Proceedings of A. Razmadze
Mathematical Institute
Vol. **158** (2012), 1–12

SOBOLEV SPACES AND LAGRANGE INTERPOLATION

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Abstract. We present a direct proof of the pointwise inequality for Lagrange interpolation remainder with equidistant colinear nodes for functions in the Sobolev space $W^{m,p}(\mathbb{R}^n)$, $p > 1$. As shown in [6] this inequality characterizes the space $W^{m,p}(\mathbb{R}^n)$.

რეზიუმე. მოყვანილია სობოლევის $W^{m,p}(\mathbb{R}^n)$ სივრცის f ფუნქციისათვის [6] შრომაში მიღებული წერტილოვანი დახასიათების პირდაპირი, გეომეტრიული დამტკიცება და კომენტირებულია ამავე შრომაში მიღებული ზოგიერთი შედეგი. ამასთანავე შემოღებულია $\mathbb{R}^n$ სივრცის G არეზე განსაზღვრული სობოლევის სივრცეების ბუნებრივი კლასი და ბოლოს, გამოთქმულია ზოგიერთი დასკვნითი შენიშვნა.

1. Let us recall [15], [19] that for an arbitrary integer $l \geq 0$ and a real or complex valued function f on $\mathbb{R}^n$ the expression

$$\Delta_h^l f(x) := \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x+jh) = (-1)^l \sum_{j=0}^{l} (-1)^j \binom{l}{j} f(x+jh) \quad (1.1)$$

is called the l-th difference of the function f at the point $x \in \mathbb{R}^n$ with step h, $h \in \mathbb{R}^n$, $h \neq 0$. We set also $\Delta_h^0 f(x) := f(x)$ and $\Delta_0^l f(x) := 0$. As is classically known, for $y = x + lh$, $h = \frac{y-x}{l}$, (1.1) has a beautiful interpretation as the difference or error in approximating the function $f(y)$ by its interpolating polynomial evaluated at y

$$\Delta^l f(x;y) = \Delta_h^l f(x) = f(y) - \mathcal{L}(y; f; x_0, \dots, x_{l-1}) \equiv (-1)^l \widetilde{\Delta}^l f(x,y) \quad (x_0 = x), \quad (1.2)$$

where $\mathcal{L}(y; f; x_0, \dots, x_{l-1}) \equiv \sum_{j=0}^{l-1} f(x_j) \ell_j(y, x_0, \dots, x_{l-1})$ is the Lagrange interpolating polynomial for the function f and the equidistant colinear nodes $x_i = x_0 + ih$, $i = 0, \dots, l-1$, [19]. Here $\ell_j(y, x_0, \dots, x_{l-1})$ stand for the fundamental Lagrange polynomials in y. $\widetilde{\Delta}^l(x,y)$ is the notation used in [6]. Let us remark that all points $x_j = x + jh$, $j = 0, \dots, l-1$, $x_l = y$, are situated on the affine line $\mathcal{R}$ in $\mathbb{R}^n$, joining x and y, which can be identified

2010 *Mathematics Subject Classification.* 46E35, 41A05.
Key words and phrases. Sobolev spaces, pointwise characterization, geometric proof, heigh order difference, Lagrange interpolation.

with the real line $\mathbb{R}^1$, making all the algebraic operations inherent in (1.2) meaningful.

In the sequel we shall use for (1.2) the term *Lagrange interpolation remainder*, or just *Lagrange remainder*, of order l at the point x evaluated at y, in analogy with the term *Taylor–Whitney remainder* centered at x,

$$R^{l-1}f(x;y) := f(y) - T_x^{l-1}f(y), \tag{1.3}$$

now common it mathematical literature. In [6] $\widetilde{\Delta}^l f(x,y)$ was also called an l-th finite difference remainder of the function f at x evaluated at y.

