that $\{Bu,\ u \in B_r\}$ is a family of equicontinuous functions.

For any $u \in B_r$ and $0 \leq t_1 < t_2 \leq T$, we get

$$\begin{aligned}
&\big\|(Bu)(t_2)-(Bu)(t_1)\big\| \leq \\
&\leq \frac{1}{\Gamma(\alpha)}\bigg\|\int\limits_0^{t_1}\big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big)g(s,u(s))ds\bigg\| + \\
&\quad + \frac{1}{\Gamma(\alpha)}\bigg\|\int\limits_{t_1}^{t_2}(t_2-s)^{\alpha-1}g(s,u(s))ds\bigg\| \leq \\
&\leq \frac{1}{\Gamma(\alpha)}\int\limits_0^{t_1}\big\|\big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big)g(s,u(s))\big\|ds + \\
&\quad + \frac{1}{\Gamma(\alpha)}\int\limits_{t_1}^{t_2}\big\|(t_2-s)^{\alpha-1}g(s,u(s))\big\|ds \leq \\
&\leq \frac{1}{\Gamma(\alpha)}\int\limits_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)m(s)ds +
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_1}^{t_2}(t_2-s)^{\alpha-1}m(s)ds\le$$

$$\le\frac{1}{\Gamma(\alpha)}\Bigg(\int\limits_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)^{\frac{1}{1-\beta}}ds\Bigg)^{1-\beta}\|m\|_{L^{\frac{1}{\beta}}[0,t_1]}+$$

$$+\frac{1}{\Gamma(\alpha)}\Bigg(\int\limits_{t_1}^{t_2}\big((t_2-s)^{\alpha-1}\big)^{\frac{1}{1-\beta}}ds\Bigg)^{1-\beta}\|m\|_{L^{\frac{1}{\beta}}[t_1,t_2]}\le$$

$$\le\frac{M}{\Gamma(\alpha)}\Bigg(\int\limits_0^{t_1}\big((t_1-s)^{q_2}-(t_2-s)^{q_2}\big)ds\Bigg)^{1-\beta}+$$

$$+\frac{M}{\Gamma(\alpha)}\Bigg(\int\limits_{t_1}^{t_2}(t_2-s)^{q_2}ds\Bigg)^{1-\beta}\le$$

$$\le\frac{M}{\Gamma(\alpha)(1+q_2)^{1-\beta}}\big((t_1)^{1+q_2}-(t_2)^{1+q_2}+(t_2-t_1)^{1+q_2}\big)^{1-\beta}+$$

$$+\frac{M}{\Gamma(\alpha)(1+q_2)^{1-\beta}}(t_2-t_1)^{(1+q_2)(1-\beta)}\le$$

$$\le\frac{2M}{\Gamma(\alpha)(1+q_2)^{1-\beta}}(t_2-t_1)^{(1+q_2)(1-\beta)}.$$

As $t_2-t_1\to 0$, the right-hand side of the above inequality tends to zero independently of $u\in B_r$. We get that $\{Bu,\ u\in B_r\}$ is a family of equicontinuous functions.

In view of the condition (g_3) and the Lemma 2.7, we know that $\overline{conv}\mathbb{S}_g$ is compact.

For any $t^*\in J$,

$$(Bu_n)(t^*)=\frac{1}{\Gamma(\alpha)}\int\limits_0^{t^*}(t^*-s)^{\alpha-1}g(s,u_n(s))ds=$$

$$=\frac{1}{\Gamma(\alpha)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\bigg(t^*-\frac{it^*}{k}\bigg)^{\alpha-1}g\bigg(\frac{it^*}{k},u_n\Big(\frac{it^*}{k}\Big)\bigg)=\frac{t^*}{\Gamma(\alpha)}\zeta_n,$$

where

$$\zeta_n=\lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}\bigg(t^*-\frac{it^*}{k}\bigg)^{\alpha-1}g\bigg(\frac{it^*}{k},u_n\Big(\frac{it^*}{k}\Big)\bigg).$$

Since $\overline{conv}\mathbb{S}_g$ is convex and compact, we know that $\zeta_n\in\overline{conv}\mathbb{S}_g$. Hence, for any $t^*\in J$, the set $\{Bu_n\}$ $(n=1,2,\dots)$ is relatively compact.

From Ascoli-Arzela theorem every $\{Bu_n(t)\}$ contains a uniformly convergent subsequence $\{Bu_{n_k}(t)\}$ $(k = 1, 2, \dots)$ on J. Thus, the set $\{Bu : u \in B_r\}$ is relatively compact.

