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SOLUTION OF A MIXED BOUNDARY VALUE PROBLEM OF THE PLANE THEORY OF ELASTIC MIXTURE FOR A MULTIPLY CONNECTED DOMAIN WITH A PARTIALLY UNKNOWN BOUNDARY HAVING THE AXIS OF SYMMETRY

K. SVANADZE

Abstract. In the present work we consider a mixed boundary value problem of the plane theory of an elastic mixture for a multiply connected domain, a square weakened by five holes with equally strong unknown boundaries. Four of the holes are equal and symmetric with respect to the segments connecting midpoints of opposite sides, while the fifth one is symmetric with respect to these segments and to the coordinate axes. The vertices of the square lie on the coordinate axes and their neighborhoods are cut out by equal smooth arcs, symmetric with respect to the coordinate axes. The linear portion of the boundary is under the action of absolutely smooth rigid punches with rectilinear bases which are acted on by forces of magnitude $p = (p_1, p_2)^T$. Unknown equally strong parts of the boundary are free from external forces. Using the method of the theory of analytic functions, the portions of equally strong boundaries as well as the stressed state of the body are found.

რეზიუმე. ნაშრომში გამოკვლეულია დრეკად ნარევთა ბრტყელი თეორიის შერეული ამოცანა ხუთი უცნობი ხვრელით შესუსტებული კვადრატისათვის, რომლის ოთხი ხვრელი ტოლი და სიმეტრიულია მოპირდაპირე გვერდების შუა წერტილების შემაერთებელი მონაკვეთების მიმართ. მეხუთე ხვრელი შეიცავს გადაკვეთის წერტილს და სიმეტრიულია ამ მონაკვეთებისა და კოორდინატთა ღერძების მიმართ. კვადრატის წვეროები მდებარეობენ კოორდინატთა ღერძებზე და მათი მიდამოები ამოჭრილია კოორდინატთა ღერძების სიმეტრიული ტოლი სიდიდის გლუვი რკალებით. საზღვრის წრფივ მონაკვეთებზე მოდებულია აბსოლუტურად გლუვი მყარი შტამპები სწორხაზოვანი ფუძეებით, რომლებზეც მოდებულია ძალა $p = (p_1, p_2)^T$. საძიებელი თანაბრადმტკიცე საზღვრის ნაწილები თავისუფალია გარეშე ზემოქმედებისაგან. ანალიზურ ფუნქციათა თეორიის მეთოდების გამოყენებით განისაზღვრება საზღვრის თანაბრადმტკიცე ნაწილები და სხეულის დაძაბული მდგომარეობა.

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The problems of the plane theory of elasticity for infinite domains weakened by equally strong holes have been studied in [1], [9] and by many other authors. The same problem for simply and doubly connected domains with partially unknown boundaries are investigated in [2], [3], [4]. The mixed boundary value problems of the plane theory of elasticity for domains with partially unknown boundaries have been studied by R. Bantsuri [5]. Analogous problems in the case of the plane theory of elastic mixtures can be found in [15].

In [14], using the method suggested by R. Bantsuri in [6], the author gives a solution of the mixed problem of the plane theory of elasticity for a finite multiply connected domain with a partially unknown boundary having the axis of symmetry.

In the present work, in the case of the plane theory of elastic mixtures we study the problem analogous to that solved in [14]. For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvili's formula [15] and the method developed in [6] and [14].

1. Some Auxiliary Formulas and Operators

The homogeneous equation of statics of the theory of elastic mixtures in a complex form looks as follows [8]:

$$\frac{\partial^2 U}{\partial z\,\partial\overline{z}} + K\frac{\partial^2\overline{U}}{\partial\overline{z}^{\,2}} = 0, \tag{1.1}$$

where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, $\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\big)$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\big)$, $U = (u_1 + iu_2, u_3 + iu_4)^T$, $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements.

$$K = -\frac{1}{2}\,\ell m^{-1},\ \ \ell = \begin{bmatrix} \ell_4 & \ell_5\\ \ell_5 & \ell_6 \end{bmatrix},\quad m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3 & -m_2\\ -m_2 & m_1 \end{bmatrix},$$

$$\Delta_0 = m_1m_3 - m_2^2,\ m_k = \ell_k + \frac{1}{2}\,\ell_{3+k},\ k=1,2,3,\ \ell_1 = a_2/d_2,\ \ell_2 = -c/d_2,$$

$$\ell_3 = a_1/d_2,\ \ d_2 = a_1a_2 - c^2,\ \ a_1 = \mu_1 - \lambda_5,\ \ a_2 = \mu_2 - \lambda_5,\ \ c = \mu_3 + \lambda_5,$$

$$\ell_1 + \ell_4 = b/d_1,\quad \ell_2 + \ell_5 = -c_0/d_1,\quad \ell_3 + \ell_6 = a/d_1,\quad a = a_1 + b_1,$$

$$b = a_2 + b_2,\quad c_0 = c + d,\quad d_1 = ab - c_0^2,\quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho,$$

$$b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho,\quad \alpha_2 = \lambda_3 - \lambda_4,\quad \rho = \rho_1 + \rho_2,$$

$$d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho.$$

Here $\mu_1, \mu_2, \mu_3, \lambda_\rho$, $p = \overline{1,5}$ are elasticity modules characterizing mechanical properties of a mixture, ρ_1 and ρ_2 are its particular densities. The elastic constants $\mu_1, \mu_2, \mu_3, \lambda_p$, $p = \overline{1,5}$ and particular densities ρ_1 and ρ_2 will be assumed to satisfy the conditions of the inequality [13].