For functions $f \in W^{m,p}(\mathbb{R}^n)$, $p > 1$, the fundamental novel inequality referred to above reads as

$$|\Delta^m f(x;y)| \leq |x-y|^m[\widehat{a}_f^m(x) + \widehat{a}_f^m(y)] \tag{1.4}$$

for some $\widehat{a}_f \in L^p(\mathbb{R}^n)$.

We skip here over the somewhat delicate point that the Sobolev functions in general do not have pointwise values and the left hand side of inequality (1.4) is meaningful only up to subsets of measure zero. The right hand side may be infinite on a non-empty set of measure zero.

The functional coefficients $\widehat{a}_f^m(x)$ in (1.4) are not uniquely defined. They are collectively called *mean maximal m-gradients* of the function f and play the role of a variable Lipschitz coefficient of f. Roughly speaking, they all can be majorized by the local maximal function of the generalized Sobolev gradient $|\nabla^m f|$ of $f \in W^{m,p}(\mathbb{R}^n)$ as will be also seen from the constructive proof of (1.4) sketched below.

Our proof is organized in a series of lemmata.

Lemma 1. *For $f \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, the following inequality holds*

$$|f(x) - f(y)| \leq |x-y|\big(a_f^\delta(x) + a_f^\delta(y)\big), \quad x, y \in \mathbb{R}^n \tag{1.5}$$

for some $a_f^\delta \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $\delta = |x-y|$.

Proof. For arbitrary $x, y \in \mathbb{R}^n$ we have

$$f(x) - f(y) = \int_0^1 \langle \nabla f(x+ht), h\rangle \, dt, \quad h = y - x, \tag{1.6}$$

hence

$$|f(x) - f(y)| \leq |x-y| \int_0^1 |\nabla f|(x+ht)\, dt. \tag{1.7}$$

Let $B(x,r)$ be the ball of radius r centered at x, and $\Sigma_r(x,y)$ the spherical segment

$$\Sigma_r = B(x,r) \cap B(y,r), \quad r = |x-y|.$$

For an arbitrary $z \in \Sigma_r$

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)|. \tag{1.8}$$

Since $|x - z| \le |x - y|$, $|y - z| \le |x - y|$, averaging (1.8) over $z \in \Sigma_r$ we get

$$\begin{aligned} |f(x) - f(y)| &\le \fint_{\Sigma_r} |f(z) - f(x)|\, d\sigma_z + \fint_{\Sigma_r} |f(z) - f(y)|\, d\sigma_z \le \\ &\le \frac{|B(x,r)|}{|\Sigma_r|}\Big(\fint_{B(x,r)} |f(z) - f(x)|\, d\sigma_z + \fint_{B(y,r)} |f(z) - f(y)|\, d\sigma_z \Big), \end{aligned} \tag{1.9}$$

where the notation $|G|$ for a subset G in $\mathbb{R}^n$ is used for the volume of G, $|B(x,r)| = |B(y,r)|$.

By elementary geometry the ratio $\frac{|B(x,r)|}{|\Sigma_r|}$ is a constant depending only on n, $\frac{|B(x,r)|}{|\Sigma_r|} = C(n)$, and, as is well known, the average $\fint_{B(x,r)} |f(z) - f(x)|\, d\sigma_z$ is estimated by the local Hardy-Littlewood maximal function at x of the gradient $|\nabla f|$, $\mathcal{M}^\delta(|\nabla f|)(x)$, [42], [43]. Here the inequality (1.7) is used. Thus in (1.5) the function $a_f^\delta(x)$ is controlled by $\mathcal{M}^\delta(|\nabla f|)(x)$: in fact $a_f^\delta(x) \le C(n)\mathcal{M}^\delta(|\nabla f|)(x)$. □

The proof above, without changes, works for vector valued functions. This proof should be compared with the proof of the basic pointwise inequality (1) in our paper with P. Hajłasz from 1993 [7]. Notice that it does not refer to Riesz potentials and Hedberg lemma as in [7].

