Proceedings of A. Razmadze
Mathematical Institute
Vol. **155** (2011), 55–72

ON COMPLEX POTENTIALS AND ON THE APPLICATION OF RELATED TO THEM M. G. KREIN'S TWO INTEGRAL EQUATIONS IN MIXED PROBLEMS OF CONTINUUM MECHANICS

S. M. MKHITARYAN

Abstract. The methods of complex potentials, in general, those of the boundary value problems of the theory of analytic functions are well-known and widely used in various problems of solid mechanics and mathematical physics. These methods are especially efficient in combination with the methods of the theory of integral transformations and integral equations.

In the present work we suggest a way different from that suggested by N.I. Muskhelishvili in his fundamental monograph [1], i.e., a way of representation of a general solution of equations of the plane theory of elasticity by means of complex potentials and consider applications of two integral M.G. Krein's equations to the solution of one class of mixed problems of continuum mechanics which are connected with complex potentials of the corresponding physical fields.

რეზიუმე. კომპლექსური პოტენციალების მეთოდები, ზოგადად, ანალიზურ ფუნქციათა თეორიის სასაზღვრო ამოცანების მეთოდები კარგადაა ცნობილი და ფართოდ გამოიყენება უწყვეტ ტანთა მექანიკისა და მათემატიკური ფიზიკის სხვადასხვა ამოცანებში. ასეთი მეთოდები განსაკუთრებით ეფექტურია ინტეგრალური გარდაქმნებისა და ინტეგრალურ განტოლებათა თეორიის მეთოდებთან კომბინაციაში.

წინამდებარე ნაშრომში შემოთავაზებულია ნ. მუსხელიშვილის ფუნდამენტურ ნაშრომში [1] განხილული მეთოდისგან განსხვავებული მიდგომა, სახელდობრ, დრეკადობის ბრტყელი თეორიის ამოცანების ზოგადი ამონახსნების კომპლექსური პოტენციალების საშუალებით წარმოდგენის მეთოდი და განხილულია კრეინის ორ ინტეგრალური განტოლების გამოყენება უწყვეტ ტანთა მექანიკის შერეული ამოცანების ერთი კლასისათვის, რომლებიც დაკავშირებული არიან შესაბამისი ფიზიკური ველის კომპლექსურ პოტენციალთან.

1. The initial equations in [1] for a complex representation of a general solution of equations of the plane theory of elasticity are differential equations of equilibrium in the absence of body forces and also equations of compatibility of Saint-Venant deformations in the form of the Beltrami-Michell

---

2010 *Mathematics Subject Classification.* 45E05, 65R20.
Key words and phrases. Mixed problems of continuum mechanics, complex potentials.

equations which allow us to introduce into consideration the function of stresses, or the Airy function satisfying a biharmonic equation. Next, using the complex representation of biharmonic function of stresses through two functions of a complex variable, the complex potentials, we introduce for components of displacements and stresses of an elastic field the well-known Kolosov-Muskhelishvili's formulas [1].

But it turns out that these formulas can be obtained by the well-known Navier-Lame equations in the absence of body forces, i.e., from differential equations of equilibrium in displacements. Indeed, let a linear elastic isotropic body related to the right rectangular system of coordinates $Oxyz$ be in the conditions of plane deformation occurring in the plane Oxy. Let, further, the domain S representing a cross-section of an elastic body with a plane of deformation be, for the sake of simplicity, simply-connected and bounded by a simple smooth or piecewise-smooth closed contour. Then the Navier-Lame equations are of the form [1,4]

$$\begin{gathered} \mu\,\Delta u + (\lambda+\mu)\frac{\partial e}{\partial x} = 0, \\ \mu\,\Delta v + (\lambda+\mu)\frac{\partial e}{\partial y} = 0, \\ \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2};\ \ (x,y)\in S\right) \end{gathered} \tag{1}$$

the Hook's law is of the form

$$\begin{gathered} \sigma_x = \lambda e + 2\mu e_x\,, \ \ \sigma_y = \lambda e + 2\mu e_y\,, \ \ \tau_{xy} = \mu\gamma_{xy}\,, \\ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\,, \quad \mu = G = \frac{E}{2(1+\nu)}\,, \end{gathered} \tag{2}$$

and the Cauchy relations are of the form

$$\begin{gathered} e_x = \frac{\partial u}{\partial x}\,, \quad e_y = \frac{\partial v}{\partial y}\,, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\,, \\ e = e_x + e_y = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\,, \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \end{gathered} \tag{3}$$

Here $u = u(x,y)$ and $v = v(x,y)$ are the components of elastic displacements along the Ox and Oy-axes, respectively; σ_x, σ_y and τ_{xy} are the components of stresses; e is relative solid extention; ω_z is the angle of rotation of material fibres about the Oz-axis, λ and μ are elastic Lame constants, and E and ν are the elasticity modulus and the Poisson ratio ($\mu = G$ is shear modulus) of the elastic body material. We assume that the components of displacements and stresses satisfy the conditions mentioned in [1].