Step 4. C is a totally bounded operator.

Using the similar argument in Step 3, we can get that C is a continuous operator, $\{Cu,\ u \in B_r\}$ is also relatively compact, which means that C is totally bounded. Therefore, C is a totally bounded operator.

Step 5. A, B and C are three monotone increasing operators.

Since $u, v \in C(J, X)$ with $u \le v$ for $t \in J$, according to (k_2), (f_2) and (G_1), we have

$$\begin{aligned}(Au)(t) &= G(u) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, u(s))ds \\ &\le G(v) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, v(s))ds \ = \ (Av)(t).\end{aligned}$$

Hence A is a monotone increasing operator.

Similarly, we can conclude that B and C are also monotone increasing operators according to (g_2) and (h_1).

Clearly, K is a normal cone. From (H_2), Definition 4.1 and Definition 4.2, we have that $a \le Aa + Ba + Ca$ and $b \ge Ab + Bb + Cb$ with $a \le b$. Thus the operators A, B and C satisfy all the conditions of Lemma 2.13 and hence the operator equation $Au + Bu + Cu = u$ has a least and a greatest solution in $[a, b]$. Therefore, system (1) has a minimal and a maximal solution on J. This completes the proof. □

Now we assume the following conditions:

(G_1') $G : C(J, X) \to X$ is continuous and compact.

(G_2') There exists a $M_G > 0$ such that $\|G(u)\| \le M_G$ for all $u \in C(J, X)$.

Theorem 4.7. *Assume that the hypotheses* (H_1)*–*(H_2)*,* (f_1)*–*(f_2)*,* (g_1)*–*(g_3)*,* (h_1)*–*(h_2)*, and* (G_1')*,* (G_2')*,* (G_3) *hold. Then system* (1) *has a minimal and a maximal solution in the order interval* $[a, b]$ *provided that*

$$\frac{LT^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}} < 1. \tag{5}$$

Proof. Define three operators $\overline{A}$, $\overline{B}$ and $\overline{C}$ on $C(J,X)$ as follows

$$\begin{cases} (\overline{A}u)(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds, \\ (\overline{B}u)(t) = G(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} g(s,u(s))ds, \qquad \text{for } t \in J, \\ (\overline{C}u)(t) = u_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} h(s,u(s))ds, \end{cases}$$

where $u \in X$. It is easy to see that $\overline{A}, \overline{B}, \overline{C}$ are well defined.

Due to condition (5), we can choose

$$\overline{r} \geq \frac{1}{1 - \dfrac{LT^{(1+q_0)(1-\delta)}}{\Gamma(\alpha)(1+q_0)^{1-\delta}}} \left(\|u_0\| + M_G + \frac{FT^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}} + \right. \\ \left. + \frac{MT^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}} + \frac{WT^{(1+q_3)(1-\gamma)}}{\Gamma(\alpha)(1+q_3)^{1-\gamma}} \right),$$

and define $B_{\overline{r}} = \{u \in C(J,X) : \|u\|_C \leq \overline{r}\}$.

Repeating the process of Step 1–5 again, one can verify that system (1) has a minimal and a maximal solution on J. □

5. Application to Biomedical Sciences

In this section we give an example to illustrate the usefulness of our main results.

Consider the following fractional Logistic equations with perturbations

$$\begin{cases} \dfrac{\partial^\alpha}{\partial t^\alpha} u(t,y) = \dfrac{1}{1+ae^t} \dfrac{|u(t,y)|}{1+|u(t,y)|} + \dfrac{e^{-\nu t}}{1+be^t} \dfrac{|u(t,y)|}{1+|u(t,y)|} + \\ \qquad\qquad + \dfrac{e^{-\nu t}}{1+ce^t} \dfrac{|u(t,y)|}{1+|u(t,y)|}, \text{ a.e. } t \in (0,T], \\ u(t,0) = u(t,1) = 0, \ t > 0, \\ u(0,y) = \displaystyle\sum_{j=1}^{p} |\lambda_j| \, |u(t_j,y)|, \ y \in [0,1], \ 0 < t_1 < t_2 < \cdots < t_p < T. \end{cases} \tag{6}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the fractional partial derivative of order $\alpha \in (0,1)$, $u(t,y)$ denote the population number of isolated species at time t and location y, $a,b,c,\nu > 0$ and $\lambda_j \in R$, $j = 1,2,\ldots,p$, $p \in Z_+$.

The first equation of system (6) describes the variation of the population number u of the species in environment. The second equation of system (6) shows that the species is isolated. The third equation of system (6) reflects the possibility of variation on the population.