When combined with Reshetnyak's trick [36] used in [7], it can be used to deduce, in a direct way, a new and simplified proof of the basic pointwise inequalities in [2, 7, 8].

Let $f \in W^{k,p}(\mathbb{R}^n)$. For $l = 0, 1, \dots, k-1$, $h \in \mathbb{R}^n$, consider the functions

$$g_h^l(x) = \underbrace{\int_0^1 \cdots \int_0^1}_{l \text{ times}} \nabla^l f\Big(x + \sum_{i=1}^{l} t_i h\Big)(h, \dots, h)\, dt_1 \dots dt_l, \tag{1.10}$$

where $\nabla^l f$ is the l-th gradient of f considered as an l-polylinear form on $\mathbb{R}^n$.

Lemma 2. *$g_h^l(x)$ as a function of $x \in \mathbb{R}^n$ is in the class $W^{k-l,p}(\mathbb{R}^n)$.*

Proof. Obvious. □

In particular, $g_h^0(x) \equiv f(x)$,

$$g_h^1(x) = \int_0^1 \langle \nabla f(x + th, h\rangle\, dt \equiv \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x + ht) h_i\, dt, \quad \text{etc.}$$

Lemma 3. *The function $g_h^l(x)$ for $l = 0, \dots, k-1$ has the integral representation*

$$g_h^l(x) = \sum_{j=0}^{l} (-1)^l \binom{l}{j} f(x+jh) =$$
$$= \int_0^1 \cdots \int_0^1 \nabla^l f\Big(x + \sum_{i=1}^{l} t_i h\Big)(\underbrace{h,\dots,h}_{l \text{ times}})\, dt_1 \dots dt_l. \quad (1.11)$$

Proof. This is the well known formula of finite difference calculus [15, 19]. For $l = k$ it is also used in the paper [11] of R. Borghol. □

Lemma 4. *For $l = k-1$ we have the formula*

$$g_h^{k-1}(x) - g_h^{k-1}(x+h) =$$
$$= \sum_{l=1}^{k} (-1)^{l-1} \binom{k-1}{l-1} \big[f(x+(l-1)h) - f(x+lh)\big] \equiv$$
$$\equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x+jh) = \widetilde{\Delta}^k f(x,y) = (-1)^k \Delta^k f \ \text{ for } h = \frac{y-x}{k}, \quad (1.12)$$

in the notation of [6].

Proof. By Lemma 3

$$g_h^{k-1}(x) = \sum_{l=1}^{k} (-1)^{l-1} \binom{k-1}{l-1} f(x+(l-1)h). \quad (1.13)$$

Hence for $h = \frac{y-x}{k}$

$$\Delta_h^k f(x) = g_h^{k-1}(x) - g_h^{k-1}(x+h) =$$
$$= \sum_{l=1}^{k} (-1)^{l-1} \binom{k-1}{l-1} [f(x+(l-1)h) - f(x+lh)] =$$
$$= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} f(x+jh) + \sum_{l=1}^{k} (-1)^l \binom{k-1}{l-1} f(x+lh) =$$
$$= f(x) + (-1)^k f(x+kh) + \sum_{j=1}^{k-1} (-1)^j \Big[\binom{k-1}{j} + \binom{k-1}{j-1}\Big] f(x+jh) =$$
$$= (-1)^k \widetilde{\Delta}^k f(x,y) = \Delta^k f(x,y). \quad (1.14)$$

The proof is complete. □

Formulas (1.10)–(1.14) are examples of a series of formulas of finite differences [15], [19] which connect the operations of vector differential operators $\nabla^l f$ with finite difference operators $\Delta_h^k f$ for various values of the parameters l, k and h. They allow to reduce the estimates of higher order difference remainders of functions to lower order remainders of higher order gradients of these functions. They are analogues to the operations in Taylor–Whitney's algebras which played an important role in the development and applications of pointwise inequalities in [4], [5], [6], [8] and in the Whitney–Glaeser–Malgrange theory [20], [23], [31], [49], of smooth functions on arbitrary closed subsets of $\mathbb{R}^n$. Deep and important papers of Glaeser [21], [22], [23] in this theory seem to be so far waiting for better understanding and exploitment.