Taking into account (2) and (3), we transform equations (1) as follows:

$$\begin{aligned} &\mu\,\Delta u+(\lambda+\mu)\frac{\partial e}{\partial x}=0; \\ &\mu\,\Delta v+(\lambda+\mu)\frac{\partial e}{\partial y}=0; \end{aligned} \quad\Rightarrow\quad \begin{aligned} &\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\left(1+\frac{\lambda}{\mu}\right)\frac{\partial e}{\partial x}=0, \\ &\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\left(1+\frac{\lambda}{\mu}\right)\frac{\partial e}{\partial y}=0, \end{aligned} \quad\Rightarrow$$

$$\Rightarrow\quad \begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial^2 v}{\partial x\,\partial y}+\left(1+\frac{\lambda}{\mu}\right)\frac{\partial e}{\partial x}=0; \\ &\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\right)-\frac{\partial^2 u}{\partial x\,\partial y}+\left(1+\frac{\lambda}{\mu}\right)\frac{\partial e}{\partial y}=0; \end{aligned} \quad\Rightarrow$$

$$\Rightarrow\ \nu_0\frac{\partial e}{\partial x}-\frac{\partial \omega_z}{\partial y}=0;\quad \nu_0\frac{\partial e}{\partial y}+\frac{\partial \omega_z}{\partial x}=0\quad \big(\nu_0=(1-\nu)/(1-2\nu)\big).$$

Thus, the differential Navier-Lame equations under plane deformation (1) can be represented in the form

$$\nu_0\frac{\partial e}{\partial x}=\frac{\partial \omega_z}{\partial y};\quad \nu_0\frac{\partial e}{\partial y}=-\frac{\partial \omega_z}{\partial x}\quad \big((x,y)\in S\big). \tag{4}$$

But equations (4) are, in fact, the Cauchy-Riemann conditions for the pair of functions $\nu_0 e=\nu_0 e(x,y)$ and $\omega_z=\omega_z(x,y)$ which generate an analytic in the domain S of a complex plane $z=x+iy$ function $\varphi_0(z)$, i.e.,

$$\nu_0 e+i\omega_z=\varphi_0(z)\quad (z\in S). \tag{5}$$

On the other hand, if the domain S of the complex plane z is mapped conformally by analytical function $\omega(\zeta)$ ($\omega'(\zeta)\neq 0$) onto the domain Σ of the complex plane $\zeta=\alpha+i\beta$, i.e., $z=x+iy=\omega(\zeta)$ ($\zeta=\alpha+i\beta$), then in the domain Σ, due to the fact that the function $\varphi_0(\omega(\zeta))$ is analytic, the Cauchy-Riemann conditions in the system of isometric orthogonal curvilinear coordinates α,β can be written as

$$\nu_0\frac{\partial e}{\partial \alpha}=\frac{\partial \omega_z}{\partial \beta},\quad \nu_0\frac{\partial e}{\partial \beta}=-\frac{\partial \omega_z}{\partial \alpha}\quad \big((\alpha,\beta)\in \Sigma\big). \tag{6}$$

Equations (6) are identical to equations (4) and hence the Navier-Lame equations under the plane deformation in the absence of body forces possess an important property of invariance under the conformal mapping.

This result follows also from general representations of equations of the theory of elasticity in arbitrary orthogonal curvilinear coordinates cited in [4] (pp. 227-229, 311-314), as applied to a particular case of isometric orthogonal curvilinear coordinates on the plane.

Next, we transform representation (5) assuming

$$\nu_0 e+i\omega_z=\varphi_0(z)=u_0+iv_0=u_0(x,y)+iv_0(x,y), \tag{7}$$

whence $e=u_0/\nu_0$. On the other hand, from the Hook's law (2),

$$e=\sigma/2(\lambda+\mu),\quad \sigma=\sigma_x+\sigma_y\,.$$

Consequently,

$$u_0 = \frac{\nu_0 \sigma}{2(\lambda+\mu)} = \frac{1-\nu^2}{E}\,\sigma. \tag{8}$$

Assume also that

$$v_0 = \frac{1-\nu^2}{E}\,\tau, \quad \sigma + i\tau = 4\varphi'(z).$$

As a result, we arrive at the representation

$$\nu_0 e + i\omega_z = \frac{4(1-\nu^2)}{E}\,\varphi'(z), \tag{9}$$

where $\varphi(z)$ is an arbitrary analytic function in the domain S. Formula (9) in somewhat modified form is given in [4] (p. 315, formula $(9.7)_1$).