In [7], M. O. Basheleishvili obtained the following representations:

$$U = (u_1 + iu_2, u_3 + iu_4)^T = m\varphi(z) + \frac{1}{2}\, ez\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{1.2}$$

$$\begin{aligned} TU &= \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \\ &= \frac{\partial}{\partial s(x)}\Big[(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\Big], \end{aligned} \tag{1.3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^T$ and $\psi(z) = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions;

$$A = 2\mu m,\ \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix},\ B = \mu\ell,\ m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix},\ E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\frac{\partial}{\partial s(x)} = -n_2\frac{\partial}{\partial x_1} + n_1\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial n(x)} = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2},\ n = (n_1, n_2)^T$$

are the unit vectors of the outer normal, $(TU)_p$, $p = \overline{1,4}$, the stress components [7]

$$\begin{aligned} (TU)_1 &= r'_{11}n_1 + r'_{21}n_2, \quad (TU)_2 = r'_{12}n_1 + r'_{22}n_2, \\ (TU)_3 &= r''_{11}n_1 + r''_{21}n_2, \quad (TU)_4 = r''_{12}n_1 + r''_{22}n_2. \end{aligned}$$

Consider the following vectors [15]:

$$\begin{aligned} \overset{(1)}{\tau} &= \begin{pmatrix} r'_{11} \\ r''_{11} \end{pmatrix} = \begin{bmatrix} a & c_0 \\ c_0 & b \end{bmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix} - 2\frac{\partial}{\partial x_2}\,\mu\begin{pmatrix} u_2 \\ u_4 \end{pmatrix}, \\ \overset{(2)}{\tau} &= \begin{pmatrix} r'_{22} \\ r''_{22} \end{pmatrix} = \begin{bmatrix} a & c_0 \\ c_0 & b \end{bmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix} - 2\frac{\partial}{\partial x_1}\,\mu\begin{pmatrix} u_1 \\ u_3 \end{pmatrix}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} \overset{(1)}{\eta} &= \begin{pmatrix} r'_{21} \\ r''_{21} \end{pmatrix} = -\begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix} + 2\frac{\partial}{\partial x_1}\,\mu\begin{pmatrix} u_2 \\ u_4 \end{pmatrix}, \\ \overset{(2)}{\eta} &= \begin{pmatrix} r'_{12} \\ r''_{12} \end{pmatrix} = \begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix} + 2\frac{\partial}{\partial x_2}\,\mu\begin{pmatrix} u_1 \\ u_3 \end{pmatrix}, \end{aligned} \tag{1.5}$$

$$\theta' = \operatorname{div} u',\ \ \theta'' = \operatorname{div} u'',\ \ \omega' = \operatorname{rot} u',\ \ \omega'' = \operatorname{rot} u''.$$

Let $(\mathbf{n}, \mathbf{S})$ be the right rectangular system, where $\mathbf{S}$ and $\mathbf{n}$ are, respectively, the tangent and the normal of the curve L at the point $t = t_1 + it_2$. Assume that $n = (n_1, n_2)^T = (\cos\alpha, \sin\alpha)^T$ and $s = (-n_2, n_1)^T = (-\sin\alpha, \cos\alpha)^T$, where α is the angle of inclination of the normal $\mathbf{n}$ to the ox_1-axis.

Introduce the vectors

$$U_n = \begin{pmatrix} u_1n_1 + u_2n_2 \\ u_3n_1 + u_4n_2 \end{pmatrix}, \qquad U_s = \begin{pmatrix} u_2n_1 - u_1n_2 \\ u_4n_1 - u_3n_2 \end{pmatrix}, \tag{1.6}$$

$$\sigma_n = \begin{pmatrix} (Tu)_1n_1 + (Tu)_2n_2 \\ (Tu)_3n_1 + (Tu)_4n_2 \end{pmatrix},\ \ \sigma_s = \begin{pmatrix} (Tu)_2n_1 - (Tu)_1n_2 \\ (Tu)_4n_1 - (Tu)_3n_2 \end{pmatrix}, \tag{1.7}$$

$$\sigma_t = \begin{pmatrix} \left[r'_{21}n_1 - r'_{11}n_2, r'_{22}n_1 - r'_{12}n_2\right]^T s^0 \\ \left[r''_{21}n_1 - r''_{11}n_2, r''_{22}n_1 - r''_{12}n_2\right]^T s^0 \end{pmatrix}. \tag{1.8}$$

After elementary calculations we obtain

$$\sigma_n = \overset{(1)}{\tau}\cos^2\alpha + \overset{(2)}{\tau}\sin^2\alpha + \eta\sin\alpha\cos\alpha,$$

$$\sigma_t = \overset{(1)}{\tau}\sin^2\alpha + \overset{(2)}{\tau}\cos^2\alpha - \eta\sin\alpha\cos\alpha,$$

$$\sigma_s = \frac{1}{2}\big(\overset{(2)}{\tau} - \overset{(1)}{\tau}\big)\sin 2\alpha + \frac{1}{2}\eta\cos 2\alpha - \frac{1}{2}\varepsilon^*,$$

where $\eta = \overset{(1)}{\eta} + \overset{(2)}{\eta}$, $\varepsilon^* = \overset{(1)}{\eta} - \overset{(2)}{\eta}$.

Direct calculations allow us to check that on L [15]

$$\sigma_n + \sigma_t = \overset{(1)}{\tau} + \overset{(2)}{\tau} = 2(2E - A - B)\operatorname{Re}\varphi'(t), \tag{1.9}$$

$$\sigma_n - i\sigma_s = (2E - A)\overline{\varphi'(t)} - B\varphi'(t) + e^{2i\alpha}\big[B\bar{t}\varphi''(t) + 2\mu\psi'(t)\big], \tag{1.10}$$

$$\sigma_n + 2\mu\bigg(\frac{\partial U_s}{\partial s} + \frac{Un}{\rho_0}\bigg) + i\bigg[\sigma_s - 2\mu\bigg(\frac{\partial U_n}{\partial s} - \frac{U_s}{\rho_0}\bigg)\bigg] = 2\varphi'(t), \tag{1.11}$$

$$\big[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big]_L = -\int_L e^{i\alpha}(\sigma_n + i\sigma_s)ds, \tag{1.12}$$

where $\det(2E - A - B) > 0$, $\frac{1}{\rho_0}$ is the curvature of L at the point t. Everywhere in the sequel it will be assumed that the components U_n and U_s are bounded [8].

Formulas (1.2),(1.3) and (1.9)–(1.11) are analogous to those of Kolosov-Muskhelishvili in the linear theory of elastic mixtures [12].

2. Statement of the Problem and the Method of its Solving

Let an isotropic elastic mixture occupy on the plane $z = x_1 + ix_2$ a multiply connected domain D, a square weakened by five equally strong holes with unknown boundaries.

Four of the holes are equal and symmetric with respect to the segments connecting midpoints of the opposite sides, while the fifth one is symmetric with respect to these segments and to the coordinate axes.

The vertices of the square lie on the coordinate axes and their neighborhoods are cut out by equal smooth arcs, symmetric with respect to the coordinate axes.