Now, combining Lemmata 1–4 we obtain the estimate (1.4), with $\widehat{a}_f(x)$ controlled by the maximal function of the vector gradient $\nabla^k f$ as required.

For convenience we summarize our discussion in the following

Proposition 5. *Let $f \in W^{m,p}(\mathbb{R}^n)$, $1 < p \le \infty$. Then there exists a function $\widehat{a}_f \in L_p(\mathbb{R}^n)$ such that the inequality* (1.4) *holds for almost all points $x, y \in \mathbb{R}^n$. The function $\widehat{a}_f$ is majorized a.e. by the local maximal function of the generalized Sobolev gradient $|\nabla^m f| \in L_p(\mathbb{R}^n)|$.*

As formulated above Proposition 5 is the "necessary part" of the main theorem in [6] for the case of Lagrange interpolation remainders. The proof of the "sufficiency part" of the theorem is left unchanged and proceeds along the argument sketched in [6].

The sketch of the proof of Proposition 5 presented above is in the convention that we work in the class of smooth functions where all the operations involved are classically meaningful. The clue of the story is that the constants appearing in the estimates depend on the parameters n and p only. This fundamental fact comes up again in §2 below where the pointwise inequality appears as "stable" under smoothing by convolution.

2. One of the advantages of the Lagrange interpolation calculus and the Lagrange remainders over the Taylor–Whitney remainders is that they interact very well with convolutions. This is immediately seen for the remainders $\Delta^1 f(x; y) = f(y) - f(x)$.

Indeed, for a normalized mollifier $\varphi_\varepsilon(x)$, $\varphi_\varepsilon \ge 0$, $\int \varphi_\varepsilon(x)\,dx = 1$, we have

$$f_\varepsilon(x) \equiv f * \varphi_\varepsilon(x) = \int f(x-\eta)\varphi_\varepsilon(\eta)\,d\eta$$

and

$$f_\varepsilon(y) - f_\varepsilon(x) = \int \big[f(y-\eta) - f(x-\eta)\big]\varphi_\varepsilon(\eta)\,d\eta,$$

hence

$$|f_\varepsilon(y)-f_\varepsilon(x)|\le |x-y|\Big(\int \widehat{a}(x-\eta)\varphi_\varepsilon(\eta)\,d\eta+\int \widehat{a}(y-\eta)\varphi_\varepsilon(\eta)\,d\eta\Big)$$

or

$$|f_\varepsilon(y)-f_\varepsilon(x)|\le |x-y|\big(\widehat{a}_\varepsilon(x)+\widehat{a}_\varepsilon(y)\big). \tag{2.1}$$

This elementary though basic fact, already formulated in our paper [7] and even earlier and repeated later in many seminar talks, holds for the higher order remainders $\Delta^m f(x;y)$ as well. Indeed, e.g. for $m=2$ we have

$$f_\varepsilon(x)-2f_\varepsilon(\tfrac{x+y}{2})+f_\varepsilon(y)=\int\big[f(x-\eta)-2f(\tfrac{x+y}{2}-\eta)+f(y-\eta)\big]\varphi_\varepsilon(\eta)\,d\eta\le$$

$$\le |x-y|^2\int\big[\widehat{a}_f(x-\eta)+\widehat{a}_f(y-\eta)\big]\varphi_\varepsilon(\eta)\,d\eta=|x-y|^2\big(\widehat{a}_{f,\varepsilon}(x)+\widehat{a}_{f,\varepsilon}(y)\big)$$

and the same calculation works for $m>2$.