Using now (3), we can calculate a complex combination

$$e + 2i\omega_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) =$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\,, \quad w = u + iv.$$

But

$$z = x + iy, \ \ \overline{z} = x - iy; \quad x = \frac{1}{2}(z+\overline{z}), \ \ y = \frac{1}{2i}(z-\overline{z}),$$

and hence

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right), \ \ w = w(x,y) = u(x,y) + iv(x,y) = w(z,\overline{z}).$$

Therefore

$$\frac{\partial w}{\partial z} = \frac{1}{2}(e + 2i\omega_z) \quad (z \in S), \tag{10}$$

where $w = u + iv$ is the complex combination of components of elastic displacements. Further, integrating (10), we obtain

$$w = \frac{1}{2}\int(e+2i\omega_z)dz + \chi(\overline{z}) = \frac{1}{2}\int(2\nu_0 e + 2i\omega_z)dz + \frac{1}{2}\int(1-2\nu_0)e\,dz + \chi(\overline{z}) =$$
$$= \int(\nu_0 e + i\omega_z)dz + \frac{1-2\nu_0}{2}\int e\,dz + \chi(\overline{z}),$$

where $\chi(\overline{z})$ is an arbitrary complex harmonic function, since $\partial^2\chi/\partial z\,\partial\overline{z} = 0$.

Taking now into account (9), we can write

$$w = \frac{4(1-\nu^2)}{E}\int\varphi'(z)dz + \frac{1-2\nu_0}{2}\int\frac{4(1-\nu^2)}{E}\,\frac{\varphi'(z)+\overline{\varphi'(z)}}{2\nu_0}\,dz + \chi(\overline{z}) =$$
$$= \frac{4(1-\nu^2)}{E}\,\varphi(z) + (1-2\nu_0)\frac{1-\nu^2}{\nu_0 E}\,\varphi(z) + (1-2\nu_0)\frac{1-\nu^2}{\nu_0 E}\,z\,\overline{\varphi'(z)} + \chi(\overline{z}).$$

After simple transformations we find ($\chi_* = 3 - 4\nu$),

$$2\mu w = 2\mu(u + iv) = \chi_*\varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)}; \quad \psi(z) = -2\mu\,\overline{\chi(\overline{z})}. \tag{11}$$

Formula (11) coincides with the well-known result from [1].

Passing to the representation of stress components, it follows from (7)–(9) that

$$\sigma = \sigma_x + \sigma_y = 2\big[\varphi'(z) + \overline{\varphi'(z)}\big] = 2\big[\Phi(z) + \overline{\Phi(z)}\big], \ \ \Phi(z) = \varphi'(z). \tag{12}$$

Moreover, from the Hook's law and the relations (3), we obtain

$$\sigma_y - \sigma_x = 2\mu\Big(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\Big), \quad \tau_{xy} = \mu\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big),$$

whence

$$\sigma_y - \sigma_x + 2i\tau_{xy} = -2\mu\Big(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\Big) + 2i\mu\Big(\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\Big). \tag{13}$$

But from (11) follows

$$\begin{aligned} 2\mu\Big(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\Big) &= \chi_*\,\overline{\Phi(z)} - \Phi(z) - \overline{z}\,\Phi'(z) - \Psi(z), \\ 2\mu\Big(\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\Big) &= -\,i\chi_*\,\overline{\Phi(z)} + i\Phi(z) - i\,\overline{z}\,\Phi'(z) - i\Psi(z), \\ \Psi(z) &= \psi'(z). \end{aligned} \tag{14}$$

Taking into account (14), formula (13) after simple transformations can be represented in the form

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\big[\overline{z}\,\varphi''(z) + \psi'(z)\big] = 2\big[\overline{z}\,\Phi'(z) + \Psi(z)\big]. \tag{15}$$

Representation (15) coincides with the well-known result given in [1].

Turning now our attention to the question of finding physical meaning of a complex potential $\varphi'(z) = \Phi(z)$, from (9) we get

$$e = \frac{\vartheta_0}{\nu_0}\,\mathrm{Re}\,\big[\varphi'(z)\big], \quad \omega_z = \vartheta_0\,\mathrm{Im}\,\big[\varphi'(z)\big], \quad \vartheta_0 = \frac{4(1-\nu^2)}{E}\,,$$

i.e., at every point of the domain S the relative volume expansion e and the angle of rotation ω_z about the Oz-axis, emanating from that point of material fibre of infinitely small length, with accuracy of coefficients depending on elastic constants of the deformable body, are expressed through the real part and through the coefficient of an imaginary part of the function $\varphi'(z) = \Phi(z)$. Thus, the physical meaning of that function is found out.