The side length of the square we denote by $2a_0$. The linear portion of the boundary is under the action of absolutely smooth rigid punches having rectilinear bases which are acted on by forces of magnitude $p = (p_1, p_2)^T$. An unknown part of the boundary is free from external forces.

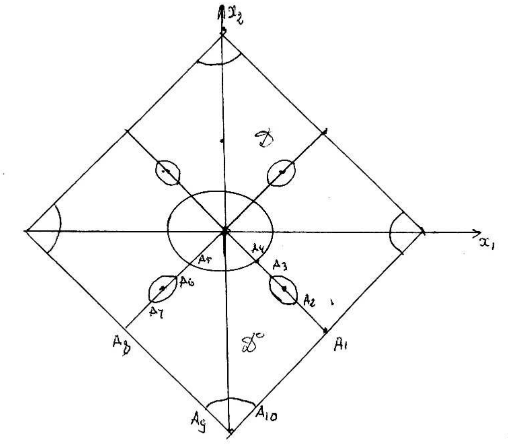

Figure 1.

Assume that the vector U_n (see $(16)_1$) takes on every segment a constant value $U_n = U^0 = \text{const}$, and σ_s, i.e., the vector (1.7), is equal to zero along the whole boundary of the domain D.

We formulate the following problem: Find a stressed state of the body and an unknown part of the boundary of D under the condition that the vector (18), i.e. σ_t, is constant, i.e., $\sigma_t = K^0$, $K^0 = (K_1^0, K_2^0)^T = \text{const}$.

Since the above-posed problem is axially symmetric, on the segments $[A_1, A_2]$, $[A_3, A_4]$, $[A_5, A_6]$, $[A_7, A_8]$, $U_n = \sigma_s = 0$.

To investigate the problem, we consider a curvilinear polygon $A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}$ and denote it by D^0.

Introduce the notation $\Gamma_j = [A_{2j-1}, A_{2j}]$, $j = 1, 4$, $\Gamma_5 = [A_8, A_9]$, $\Gamma_6 = [A_{10}, A_1]$, $\Gamma = \bigcup\limits_{j=1}^{6} \Gamma_j$, $\gamma_1 = \overset{\smile}{A_2A_3}$, $\gamma_2 = \overset{\smile}{A_4A_5}$, $\gamma_3 = \overset{\smile}{A_6A_7}$, $\gamma_4 = \overset{\smile}{A_9A_{10}}$, $\gamma = \bigcup\limits_{j=1}^{4} \gamma_j$.

By q^0 and q we denote

$$\int\limits_{\Gamma_1} \sigma_n ds = -\frac{1}{2}\, q_0 \quad \text{and} \quad \int\limits_{\Gamma_2} \sigma_n ds = -\frac{1}{2}\, q.$$

Since $\Gamma_1 \square \Gamma_5 \square \Gamma_2$ and $\int\limits_{\Gamma_5} \sigma_n ds = -\frac{1}{2}\, p$, because of equilibrium of the body D^0, we can write

$$\int\limits_{\Gamma_1} \sigma_n ds + \int\limits_{\Gamma_2} \sigma_n ds = \int\limits_{\Gamma_5} \sigma_n ds = -\frac{1}{2}\, p, \quad \text{i.e.,} \quad q_0 + q = p.$$

Owing to the symmetry of the domain D^0, we have

$$\int\limits_{\Gamma_2} \sigma_n ds = \int\limits_{\Gamma_3} \sigma_n ds = -\frac{1}{2}\, q \quad \int\limits_{\Gamma_1} \sigma_n ds = \int\limits_{\Gamma_4} \sigma_n ds = -\frac{1}{2}\, q_0.$$

The boundary conditions of the problem are of the form

$$U_n(t) = \begin{cases} 0, & t \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ U^0, & t \in \Gamma_5 \cup \Gamma_6, \end{cases} \tag{2.1}$$

$$\sigma_s = 0, \quad t \in \Gamma \cup \gamma, \tag{2.2}$$

$$\sigma_t = K^0, \quad t \in \gamma, \tag{2.3}$$

$$\int\limits_{\Gamma_5} \sigma_n ds = \int\limits_{\Gamma_6} \sigma_n ds = -\frac{1}{2}\, p. \tag{2.4}$$

To simplify our writing, we denote the geometrical point A_k and its affix by the same symbol.

Relying on the analogous Kolosov-Muskhelishvili's formulas (1.9), (1.1) and (1.12), the above-posed problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in D^0 by the boundary conditions

$$\operatorname{Im} \varphi'(t) = 0, \quad t \in \Gamma, \tag{2.5}$$

$$\operatorname{Re} \varphi'(t) = H, \quad t \in \gamma, \quad H = \frac{1}{2}(2E - A - B)^{-1} K^0, \tag{2.6}$$

$$\operatorname{Re} \left[e^{-\alpha(t)} (A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} \right] = c(t), \quad t \in \Gamma, \tag{2.7}$$

$$(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} = B^0(t), \quad t \in \gamma, \tag{2.8}$$

where $\alpha(t)$ is the angle made by the outer normal n and the ox_1-axis. The arc coordinate of the point t counting from the point A_1 we denote by S.

$$\begin{gathered} \alpha(t) = \alpha_k, \quad t \in \Gamma_k, \quad k = \overline{1,6}, \quad \alpha_1 = \alpha_2 = \frac{\pi}{4}\,, \\ \alpha_3 = \alpha_4 = \frac{3}{4}\,\pi, \quad \alpha_5 = \frac{5}{4}\,\pi, \quad \alpha_6 = \frac{7}{4}\,\pi, \end{gathered} \tag{2.9}$$

$$\begin{gathered} C(t) = \operatorname{Re}\left(e^{-i\alpha(t)} B^0(t)\right), \\ B^0(t) = -i \int\limits_{A_1}^{t} \sigma_n(t_0) e^{i\alpha(t_0)} ds_0 - \frac{1}{2}\, p e^{\frac{\pi}{4} i}. \end{gathered} \tag{2.10}$$

Taking into account (2.9) and (2.10), we obtain

$$B^0(t)=\begin{cases}-\dfrac{1}{2}\,iqe^{\frac{\pi}{4}\,i}, & t\in\gamma_1,\\ 0, & t\in\gamma_2,\end{cases}$$
$$B^0(t)=\begin{cases}-\dfrac{1}{2}\,qe^{\frac{\pi}{4}\,i}, & t\in\gamma_3,\\ -\dfrac{1}{2}\,(1+i)pe^{\frac{\pi}{4}\,i}, & t\in\gamma_4,\end{cases}\tag{2.11}$$

$$C(t)=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4,\\ \dfrac{1}{2}\,p, & t\in\Gamma_5\cup\Gamma_6.\end{cases}\tag{2.12}$$

Moreover, if $t\in\Gamma$, then we can write

$$\operatorname{Re}e^{-i\alpha(t)}t=\operatorname{Re}e^{-i\alpha(t)}A(t),\tag{2.13}$$

where $A(t)=A_k$ for $t\in\Gamma_k$, $k=\overline{1,6}$.