Pointwise inequalities for the Lagrange remainders $\Delta^m f(x,y)$ appeared also in the recent papers of H. Triebel and his school [26], [45]. When the paper [6] was written the papers of Triebel [45] and Haroske–Triebel [26] were unknown to the author. Geometrically for $m>1$ they differ from ours by introducing the intermediate nodes $x_i=x+ih$, $i=1,\dots,m-1$, in the right hand side of (1.4).

For arbitrary $0<p\le\infty$, $s>0$ and $m\in\mathbb{N}$ with $s\le m$ in [45] is introduced the class $\mathbb{L}_p^{s,m}(\mathbb{R}^n)$ of all $f\in L_p(\mathbb{R}^n)$ for which there exists a nonnegative function $g\in L_p(\mathbb{R}^n)$ such that for all $h\in\mathbb{R}^n$, $0<|h|\le 1$ the inequality

$$|\Delta_n^m f(x)|\le |h|^s\sum_{l=0}^{m} g(x+lh)\quad \text{a.e. in } \mathbb{R}^n \tag{2.2}$$

holds.

With the norm

$$\|f\|_{s,m,p}=\|f\|_{L_p(\mathbb{R}^n)}+\inf\|g\|_{L_p(\mathbb{R}^n)},$$

where the infimum is taken over all g admissible in (2.2), the space $\mathbb{L}_p^{s,m}(\mathbb{R}^n)$ is a quasi-Banach space [45].

For $s=m$, $p\ge 1$, the inequality (2.2) is stronger than (1.4), i.e. (1.4) implies (2.2). Now it is clear that one of immediate conclusions from the main theorem in [6] is that for these values of s and p (1.4) is equivalent to (2.2). Thus we conclude that the intermediate nodes in (2.2) can be discarded.

For a while, let us introduce the notation: $\mathbb{W}^{m,p}(\mathbb{R}^n)$ is the class of all functions $f\in L_p(\mathbb{R}^n)$ for which there exists an $\widehat{a}_f\in L_p(\mathbb{R}^n)$ such that inequality (1.4) holds a.e. in $\mathbb{R}^n$.

In the general context of (1.4) and (2.2) we have the following

Proposition 6. *For $f \in \mathbb{W}^{m,p}(\mathbb{R}^n)$ the mollified function $f_\varepsilon = f * \varphi_\varepsilon$ is in the class $\mathbb{W}^{m,p}(\mathbb{R}^n)$ and*

$$|\Delta^m f_\varepsilon(x;y)| \le |x-y|^m \left(\widehat{a}^m_{f,\varepsilon}(x) + \widehat{a}^m_{f,\varepsilon}(y)\right) \tag{2.3}$$

with $\widehat{a}^m_{f,\varepsilon}(x) = \widehat{a}^m_f * \varphi_\varepsilon$.

Moreover, by the known properties of convolutions the L_p-norms of the mean maximal gradients $\widehat{a}^m_{f,\varepsilon}$ are uniformly controlled by the L_p-norms of mean maximal gradients of f:

$$\|\widehat{a}^m_{f,\varepsilon}\|_{L_p(\mathbb{R}^n)} \le \|\widehat{a}^m_f\|_{L_p(\mathbb{R}^n)} \tag{2.4}$$

for all $\varepsilon > 0$.

In particular, we conclude that smooth functions are dense in $\mathbb{W}^{m,p}(\mathbb{R}^n)$.

For the same values of the parameters s, m, p as for $\mathbb{L}^{s,m}_p(\mathbb{R}^n)$ we can also consider the class $\widehat{\mathbb{L}}^{s,m}_p(\mathbb{R}^n)$ defined by the pointwise inequality

$$|\Delta^m f(x;y)| \le |x-y|^s(g(x)+g(y)), \quad y = x + mh, \tag{2.5}$$

for some $g \in L_p(\mathbb{R}^n)$.