On the other hand, the conformal mapping $w = \varphi(z)$ of S realized by the function $\varphi(z)$ can be interpreted as a certain elastic deformation under which the circles of infinitely small radii turn again into circles of infinitely small radii, and all linear elements of infinitely small length emanating from the point $z_0 \in S$ rotate about the vertical axis by one and the same angle $\alpha =$

$\arg[\varphi'(z)]$. Then the cylinder of unit height and circular base of infinitely small radius δ_0 with center at the point z_0 turns into a circular cylinder of unit height and base center at the point $w_0 = f(z_0)$ and of infinitely small radius δ, where $\delta/\delta_0 = r = |\varphi'(z)|$. Under such a deformation, for the relative volume expansion $\overline{e}$ we will have

$$\overline{e} = \frac{V - V_0}{V_0} = \frac{\pi\delta^2 \cdot 1 - \pi\delta_0^2 \cdot 1}{n\delta_0^2} = r^2 - 1 = |\varphi'(z)|^2 - 1.$$

Thus, characteristics $\overline{e}$ and α for this deformation are expressed in terms of the real physical values e and ω_z by the formulas

$$\overline{e} = \frac{1}{\vartheta_0^2}\left(\nu_0^2 e^2 + \omega_z^2\right) - 1,$$

$$\alpha = \arg\varphi'(z) = \begin{cases} \operatorname{arctg}(\omega_z/\nu_0 e) & (e > 0), \\ \pi + \operatorname{arctg}(\omega_z/\nu_0 e) & (e < 0), \\ (\pi/2)\operatorname{sign}\omega_z & (e = 0,\ \ \omega_z \neq 0). \end{cases}$$

Assuming

$$v_x = p(x, y) = \frac{\nu_0 e}{\vartheta_0}, \quad v_y = q(x, y) = -\frac{\omega_z}{\vartheta_0},$$

we consider a definable in the domain S vector field of the vector $\overline{v} = \{v_x, v_y\}$. This vector field can be interpreted hydrodynamically assuming it as velocity distribution in an ideal incompressible liquid for its steady plane flow. If the liquid flow is potential and it has no sources, we can introduce the potential of velocities $\varphi_1(x, y)$, where

$$v_x = p = \frac{\partial\varphi_1}{\partial x}, \quad v_y = q = \frac{\partial\varphi_1}{\partial y},$$

and the flow function $\psi_1(x, y)$ which is connected with the velocity potential by the Cauchy-Riemann conditions [5,6]

$$\frac{\partial\varphi_1}{\partial x} = \frac{\partial\psi_1}{\partial y}, \quad \frac{\partial\varphi_1}{\partial y} = -\frac{\partial\psi_1}{\partial x}.$$

Then the analytic function $\widetilde{\varphi}(z) = \varphi_1 + i\psi_1$ will be the characteristic function of flow [5,6], and hence in such a way we can draw hydrodynamical analogy of the complex potential $\Phi(z) = \varphi'(z)$ which together with the representation (9) may turn out to be useful for the solution of boundary value problems of elasticity in systems of isometric orthogonal curvilinear coordinates, for example, in a bipolar system of coordinates.

2. Let us now pass to the complex potentials of another physical fields and consider basic equations of the plane theory of filtration. It is well-known [7,8] that in the case of steady plane filtration of ground waters into a porous ground massive related to the rectangular system of coordinates

Oxy, where the vertical Oy-axis is directed downwards, for the Darcy law of filtration the basic equations are

$$\begin{aligned} v_x &= -k\frac{\partial h}{\partial x}\,, \quad v_y = -k\frac{\partial h}{\partial y}\,; \quad h = \frac{p}{\gamma} - y; \\ \Delta h &= \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad \big((x,y)\in S\big). \end{aligned} \tag{16}$$

Here v_x, v_y are the components of the vector of filtration velocity with respect to the coordinate axes, k is the filtration coefficient, $p = p(x,y)$ is pressure, γ is specific liquid weight, $h = h(x,y)$ is piezometric pressure, S is the domain of filtration. If we introduce a velocity potential, then the basic equations of the plane theory of filtration for a steady regime are

$$\begin{aligned} \varphi &= -kh; \quad v_x = \frac{\partial \varphi}{\partial x}\,, \quad v_y = \frac{\partial \varphi}{\partial y}\,; \\ \Delta\varphi &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \big((x,y)\in S\big), \end{aligned} \tag{17}$$

where $\varphi = \varphi(x,y)$ is the potential of velocities. Introduce also a conjugate to $\varphi(x,y)$ function $\psi(x,y)$ satisfying in the domain S the Cauchy-Riemann conditions

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}\,, \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}\,.$$

The function $\psi(x,y)$ is called a flow function. These two functions allow one to define the complex potential

$$W = \varphi + i\psi \quad (z = x + iy \in S).$$

Then solutions of the plane theory of filtration which in the main are the mixed boundary value problems of the theory of harmonic functions are defined completely by means of the characteristic flow functions

$$z = F(W) = F(\varphi + i\psi)$$

of complex potentials W. To construct a characteristic flow function, in [7,8] are widely used the methods of conformal mappings.