Assume finally, that the vector-functions $\varphi'(t)$ and $\psi(z)$ are continuously extendable on the boundary of D^0, except possibly the points $A_2,A_3,A_4,A_5,A_6,A_7,A_9,A_{10}$ in the neighborhood of which they admit the estimate of the type

$$|\varphi_j'(z)|,|\psi_j(z)|<M|z-A_k|^{-\delta_k},\quad j=1,2,\tag{2.14}$$

where $0\le\delta_k<\frac{1}{2}$, $k=2,3,4,5,6,7,9,10$, $M=\mathrm{const}>0$.

The equalities (2.5)–(2.6) are in fact the Keldysh-Sedov problem for the domain D^0,

$$\operatorname{Re}\varphi'(t)=H,\quad t\in\gamma,\quad \operatorname{Im}\varphi'(t)=0,\quad t\in\Gamma.\tag{2.15}$$

By virtue of the condition (2.14), the problem (2.15) has a unique solution $\varphi'(z)=H$ [10]. Consequently, leaving out of account the constant summand, we get

$$\varphi(z)=Hz=\frac{1}{2}\,(2E-A-B)^{-1}K^0z.\tag{2.16}$$

Here K^0 is to be defined in the course of solving the problem.

Substituting the values $B^0(t),C(t),\varphi'(t)$ defined by formulas (2.11), (2.12) and (2.13) into the boundary conditions (2.7) and (2.8), we obtain

$$\operatorname{Re}e^{-i\alpha(t)}\Big(\frac{1}{2}\,K^0t+2\mu\overline{\psi(t)}\Big)=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4,\\ \dfrac{1}{2}\,p, & t\in\Gamma_5\cup\Gamma_6,\end{cases}\tag{2.17}$$

$$e^{-\frac{\pi}{4}i}\Big(\frac{1}{2}K^0t+2\mu\overline{\psi(t)}\Big)=B^0(t)=\begin{cases}0, & t\in\gamma_2,\\ -\dfrac{1}{2}iq, & t\in\gamma_1,\\ -\dfrac{1}{2}q, & t\in\gamma_3,\\ -\dfrac{1}{2}(1+i)p, & t\in\gamma_4,\end{cases}\tag{2.18}$$

$$\mathrm{Re}\,\big(te^{-i\alpha(t)}\big)=\begin{cases}0, & t\in\Gamma_j,\ \ j=\overline{1,4},\\ -a_0, & t\in\Gamma_5\cup\Gamma_6.\end{cases}\tag{2.19}$$

On the basis of formula (2.9), it follows from the conditions (2.17)–(2.19) that

$$\mathrm{Re}\Big[\frac{1}{2}K^0te^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(t)\Big]=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2\cup\gamma_1\cup\gamma_2,\\ -\dfrac{1}{2}p, & t\in\Gamma_5\cup\gamma_4,\\ -\dfrac{1}{2}q, & t\in\gamma_3,\end{cases}\tag{2.20}$$

$$\mathrm{Im}\Big[\frac{1}{2}K^0te^{-\frac{\pi}{4}i}-2\mu e^{\frac{\pi}{4}i}\psi(t)\Big]=\begin{cases}0, & t\in\Gamma_3\cup\Gamma_4\cup\gamma_2\cup\gamma_3,\\ -\dfrac{1}{2}p, & t\in\Gamma_6\cup\gamma_4,\\ -\dfrac{1}{2}q, & t\in\gamma_1,\end{cases}\tag{2.21}$$

$$\mathrm{Re}\big[te^{-\frac{\pi}{4}i}\big]=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2,\\ -a_0, & t\in\Gamma_5,\end{cases}\tag{2.22}$$

$$\mathrm{Im}\big[te^{-\frac{\pi}{4}i}\big]=\begin{cases}0, & t\in\Gamma_3\cup\Gamma_4,\\ -a_0, & t\in\Gamma_6.\end{cases}\tag{2.23}$$

Multiplying equalities (2.23) and (2.21) by, respectively, K^0 and -1 and then summing up, we find that

$$\mathrm{Im}\Big[\frac{1}{2}K^0te^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(t)\Big]=\begin{cases}0, & t\in\Gamma_3\cup\Gamma_4,\\ \dfrac{1}{2}p-a_0K^0, & t\in\Gamma_6,\end{cases}\tag{2.24}$$

Analogously, equalities (2.20) and (2.22) result in

$$\mathrm{Re}\Big[\frac{1}{2}K^0te^{-\frac{\pi}{4}i}-2\mu e^{\frac{\pi}{4}i}\psi(t)\Big]=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2,\\ \dfrac{1}{2}p-a_0K^0, & t\in\Gamma_5,\end{cases}\tag{2.25}$$

Let the function $z=\omega(\zeta)$, $\zeta=\xi_1+i\xi_2$ map conformally the domain D^0 onto the upper plane $\mathrm{Im}\,\zeta>0$. By β_j we denote the image of the point A_j, $j=\overline{1,10}$. Assume that $\beta_9=-1$, $\beta_{10}=1$ and, moreover, that the midpoint of the arc $\gamma_2=\overset{\frown}{A_4A_5}$ turns into $\zeta=\infty$.

Because of the fact that the domain D^0 is symmetric with respect to the axis ox_2, we can suppose that $\beta_8=-\beta_1$, $\beta_7=-\beta_2$, $\beta_6=-\beta_3$, $\beta_5=-\beta_4$,.