In [45] the quasi-Banach spaces $\mathbb{L}^{s,m}_p(\mathbb{R}^n)$ are used to identify some Besov spaces $B^{s\theta}_{p,q}(\mathbb{R}^n)$, $0 < \theta < 1$, $0 < q \le \infty$, as real interpolation spaces ([45], the main theorem). For $s = m$ in [6], as well as in the paper [45], the spaces $\mathbb{W}^{m,p}(\mathbb{R}^n)$ are identified with classical Sobolev spaces $W^{m,p}(\mathbb{R}^n)$.

The natural interesting question is to characterize the spaces $\widehat{\mathbb{L}}^{s,m}_p(\mathbb{R}^n)$, for s not integer, as some Sobolev–Besov type spaces.

Proposition 6 and its proof are also valid for the class $\widehat{\mathbb{L}}^{s,m}_p(\mathbb{R}^n)$. Thus smooth functions are also dense in $\widehat{\mathbb{L}}^{s,m}_p(\mathbb{R}^n)$.

The described characterization of Sobolev spaces $W^{m,p}(\mathbb{R}^n)$, $p > 1$, by pointwise inequalities (1.4) obviously holds for open subdomains $G \subset \mathbb{R}^n$ as well, if they have sufficiently regular boundary, e.g. for extension domains [38]. However this characterization is definitely not true for arbitrary subdomains. In this context it seems legitimate to introduce the (maximal) class of subdomains which admit global pointwise characterization.

Definition 7. An open subdomain $G \subset \mathbb{R}^n$ is called a *natural Sobolev (p,s), $1 < p \le \infty$, $s > 0$, domain* if the pointwise inequality

$$|\Delta^m f(x,y)| \le |x-y|^s(a(x)+a(y)), \quad s \le m, \tag{2.6}$$

for all pairs of points $x, y \in G$ such that the segment $[x,y] \subset G$, defines a Sobolev type Banach space $\widetilde{W}^{s,p}(G)$.

For $s \le m$, $0 < p \le \infty$ a related class of spaces $L^s_p(\mathbb{R}^n)^m$ has been introduced by H. Triebel in [45] and identified with subspaces of Besov spaces $B^s_{p,\infty}(\mathbb{R}^n)$ for $G = \mathbb{R}^n$.

It is natural to ask in particular in what sense our Sobolev–Besov type spaces coincide with Triebel–Besov spaces in [45]. More generally we can ask in what sense and for what subdomains the spaces $\widetilde{W}^{s,p}(G)$ coincide with Besov–Sobolev type spaces for non-integer s.

3. The pointwise inequality (1.4) actually can be used to characterize rather the homogeneous Sobolev spaces $\dot{W}^{m,p}(G)$ for a subdomain $G \subset \mathbb{R}^n$ with the seminorm $\|\nabla^m f\|_{L^p(G)}$. The classical inhomogeneous Sobolev spaces arise then as subspaces of $L^p(G)$. For the model case $G = \mathbb{R}^n$ various delicate phenomena are related to the asymptotic behavior of functions in $\dot{W}^{m,p}(\mathbb{R}^n)$ for $|x| \to \infty$ which seem to be so far only partially understood (see [3] and numerous other references. See also [1] for the case $n = 1$, i.e. on the line $\mathbb{R}^1$).

In our presentation here the pointwise inequality (1.4) as well as somewhat more sophisticated inequality for Taylor–Whitney remainders $R^{m-1} f(x; y)$ should be considered as elementary, though fundamental, facts appearing at the first steps of any discussion of Sobolev spaces.

What seems to be still lacking in this elementary discussion of Sobolev space theory, is the deeper, geometric and analytical, understanding of the trace (projection) operator $W^{m,p}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^k)$ for the corresponding values of the parameters s, p. As is classically known, this question led to the introduction of Sobolev fractional spaces, $W^{s,p}(\mathbb{R}^n)$, s — real, $\mathcal{H}^{s,p}(\mathbb{R}^n)$ and Besov spaces ([3], [42], [43] and many references therein).

It seems also that special attention should be directed to “*les schémas d'interpolation*” of G. Glaeser [20], [23], [27], [34], and their role in the theory of Sobolev spaces, see also [37], [38].