But efficient solutions of such problems can also be constructed by the methods of integral transformations and integral equations. As an example, we consider the following boundary value problem of the plane theory of filtration for steady flow which is connected with calculation of a hydrotechnical dam type structure. Assume that the upstream liquid is of height H_1, the downstream one is of height H_2 ($H_1 > H_2$) and the liquid filtration occurs in a porous ground massive in the form of a strip $\Omega = \{-\infty < x < \infty;\ 0 < y < T\}$ of capacity T, and the apron, the underground dam flat base, together with the water confining layer $y = T$ are water-impermeable. The above-described problem by means of (16) is formulated mathematically as the following mixed boundary value problem for a harmonic function in the strip Ω [8]:

$$\begin{cases} \dfrac{\partial^2 \overline{h}}{\partial \xi^2} + \dfrac{\partial^2 \overline{h}}{\partial \eta^2} = 0 \quad (-\infty < \xi < \infty;\ 0 < \eta < T); \\ \overline{h}\,\Big|_{\eta=0} = 1 \ \ (-\infty < \xi < -b); \quad \overline{h}\,\Big|_{\eta=0} = 0 \ \ (b < \xi < \infty); \\ \dfrac{\partial \overline{h}}{\partial \eta}\bigg|_{\eta=0} = 0 \ \ (-b < \xi < b); \quad \dfrac{\partial \overline{h}}{\partial \eta}\bigg|_{\eta=T} = 0 \ \ (-\infty < \xi < \infty). \end{cases} \tag{18}$$

We have here introduced a dimensionless piezometric pressure

$$\overline{h} = \overline{h}(\xi, \eta) = \frac{h(\xi, \eta) - H_2}{H}, \quad H = H_1 - H_2 \tag{19}$$

and for the coordinates the notations ($\xi = x,\ \eta = y$) from [8] are maintained. We are required to define basic filtration characteristics of the problem which are the distribution of piezometric pressure h along the underground apron, the component of outlet velocity v_η with respect to the line of downstream and a full filtration discharge.

We reformulate the boundary value problem (18) for the conjugate to $\varphi_0(\xi, \eta) = \overline{h}(\xi, \eta)$ function $\psi_0(\xi, \eta)$ satisfying the Cauchy-Riemann conditions. Then the problem (18) transforms into the following problem for the function $\psi_0(\xi, \eta)$

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial \xi^2} + \dfrac{\partial^2 \psi_0}{\partial \eta^2} = 0 \quad (-\infty < \xi < \infty;\ 0 < \eta < T), \\ \dfrac{\partial \psi_0}{\partial \eta}\bigg|_{\eta=0} = 0 \quad (b < |\xi| < \infty), \\ \psi_0\Big|_{\eta=0} = c \quad (-b < \xi < b); \quad \psi_0\Big|_{\eta=T} = 0 \quad (-\infty < \xi < \infty), \end{cases} \tag{20}$$

where c is the constant to be defined. Note that according to (16)-(17) and (19), the real velocity potential and the real flow function will, respectively, be the functions

$$\begin{aligned} &\varphi(\xi, \eta) = -k\,H\,\overline{h}(\xi, \eta) - k\,H_2 = -k\,H\,\varphi_0(\xi, \eta) - k\,H_2, \\ &\psi(\xi, \eta) = -k\,H\,\psi_0(\xi, \eta) \quad ((\xi, \eta) \in \Omega). \end{aligned} \tag{21}$$

To solve the boundary value problem (20), we put

$$g(\xi) = \begin{cases} 0 \quad (b < |\xi| < \infty), \\ \chi(\xi) \quad (|\xi| < b), \end{cases} \tag{22}$$

and consider an auxiliary mixed boundary value problem

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial \xi^2} + \dfrac{\partial^2 \psi_0}{\partial \eta^2} = 0 \quad (-\infty < \xi < \infty;\ 0 < \eta < T), \\ \dfrac{\partial \psi_0}{\partial \eta}\bigg|_{\eta=0} = g(\xi) \ \ (-\infty < \xi < \infty); \quad \psi_0\Big|_{\eta=T} = 0 \ \ (-\infty < \xi < \infty). \end{cases} \tag{23}$$