Consider the vector-functions

$$\Phi(\zeta)=\frac{1}{2}\,K^0\omega(\zeta)e^{-\frac{\pi}{4}\,i}+2\mu e^{\frac{\pi}{4}\,i}\psi(\omega(\zeta)), \tag{2.26}$$

$$\Psi(\zeta)=\frac{1}{2}\,K^0\omega(\zeta)e^{-\frac{\pi}{4}\,i}-2\mu e^{\frac{\pi}{4}\,i}\psi(\omega(\zeta)). \tag{2.27}$$

If we take into account (2.26) and (2.27), then the boundary conditions (2.20), (2.21), (2.24) and (2.25) can be rewritten in the form

$$\Phi(\xi_1)+\overline{\Phi(\xi_1)}=\begin{cases}0, & \xi_1\in(-\infty,-\beta_4)\cup(\beta_1,\infty),\\ -p, & \xi_1\in(-\beta_1,1),\\ -q, & \xi_1\in(-\beta_3,-\beta_2),\end{cases} \tag{2.28}$$

$$\Phi(\xi_1)-\overline{\Phi(\xi_1)}=\begin{cases}0, & \xi_1\in(-\beta_4,-\beta_3)\cup(-\beta_2,-\beta_1),\\ (p-2a_0K^0)i, & \xi_1\in(1,\beta_1),\end{cases} \tag{2.29}$$

$$\Psi(\xi_1)+\overline{\Psi(\xi_1)}=\begin{cases}0, & \xi_1\in(\beta_1,\beta_2)\cup(\beta_3,\beta_4),\\ p-2a_0K^0, & \xi_1\in(-\beta_1,-1),\end{cases} \tag{2.30}$$

$$\Psi(\xi_1)-\overline{\Psi(\xi_1)}=\begin{cases}0, & \xi_1\in(-\infty,-\beta_1)\cup(\beta_4,\infty),\\ -ip, & \xi_1\in(-1,\beta_1),\\ -iq, & \xi_1\in(\beta_2,\beta_3).\end{cases} \tag{2.31}$$

The above problems are the vector forms of the Keldysh-Sedov problems.

A solution of the problems (2.28)-(2.29) and (2.30)-(2.31) can be represented as follows [10]:

$$\begin{aligned}\Phi(\zeta)=&-\frac{\chi_1(\zeta)}{2\pi i}\Bigg[\int\limits_{-\beta_3}^{-\beta_2}\frac{qd\xi_1}{\chi_1(\xi_1)(\xi_1-\zeta)}+\int\limits_{-\beta_1}^{1}\frac{pd\xi_1}{\chi_1(\xi_1)(\xi_1-\zeta)}-\\ &-\int\limits_{1}^{\beta_1}\frac{(p-2a_0K^0)id\xi_1}{\chi_1(\xi_1)(\xi_1-\zeta)}+C\Bigg],\ \ \operatorname{Im}\zeta>0.\end{aligned} \tag{2.32}$$

$$\begin{aligned}\Psi(\zeta)=&\frac{\chi_2(\zeta)}{2\pi i}\Bigg[\int\limits_{-\beta_1}^{1}\frac{(p-2a_0K^0)id\xi_1}{\chi_2(\xi_1)(\xi_1-\zeta)}-\int\limits_{-1}^{\beta_1}\frac{pid\xi_1}{\chi_2(\xi_1)(\xi_1-\zeta)}-\\ &-\int\limits_{\beta_2}^{\beta_1}\frac{ip_2d\xi_1}{\chi_2(\xi_1)(\xi_1-\zeta)}+Ci\Bigg],\ \ \operatorname{Im}\zeta>0,\end{aligned} \tag{2.33}$$

where

$$\chi_1(\zeta)=\sqrt{\frac{(\zeta-1)(\zeta+\beta_2)(\zeta+\beta_4)}{(\zeta-\beta_1)(\zeta+\beta_1)(\zeta+\beta_3)}},\ \ \operatorname{Im}\zeta>0, \tag{2.34}$$

$$\chi_2(\zeta)=\sqrt{\frac{(\zeta+1)(\zeta-\beta_2)(\zeta-\beta_4)}{(\zeta+\beta_1)(\zeta-\beta_1)(\zeta-\beta_3)}},\ \ \operatorname{Im}\zeta>0, \tag{2.35}$$

By $\chi_1(\zeta)$ and $\chi_2(\zeta)$ is meant a branch of the function which turns into unity as $\zeta\to\infty$ ($\operatorname{Im}\zeta>0$)

$$\chi_1(\infty)=\chi_2(\infty)=1.$$

It is not difficult to state that

$$\chi_1(\xi_1)=\begin{cases}|\chi_1(\xi_1)|, & \xi_1\in(-\infty,-\beta_4)\cup(-\beta_3,-\beta_2)\cup\\ & \qquad\cup(-\beta_1,1)\cup(\beta_1,\infty),\\ -i|\chi_1(\xi_1)|, & \xi_1\in(-\beta_4,-\beta_3)\cup(-\beta_2,-\beta_1)\cup(1,\beta_1),\end{cases} \tag{2.36}$$

$$\chi_2(\xi_1)=\begin{cases}|\chi_2(\xi_1)|, & \xi_1\in(-\infty,-\beta_1)\cup(-1,\beta_1)\cup\\ & \qquad\cup(\beta_2,\beta_3)\cup(\beta_4,\infty),\\ i|\chi_2(\xi_1)|, & \xi_1\in(-\beta_1,-1)\cup(\beta_1,\beta_2)\cup(\beta_3,\beta_4),\end{cases} \tag{2.37}$$

$$|\chi_1(\xi_1)|=|\chi_2(-\xi_1)|. \tag{2.38}$$

By virtue of (2.36) and (2.37), formulas (2.32) and (2.33) can be written as

$$\begin{aligned}\Phi(\zeta)=&-\frac{\chi_1(\zeta)}{2\pi i}\Bigg[\int\limits_{-\beta_3}^{-\beta_2}\frac{q\,d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\zeta)}+\int\limits_{-\beta_1}^{\beta_1}\frac{p\,d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\zeta)}-\\ &-2a_0K^0\int\limits_{1}^{\beta_1}\frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\zeta)}+C\Bigg],\ \ \operatorname{Im}\zeta>0,\end{aligned} \tag{2.39}$$

$$\begin{aligned}\Psi(\zeta)=&-\frac{\chi_2(\zeta)}{2\pi}\Bigg[\int\limits_{-\beta_1}^{\beta_1}\frac{p\,d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\zeta)}+\int\limits_{\beta_2}^{\beta_3}\frac{p\,d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\zeta)}-\\ &-2a_0K^0\int\limits_{-\beta_1}^{-1}\frac{d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\zeta)}+C\Bigg],\ \ \operatorname{Im}\zeta>0.\end{aligned} \tag{2.40}$$