As already remarked in [6], and earlier even in [4], [7], [8], the pointwise inequality (1.4) above, together with the related inequality for the Taylor–Whitney remainder $R^{m-1}f(x; y)$ (precisely, inequality (1.2) in [6]) may serve as natural starting points and effective tools in the study of fundamental structural properties of Sobolev functions. Let us briefly recall some of them without going here into details (postponed to the activity and exposition plan foreseen in the last lines of [6]).

a) Lusin's approximation of Sobolev functions, i.e. interpolation by smooth functions on closed subsets, up to open complements of arbitrary small measure.
b) Stability under the convolution with compactly supported C^∞ kernels; density of subspaces of smooth, C^∞ functions.
c) S. M. Nikolskii's [3] fundamental theorems describing the characterization of Sobolev functions by their behavior on typical (almost all, in some natural sense) hyperplanes of positive codimensions less than n. The inequalities (1.4) reduce the characterization of Sobolev functions to their behavior on affine segments in their domain of definition. In particular the functions in $W^{m,p}(\mathbb{R}^n)$ for $p > 1$ are Hölder continuous on a.e. hyperplanes $\mathbb{R}^k \subset \mathbb{R}^n$, $k < p$,

the only global condition binding the variable Lipschitz coefficients on hyperplanes is the Fubini theorem for Lebesgue spaces $L^p(\mathbb{R}^n)$ and the factorizations $\mathbb{R}^n \sim \mathbb{R}^{n-k} \times \mathbb{R}^k$.

d) Characterization of compact subsets of $W^{m,p}(\mathbb{R}^n)$ by some conditions on the corresponding variable Lipschitz coefficients.
e) Differentiability properties of Sobolev functions, Calderón differentiability theorems, approximate and Peano differentiability, [3], [7], [8], [13], [14], [25], [33], [35], [44].
f) Extension of Hajłasz–Sobolev imbedding theorems of Sobolev spaces into higher exponent Lebesgue spaces: $W^{l,p}(G) \subset L^q(G)$, $q > p$, $l > 1$, for suitable values of the parameters l, p, q and $\dim G = n$, as in the classical Sobolev theory, modeled on Hajłasz's proof for measure metric spaces, [24].
g) Extension of the classical Hermite interpolation formulas [15], [19] to the general multidimensional context of "multiple" nodes as a theory intermediate between the Taylor–Whitney and Lagrange ("simple" nodes) interpolation theory ([20], [23], [31]).
h) If instead of the colinear equidistant nodes other configurations of interpolation nodes are used, more complicated algebraic and geometric phenomena occur, e.g. in the case of colinear not equidistant simple nodes the classical divided difference calculus comes up [14], [18]. New interpolation concepts appear in the Glaeser papers [20], [22], and, more recent, [24], [31]. Combining these ideas with the Sobolev's averaging procedure, [39], [40], [41], defines an apparently new interesting research direction.
i) Also the natural inclusions, e.g. $W^{m+1,p}(\mathbb{R}^n) \subset W^{m,p}(\mathbb{R}^n)$ when interpreted in terms of pointwise inequalities (1.4) suggest direct implications between inequalities (1.4) for various admissible values of the parameters (m,p). These lead to interesting and non-trivial arguments of geometric and analytic character. Probably the first beautiful example of this type of argument was given by Y. Zhou [53].

Let us remark at last that the "pointwise" approach to the Sobolev space theory seems to bring out more clearly than usually presented in the literature (e.g. [30], [32], [33], [15], [13]) close and natural connections between the general concepts of functional spaces and approximation theory on the real line $\mathbb{R}^1$ and in n-dimensional, $n > 1$, euclidean spaces.

Acknowledgement

Partially supported by the Polish Ministry of Science grant no. N N201 397837 (years 2009–2012) and the Academy of Finland.

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(Received 12.01.2012)

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