A solution of the problem (23) is constructed in terms of the integral Fourier transformation by introducing the Fourier transformers

$$\big\{\overline{\psi}_0(\lambda,\eta);\overline{g}(\lambda)\big\}=\int\limits_{-\infty}^{\infty}\big\{\psi_0(\lambda,\eta);g(\xi)\big\}e^{i\lambda\xi}d\xi.$$

As a result, the boundary value problem (23) in the Fourier transformers can be written in the form of the problem

$$\begin{cases}\dfrac{\partial^2\overline{\psi}_0}{\partial\xi^2}-\lambda^2\overline{\psi}_0=0 \quad (0<\eta<T),\\ \dfrac{\partial\overline{\psi}_0}{\partial\eta}\Big|_{\eta=0}=\overline{g}(\lambda); \quad \overline{\psi}_0\big|_{\eta=T}=0,\end{cases}$$

whose solution is expressed by the formula

$$\overline{\psi}_0(\lambda,\eta)=\operatorname{sh}\big[\lambda(\eta-T)\big]\overline{g}(\lambda)/\lambda\operatorname{ch}(\lambda T) \quad (0\le\eta\le T).$$

Hence by the formula of inverse Fourier transformation, in view of (22), we find

$$\begin{aligned}\psi_0(\xi,\eta)&=\int\limits_{-b}^{b}L\big(|\xi-\tau|,\eta\big)\chi(\tau)d\tau \quad (-\infty<\xi<\infty;\ 0\le\eta\le T),\\ L(\tau,\eta)&=\frac{1}{\pi}\int\limits_{0}^{\infty}\frac{\operatorname{sh}[\lambda(\eta-T)]}{\lambda\operatorname{ch}(\lambda T)}\cos(\lambda\tau)d\lambda.\end{aligned} \tag{24}$$

Further, assuming in (24) $\eta=0$ and taking into account the expression of the known integral ([9], p. 39, formula (28)),

$$K(\xi)=\frac{1}{\pi}\int\limits_{0}^{\infty}\frac{\operatorname{th}(\lambda T)}{\lambda}\cos(\lambda\xi)d\lambda=\frac{1}{\pi}\ln\operatorname{cth}\left(\frac{\pi|\xi|}{4T}\right) \quad (-\infty<\xi<\infty),$$

we have

$$\begin{aligned}\psi_0(\xi,0)&=-\int\limits_{-b}^{b}K(|\tau-\xi|)\chi(\tau)d\tau=\\ &=-\frac{1}{\pi}\int\limits_{-b}^{b}\ln\operatorname{cth}\left[\frac{\pi|\tau-\xi|}{4T}\right]\chi(\tau)d\tau \quad (-\infty<\xi<\infty).\end{aligned} \tag{25}$$

Using now (25) and realizing the boundary condition from (20), we arrive at the integral M.G. Krein's equation [2,3] for the partial right-hand side

$$\int_{-a}^{a} \ln \operatorname{cth} \frac{|t-s|}{4} \chi_0(s) ds = -\frac{\pi^2 c}{T}, \tag{26}$$

where we have adopted the following notation

$$t = \pi\xi/T, \quad s = \pi\tau/T, \quad a = \pi b/T, \quad \chi_0(s) = \chi(Ts/\pi).$$

A solution of the integral equation (26) is given by the formula [2,3]

$$\chi_0(t) = -\pi c \Big[T Q_{-1/2}(\operatorname{ch} a)\sqrt{2(\operatorname{ch} a - \operatorname{ch} t)}\Big]^{-1} \quad (-a < t < a),$$

where $Q_{-1/2}(x)$ is the Legendre function of second kind.

Getting back to the aforementioned values, we have $(-b < \xi < b)$

$$\chi(\xi) = -\pi c \left\{ T Q_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)\right] \sqrt{2\left[\operatorname{ch}\left(\frac{\pi b}{T}\right) - \operatorname{ch}\left(\frac{\pi \xi}{T}\right)\right]} \right\}^{-1}. \tag{27}$$

Next, proceeding from the Cauchy-Riemann conditions

$$\frac{\partial \varphi_0}{\partial \xi} = \frac{\partial \psi_0}{\partial \eta}, \quad \frac{\partial \varphi_0}{\partial \eta} = -\frac{\partial \psi_0}{\partial \xi}$$

by virtue of (24), we find the congugate to $\psi_0(\xi,\eta)$ function $\varphi_0(\xi,\eta) = \overline{h}(\xi,\eta)$, the solution of the boundary value problem (18), and the constant c. Omitting intermediate calculations, we obtain $(-\infty < \xi < \infty; 0 \le \eta \le T)$