3. Investigation of a Solution of the Problem and Construction of Charts for a Partially Unknown Boundary

Since the functions $\chi_1(\xi_1)$ and $\chi_2(\xi_1)$ (see (2.34) and (2.35)) at the points $\xi_1=\pm\beta_1$ and $\xi_1=\pm\beta_3$ have singularities of order 1/2, therefore for the vector-functions $\Phi(\zeta)$ and $\Psi(\zeta)$ to be bounded in the neighborhood of the points $\pm\beta_1$ and $\pm\beta_3$, it is necessary and sufficient that the conditions

$$\begin{aligned}
&\int_{-\beta_3}^{-\beta_2} \frac{q d\xi_1}{|\chi_1(\xi_1)|(\xi_1+\beta_1)} + \int_{-\beta_1}^{\beta_1} \frac{p d\xi_1}{|\chi_1(\xi_1)|(\xi_1+\beta_1)} - \\
&\qquad - \int_{1}^{\beta_1} \frac{2a_0 K^0 d\xi_1}{|\chi_1(\xi_1)|(\xi_1+\beta_1)} + C = 0, \\
&\int_{-\beta_3}^{-\beta_2} \frac{q d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\beta_1)} + \int_{-\beta_1}^{\beta_1} \frac{p d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\beta_1)} - \\
&\qquad - \int_{1}^{\beta_1} \frac{2a_0 K^0 d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\beta_1)} + C = 0, \\
&\int_{-\beta_3}^{-\beta_2} \frac{q d\xi_1}{|\chi_1(\xi_1)|(\xi_1+\beta_3)} + \int_{-\beta_1}^{\beta_1} \frac{p d\xi_1}{|\chi_1(\xi_1)|(\xi_1+\beta_3)} - \\
&\qquad - \int_{1}^{\beta_1} \frac{2a_0 K^0 d\xi_1}{|\chi_1(\xi_1)|(\xi_1+\beta_3)} + C = 0,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&\int_{-\beta_1}^{\beta_1} \frac{p d\xi_1}{|\chi_2(\xi_1)|(\xi_1\mp\beta_1)} + \int_{\beta_2}^{\beta_3} \frac{q d\xi_1}{|\chi_2(\xi_1)|(\xi_1\mp\beta_1)} - \\
&\qquad - 2a_0 K^0 \int_{-\beta_1}^{-1} \frac{d\xi_1}{|\chi_2(\xi_1)|(\xi_1\mp\beta_1)} - C = 0, \\
&\int_{-\beta_1}^{\beta_1} \frac{p d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\beta_3)} + \int_{\beta_2}^{\beta_3} \frac{q d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\beta_3)} - \\
&\qquad - 2a_0 K^0 \int_{1}^{\beta_1} \frac{d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\beta_1)} - C = 0.
\end{aligned} \tag{3.2}$$

are fulfilled.

If in (3.2) we replace ξ_1 by $-\xi_1$ and take into account (2.38), then we will get the condition coinciding with (3.1).

Note that the system of the above three equations involves $\beta_1, \beta_2, \beta_3$, $C=(c_1,c_2)^T$, $K^0=(k_1^0,k_2^0)^T$, $q=(q_1,q_2)^T$ as unknown parameters.

Having fixed β_1, β_2 and β_3 and solving the system with respect to K^0, C and q, we find a solution of the unknown problem, obtain equations for unknown parts of the boundary and hence the value $\psi(\omega(\zeta))$.

It follows from (2.26) and (2.27) that

$$K^0\omega(\zeta)=\big[\Phi(\zeta)+\Psi(\zeta)\big]e^{\frac{\pi}{4}i}, \quad \operatorname{Im}\zeta\geq 0, \tag{3.3}$$

$$\psi(\omega(\zeta))=\frac{1}{4}\mu^{-1}\big(\Phi(\zeta)-\Psi(\zeta)\big)e^{-\frac{\pi}{4}i}, \quad \operatorname{Im}\zeta\geq 0. \tag{3.4}$$

Passing in (2.32) and (2.33) to the limit as $\zeta\to\xi_1^0\in G$, where $G=(-\beta_3,-\beta_2)\cup(-1,1)\cup(\beta_2,\beta_3)\cup(-\infty,-\beta_4)\cup(\beta_4,\infty)$, by the Sokhotskii-Plemelj formulas [11] we obtain

$$\begin{aligned}
&\Phi(\xi_1^0)=i\Phi^0(\xi_1^0)-\frac{1}{2}\,p,\ \ \Psi(\xi_1^0)=\Psi^0(\xi_1^0)-\frac{1}{2}\,pi,\ \ \xi_1^0\in(-1,1),\\
&\Phi(\xi_1^0)=i\Phi^0(\xi_1^0),\ \ \Psi(\xi_1^0)=\Psi^0(\xi_1^0)-\frac{1}{2}\,qi,\ \ \xi_1^0\in(\beta_2,\beta_3),\\
&\Phi(\xi_1^0)=i\Phi^0(\xi_1^0)-\frac{1}{2}\,q,\ \ \Psi(\xi_1^0)=\Psi^0(\xi_1^0),\ \ \xi_1^0\in(-\beta_3,-\beta_2),\\
&\Phi(\xi_1^0)=i\Phi^0(\xi_1^0),\ \ \Psi(\xi_1^0)=\Psi^0(\xi_1^0),\ \ \xi_1^0\in(-\infty,-\beta_4)\cup(\beta_4,\infty),
\end{aligned} \tag{3.5}$$

where $\Phi^0=(\Phi_1^0,\Phi_2^0)^T$, $\Psi^0=(\Psi_1^0,\Psi_2^0)^T$,

$$\begin{aligned}
\Phi^0(\xi_1^0)=\frac{\chi_1(\xi_1^0)}{2\pi}\Bigg[\int\limits_{-\beta_3}^{-\beta_2}&\frac{q\,d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\xi_1^0)}+\int\limits_{-\beta_1}^{\beta_1}\frac{p\,d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\xi_1^0)}-\\
&-2a_0K^0\int\limits_{1}^{\beta_1}\frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\xi_1^0)}+C\Bigg],\ \ \xi_1^0\in G,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\Psi^0(\xi_1^0)=-\frac{\chi_2(\xi_1^0)}{2\pi}\Bigg[\int\limits_{-\beta_1}^{\beta_1}&\frac{p\,d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\xi_1^0)}+\int\limits_{\beta_2}^{\beta_3}\frac{q\,d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\xi_1^0)}-\\
&-2a_0K^0\int\limits_{1}^{\beta_1}\frac{d\xi_1}{|\chi_2(\xi_1)|(\xi_1-\xi_1^0)}-C\Bigg],\ \ \xi_1^0\in G.
\end{aligned} \tag{3.7}$$