Let $X = L^2[0,1]$ equipped with its natural norm and inner product defined respectively for all $u, v \in L^2[0,1]$ by

$$\|u\|_{L^2[0,1]} = \left(\int_0^1 |u(x)|^2 dx \right)^{\frac{1}{2}} \text{ and } \langle u, v \rangle = \int_0^1 u(x)\overline{v(x)}dx.$$

For $(t,u) \in J \times X$, set ${}^cD^\alpha u(t)(y) = \frac{\partial^\alpha}{\partial t^\alpha} u(t,y)$, $u(t)(y) = u(t,y)$, $G(u)(y) = \sum_{j=1}^{p} |\lambda_j||u(t_j,y)|$,

$$\begin{aligned} f(t,u(t))(y) &= \frac{1}{1+ae^t} \frac{|u(t,y)|}{1+|u(t,y)|}, \\ g(t,u(t))(y) &= \frac{e^{-\nu t}}{1+be^t} \frac{|u(t,y)|}{1+|u(t,y)|}, \\ h(t,u(t))(y) &= \frac{e^{-\nu t}}{1+ce^t} \frac{|u(t,y)|}{1+|u(t,y)|}. \end{aligned}$$

Then, the system (1) is the abstract formulation of the problem (6).

It is obvious that $f(t,u)$, $g(t,u)$, $h(t,u)$ are nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$. For $u_1, u_2 \in X$ and $t \in J$, we have

$$\begin{aligned} \|f(t,u_1) - f(t,u_2)\| &\le \frac{1}{1+ae^t} \|u_1 - u_2\| \le \\ &\le l(t)\|u_1 - u_2\|, \; l(t) := \frac{1}{1+a} \in L^{\frac{1}{\delta}}(J, R_+), \\ \|G(u_1) - G(u_2)\| &\le \sum_{j=1}^{p} |\lambda_j| \, \|u_1(t_j) - u_2(t_j)\| \; \le \; \sum_{j=1}^{p} |\lambda_j| \, \|u_1 - u_2\|_C. \end{aligned}$$

Further, for all $u \in X$ and each $t \in J$,

$$\begin{aligned} \|g(t,u)\| &\le \frac{1}{1+b} := m(t) \in L^{\frac{1}{\beta}}(J, R_+), \\ \|h(t,u)\| &\le \frac{1}{1+c} := w(t) \in L^{\frac{1}{\gamma}}(J, R_+), \\ \|G(u)\| &\le \sum_{j=1}^{p} |\lambda_j| \, \|u\|_C. \end{aligned}$$

One can easily check that $u_{min}(t) = 0$ is a lower solution of system (6). On the other hand, let $u_{max}(t) = 3 \max\left\{ \frac{1}{1+a}, \frac{1}{1+b}, \frac{1}{1+c} \right\} u(t)$. Then, $u_{max} \in C(J,X)$ is a upper solution of system (6). Moreover, we suppose that $\overline{\mathbb{S}_g}$

and $\overline{\mathbb{S}_h}$ are compact in $C(J,X)$ where

$$\mathbb{S}_g = \left\{ (t-s)^{\alpha-1} \frac{e^{-\nu s}}{1+be^s} \frac{u(t)}{1+u(t)} : u \in C(J,X),\ s \in [0,t] \right\},$$

and

$$\mathbb{S}_h = \left\{ (t-s)^{\alpha-1} \frac{e^{-\nu s}}{1+ce^s} \frac{u(t)}{1+u(t)} : u \in C(J,X),\ s \in [0,t] \right\}.$$

From the above discussion, all the assumptions in Theorem 4.6 are satisfied by choosing a small enough $T > 0$, a large enough $a > 0$ and suitable $\delta \in [0,\alpha)$, λ_i, p such that

$$\sum_{j=1}^{p} |\lambda_j| + \frac{\|\frac{1}{1+a}\|_{L^{\frac{1}{\delta}}J} \times T^{(1+\frac{\alpha-1}{1-\delta})(1-\delta)}}{\Gamma(\alpha)(1+\frac{\alpha-1}{1-\delta})^{1-\delta}} < 1,$$

our results can be applied to the problem (6), that is, we can use a biological approach to regulate the maximal or minimal quantity of a single, isolated species or eradicate pests. It provides us a reliable method for managing the single and isolated species in the nature.

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Authors' address:

Department of Mathematics, Guizhou
University, Guiyang, Guizhou 550025
P.R. China
E-mail: wjr9668@126.com
jzdxw@yahoo.com.cn
wei66@yahoo.com