$$\overline{h}(\xi,\eta) = \varphi_0(\xi,\eta) =$$
$$= \frac{1}{2} - \frac{1}{\pi} \int_{-b}^{b} \left\{ \int_{0}^{\infty} \frac{\operatorname{ch}[\lambda(\eta - T)]}{\lambda \operatorname{ch}(\lambda T)} \sin\left[\lambda(\tau - \xi) d\lambda\right] \right\} \chi(\tau) d\tau, \tag{28}$$
$$c = Q_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)\right] \Big/ \pi P_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)\right],$$

where $P_{-1/2}$ is the Legendre function of the first kind. Consequently, with regard for (28), the function $\chi(\xi)$ from (27) takes the form $(-b < \xi < b)$

$$\chi(\xi) = -\left\{ T P_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)\right] \sqrt{2\left[\operatorname{ch}\left(\frac{\pi b}{T}\right) - \operatorname{ch}\left(\frac{\pi \xi}{T}\right)\right]} \right\}^{-1}. \tag{29}$$

Note that the above-mentioned Legendre functions can be expressed through the full elliptic integral of first kind $K(d)$ ($d = e^{-\alpha}$, $d' = \sqrt{1-d^2} = \sqrt{1-e^{-2\alpha}}$):

$$Q_{-1/2}(\operatorname{ch}\alpha) = 2\sqrt{d}\, K(d), \quad P_{-1/2}(\operatorname{ch}\alpha) = \frac{2}{\pi}\sqrt{d}\, K(d').$$

Proceeding now from (16) or (17) and using (21),(24), or (28), we calculate components of velocity. Taking into account the expressions of the known from [10] integrals (p. 518, formulas (6), (10)), we obtain

$$
\begin{aligned}
v_\xi &= \frac{\partial\varphi}{\partial\xi} = -kH\frac{\partial\varphi_0}{\partial\xi} = -kH\frac{\partial\psi_0}{\partial\eta} = -\frac{kH}{T}\cos\left[\frac{\pi(\eta-T)}{2T}\right]\times\\
&\times\int\limits_{-b}^{b}\frac{\operatorname{ch}\left[\frac{\pi(\tau-\xi)}{2T}\right]\chi(\tau)d\tau}{\cos\left[\frac{\pi(\eta-T)}{T}\right]+\operatorname{ch}\left[\frac{\pi(\tau-\xi)}{T}\right]},\\
v_\eta &= \frac{\partial\varphi}{\partial\eta} = -kH\frac{\partial\varphi_0}{\partial\eta} = kH\frac{\partial\psi_0}{\partial\xi} = \frac{kH}{T}\sin\left[\frac{\pi(\eta-T)}{2T}\right]\times\\
&\times\int\limits_{-b}^{b}\frac{\operatorname{sh}\left[\frac{\pi(\tau-\xi)}{2T}\right]\chi(\tau)d\tau}{\cos\left[\frac{\pi(\eta-T)}{T}\right]+\operatorname{ch}\left[\frac{\pi(\tau-\xi)}{T}\right]}\quad(-\infty<\xi<\infty;\ 0\le\eta\le H).
\end{aligned}
\tag{30}
$$

In terms of the calculated function $\chi(\xi)$ from (29) we express likewise the reduced pressure under the apron. Namely, substituting $\eta=0$ in (28) and taking into account formula (29), we have $(-b<\xi<b)$

$$
\overline{h}(\xi,0)=\varphi_0(\xi,0)=\frac{1}{2}\left\{1+\frac{1}{TP_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)\right]}\int\limits_{-b}^{b}\frac{\operatorname{sign}(\xi-\tau)d\tau}{\sqrt{2\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)-\operatorname{ch}\left(\frac{\pi\tau}{b}\right)\right]}}\right\}.
$$

Finally, we calculate the liquid discharge in the ground $0<\eta<T$ through its vertical cross-sections

$$
Q=\int\limits_0^T v_\xi\,d\eta=\int\limits_0^T\frac{\partial\varphi}{\partial\xi}\,d\eta=\int\limits_0^T\frac{\partial\psi}{\partial\eta}\,d\eta\quad(-\infty<\xi<\infty).
$$

Hence taking into account (30), after simple transformations we find that

$$
Q=Q(\xi)=-\frac{KH}{\pi}\int\limits_{-b}^{b}\ln\operatorname{cth}\left[\frac{\pi|\tau-\xi|}{4T}\right]\chi(\tau)d\tau\ \ (-\infty<\xi<\infty). \tag{31}
$$

If in (31) we assume $-b<\xi<b$ and take into account (26) and (29), we will get

$$
\begin{aligned}
Q=Q(\xi)&=\frac{KH}{\pi}\,\frac{Q_{-1/2}\left[\operatorname{ch}(\pi b/T)\right]}{P_{-1/2}\left[\operatorname{ch}(\pi b/T)\right]}=\frac{KH}{\pi}\,\frac{Q_{-1/2}(\operatorname{ch}a)}{P_{-1/2}(\operatorname{ch}a)}=\frac{kH}{\pi M(a)},\\
M(a)&=\frac{P_{-1/2}(\operatorname{ch}a)}{Q_{-1/2}(\operatorname{ch}a)},
\end{aligned}
$$

where $M(a)$ is the known M.G. Krein's function [2,3]. Introducing dimensionless liquid discharge $\overline{Q}(\xi)$ through the vertical cross-sections of the ground under the apron, we can write

$$\overline{Q}(\xi) = \frac{\pi Q}{kH} = \frac{1}{M(a)} \quad (-b < \xi < b) \tag{32}$$

and the function $M(a)$ in the given problem acquires physical meaning.