The both integrals

$$\int\limits_{-1}^{1}\frac{p\,d\xi_1}{|\chi_j(\xi_1)|(\xi_1-\xi_1^0)}\ \text{ and }\ \int\limits_{-\beta_3}^{-\beta_2}\frac{q\,d\xi_1}{|\chi_j(\xi_1)|(\xi_1-\xi_1^0)},\ \ j=1,2,$$

exist in the sense of the Cauchy principal value.

In (3.6), we replace ξ_1 and ξ_1^0, respectively, by $-\xi_1$ and $-\xi_1^0$ and hence by virtue of (2.36)-(2.38) we get

$$\Psi^0(\xi_1^0) = \Phi_0(-\xi_1^0), \quad \xi_1^0 \in G. \tag{3.8}$$

Equality (3.3) allows us to find equations for the arc γ_j, $j = \overline{1,4}$:

$$t = \omega(\xi_1^0) = \frac{\Phi_1(\xi_1^0) + \Psi_1(\xi_1^0)}{k_1^0}\, e^{\frac{\pi}{4}\, i} = \frac{\Phi_2(\xi_1^0) + \Psi_2(\xi_1^0)}{k_2^0}\, e^{\frac{\pi}{4}\, i},\ \xi_1^0 \in G. \tag{3.9}$$

Substituting in (3.9) the values γ_2 and γ_4 defined by formulas (3.5) and taking into account equalities (3.8), we find that the equation for the arc ox_2 is given by the formula

$$\begin{aligned} t = \omega(\xi_1^0) &= \frac{\Phi_1^0(-\xi_1^0) - \Phi_1^0(\xi_1^0) + 0,5q_1}{\sqrt{2}k_1^0} + i\frac{\Phi_1^0(\xi_1^0) + \Phi_1^0(-\xi_1^0) - 0,5q_1}{\sqrt{2}k_1^0} = \\ &= \frac{\Phi_2^0(-\xi_1^0) - \Phi_2^0(\xi_1^0) + 0,5q_2}{\sqrt{2}k_2^0} + i\frac{\Phi_2^0(\xi_1^0) + \Phi_2^0(-\xi_1^0) - 0,5q_2}{\sqrt{2}k_2^0}, \\ &\qquad \xi_1^0 \in (\beta_2, \beta_3), \quad t \in \gamma_1, \end{aligned} \tag{3.10}$$

$$\begin{aligned} t = \omega(\xi_1^0) &= \frac{\Phi_1^0(-\xi_1^0) - \Phi_1^0(\xi_1^0)}{\sqrt{2}k_1^0} + i\frac{\Phi_1^0(\xi_1^0) + \Phi_1^0(-\xi_1^0)}{\sqrt{2}k_1^0} = \\ &= \frac{\Phi_2^0(-\xi_1^0) - \Phi_2^0(\xi_1^0)}{\sqrt{2}k_2^0} + i\frac{\Phi_2^0(\xi_1^0) + \Phi_2^0(-\xi_1^0)}{\sqrt{2}k_2^0}, \\ &\qquad \xi_1^0 \in (-\infty, -\beta_4) \cup (\beta_4, \infty), \quad t \in \gamma_2, \end{aligned} \tag{3.11}$$

$$\begin{aligned} t = \omega(\xi_1^0) &= \frac{\Phi_1^0(-\xi_1^0) - \Phi_1^0(\xi_1^0) - 0,5q_1}{\sqrt{2}k_1^0} + i\frac{\Phi_1(\xi_1^0) + \Phi_1(-\xi_1^0) - 0,5q_1}{\sqrt{2}k_1^0} = \\ &= \frac{\Phi_2^0(-\xi_1^0) - \Phi_2^0(\xi_1^0) - 0,5q_2}{\sqrt{2}k_2^0} + i\frac{\Phi_2(\xi_1^0) + \Phi_2(-\xi_1^0) - 0,5q_2}{\sqrt{2}k_2^0}, \\ &\qquad \xi_1^0 \in (-\beta_3, -\beta_2), \quad t \in \gamma_3, \end{aligned} \tag{3.12}$$

$$\begin{aligned} t = \omega(\xi_1^0) &= \frac{\Phi_1^0(-\xi_1^0) - \Phi_1^0(\xi_1^0)}{\sqrt{2}k_1^0} + i\frac{\Phi_1(\xi_1^0) + \Phi_1(-\xi_1^0)}{\sqrt{2}k_1^0} = \\ &= \frac{\Phi_2^0(-\xi_1^0) - \Phi_2^0(\xi_1^0)}{\sqrt{2}k_2^0} + i\frac{\Phi_2(\xi_1^0) + \Phi_2(-\xi_1^0) - p}{\sqrt{2}k_2^0}, \\ &\qquad \xi_1^0 \in (-1, 1), \quad t \in \gamma_4, \end{aligned} \tag{3.13}$$

Clearly, the arcs γ_2 and γ_4 are symmetric with respect to the ox_2-axis.

Let us consider a particular case when the square is weakened by only one central hole, i.e., $\beta_2 = \beta_3$.

In this case we have (see (2.39) and (2.40))

$$\Phi(\zeta)=-\frac{\chi_1^0(\zeta)}{2\pi i}\Bigg[\int\limits_{-\beta_1}^{\beta_1}\frac{p d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1-\zeta)}-$$

$$-2a_0K^0\int\limits_{1}^{\beta_1}\frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1-\zeta)}+C\Bigg],\quad \operatorname{Im}\zeta>0, \tag{3.14}$$

$$\Psi(\zeta)=-\frac{\chi_2^0(\zeta)}{2\pi}\Bigg[\int\limits_{-\beta_1}^{\beta_1}\frac{p d\xi_1}{|\chi_2^0(\xi_1)|(\xi_1-\zeta)}-$$

$$-2a_0K^0\int\limits_{-\beta_1}^{-1}\frac{d\xi_1}{|\chi_2^0(\xi_1)|(\xi_1-\zeta)}-C\Bigg],\quad \operatorname{Im}\zeta>0, \tag{3.15}$$

where

$$\chi_1^0(\zeta)=\sqrt{\frac{(\zeta-1)(\zeta+\beta_4)}{\zeta^2-\beta_1^2}},\quad \chi_2^0(\zeta)=\sqrt{\frac{(\zeta+1)(\zeta-\beta_4)}{\zeta^2-\beta_1^2}},\quad \operatorname{Im}\zeta>0.$$

Finally, we notice that if the square is weakened by only one central hole, then a solution of the problem is represented by means of formulas (3.14) and (3.15) (see (3.3) and (3.4)).

Since the functions $\chi_1(\zeta)$ and $\chi_2(\zeta)$ at the points $\mp\beta_1$ have singularities of order $1/2$, therefore for the vector-functions $\Phi(\zeta)$ and $\Psi(\zeta)$ to be bounded in the neighborhood of the points β_1 and $-\beta_2$, it is necessary and sufficient that the conditions

$$p\int\limits_{-\beta_1}^{\beta_1}\frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1-\beta_1)}-2a_0K^0\int\limits_{1}^{\beta_1}\frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1-\beta_1)}+C=0,$$

$$p\int\limits_{-\beta_1}^{\beta_1}\frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1+\beta_1)}-2a_0K^0\int\limits_{1}^{\beta_1}\frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1+\beta_1)}+C=0.$$

are fulfilled.

Note that the system of the above two equations involves β_1,β_4,C and K^0 as unknown parameters.

Solving these systems with respect to K^0 and C, we obtain

$$K^0 = \frac{p}{2a_0} \frac{\int\limits_{-\beta_1}^{\beta_1} \frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1^2-\beta_1^2)}}{\int\limits_{1}^{\beta_1} \frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1^2-\beta_1^2)}}, \quad K_j^0 p_j > 0, \quad j=1,2,$$

$$C = 2a_0 K^0 \int\limits_{1}^{\beta_1} \frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1+\beta_1)} - p \int\limits_{-\beta_1}^{\beta_1} \frac{d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1+\beta_1)}.$$

In this case to draw the charts for the arcs γ_2 and γ_4, we use formulas (3.11) and (3.13), where

$$\Phi_j(\xi_1^0) = \frac{\chi_1^0(\xi_1^0)}{2\pi} \left[\int\limits_{-\beta_1}^{\beta_1} \frac{p_j d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1-\xi_1^0)} - \int\limits_{1}^{\beta_1} \frac{2a_0 K_j^0 d\xi_1}{|\chi_1^0(\xi_1)|(\xi_1-\xi_1^0)} + C_j \right], \ j=1,2.$$

To construct the diagrams, we have the Mathcad method used.

Finally, to construct the charts for the rest parts of the arcs γ_2 and γ_4, we have, owing to the cyclic symmetry of the problem, to turn the chart for the function $t \in \omega(\xi_1^0)$ by the angle $\frac{\pi}{2}$.

In Figures 2–6 we can see the charts of the arcs γ_2 and γ_4 for the given values a_0 and p and for different values β_1 and β_4. As is seen, γ_2 and γ_4 are of the same form and, moreover, the size of the central hole, i.e., the value of the contour of γ_2, decreases as the length of the segment $[-\beta_1, \beta_1]$ increases.

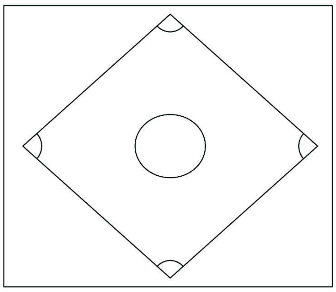

Figure 2.

$p_1 = -10$, $a_0 = 1$, $\beta_1 = 9$, $\beta_4 = 24$,
$K_1^0 = -11,289$, $C_1 = 16,802$.

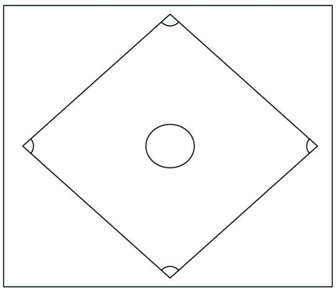

Figure 3.

$p_1 = -10$, $a_0 = 1$, $\beta_1 = 19$, $\beta_4 = 104$,
$K_1^0 = -10,499$, $C_1 = 10,949$.

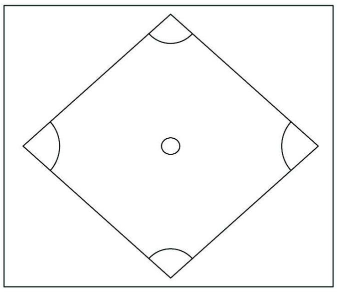

Figure 4.

$p_1 = -10$, $a_0 = 1$, $\beta_1 = 3$, $\beta_4 = 124$,
$K_1^0 = -10,914$, $C_1 = 4,007$.

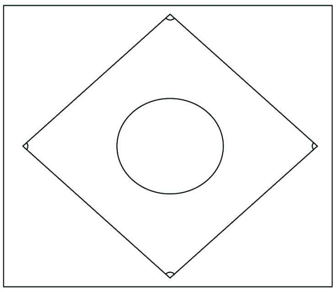

Figure 5.

$p_1 = -10$, $a_0 = 1$, $\beta_1 = 79$, $\beta_4 = 104$,
$K_1^0 = -12,754$, $C_1 = 29,642$.

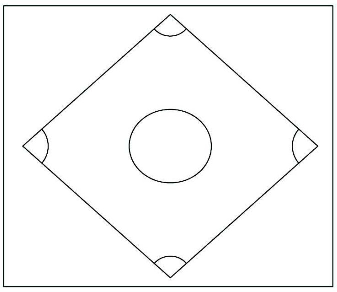

Figure 6.

$$p_1 = -10,\ a_0 = 1,\ \beta_1 = 10,\ \beta_4 = 44,$$
$$K_1^0 = 10,85,\ C_1 = -16,85.$$

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Author's address:

Department of Mathematics
Akaki Tsereteli State University
59, Queen Tamar Ave., Kutaisi 4600
Georgia