In a general case, we substitute in (31) the expression of the calculated from (29) function $\chi(\xi)$ and calculate the filtration liquid discharge through vertical cross-sections of the ground in the downstream ($b < \xi < \infty$). Using the results of [11], after transformations we obtain

$$\overline{Q}(t) = \frac{2\pi \exp(-a/2)}{\pi P_{-1/2}(\operatorname{ch} a)} \int\limits_0^{\exp(\frac{a-t}{2})} \frac{du}{\sqrt{(1-u^2)(1-e^{-2a}u^2)}} \tag{33}$$

$$\big(a = \pi b/T, \ \ t = \pi\xi/T, \quad \overline{Q}(t) = \frac{\pi}{kH}\, Q(Tt/\pi), \ \ t > a\big).$$

Thus, for the dimensionless liquid discharge formulas (32) and (33) hold. For the outlet velocity, i.e., for that in the downstream, by virtue of (30) we have

$$v_\eta = \frac{\partial \varphi}{\partial \eta}\bigg|_{\eta=0} = -kH \frac{\partial \varphi_0}{\partial \eta}\bigg|_{\eta=0} = -\frac{kH}{2T} \int\limits_{-b}^{b} \frac{\chi(\tau)d\tau}{\operatorname{sh}\left[\pi(\tau-\xi)/2T\right]} \quad (\xi > b).$$

Next, substituting here the expression of the calculated function from (29) and using again the results of [11] or the formula from [12] (p. 177, formula 32 for $\nu = 1/2$), after transformations we find ($\xi > b$)

$$\overline{v} = \frac{T}{KH}\, v_\eta\bigg|_{\eta=0} = $$

$$= -\left\{ P_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)\right] \sqrt{2\left[\operatorname{ch}\left(\frac{\pi \xi}{T}\right) - \operatorname{ch}\left(\frac{\pi b}{T}\right)\right]} \right\}^{-1}, \tag{34}$$

where we have introduced the dimensionless velocity $\overline{v}$.

In addition, for partial filtration liquid discharge on the downstream segment $[\xi_1, \xi_2]$ we obtain

$$Q_{12} = -\int\limits_{\xi_1}^{\xi_2} v_\eta\big|_{\eta=0} d\xi = \frac{KH}{\pi P_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{T}\right)\right]} \int\limits_{\xi_1}^{\xi_2} \frac{d\xi}{\sqrt{2\left[\operatorname{ch}\left(\frac{\pi\xi}{T}\right) - \operatorname{ch}\left(\frac{\pi b}{T}\right)\right]}}\,. \tag{35}$$

In the limiting case $T \to \infty$, the filtrating base in the form of a strip Ω goes over to the lower half-plane $\eta > 0$, and appreciably simplified formulas

(34) and (35) take, respectively, the form

$$v_\eta\big|_{\eta=0} = -\frac{KH}{\pi\sqrt{\xi^2-b^2}} \quad (\xi > b); \quad Q_{12} = \frac{KH}{\pi} \ln\left(\frac{\xi_2+\sqrt{\xi_2^2-b^2}}{\xi_1+\sqrt{\xi_1^2-b^2}}\right).$$

In particular, the liquid discharge on the downstream segment $[b, \xi]$ will be

$$Q_{b\xi} = \frac{KH}{\pi} \ln\left(\frac{\xi+\sqrt{\xi^2-b^2}}{b}\right).$$

3. In this section we will point out the mixed problems of elasticity whose solution is reduced to the solution of the above-mentioned M.G. Krein's integral equation. Let the related to the right rectangular system of coordinates $Oxyz$ elastic layer $\Omega = \{-\infty < x,\ z < \infty;\ -H \le y \le 0\}$ of height H and with shear modulus G be in the conditions of antiplane deformation in the direction of the Oz-axis with the base plane Oxy. Moreover, the lower bound of the layer $y = -H$ is rigidly fixed, and to the upper bound of the layer $y = 0$ is applied a punch having the form of a strip $\omega_0 = \{-b < x < b,\ y = 0,\ -\infty < z < \infty\}$ along whose median in the direction of the Oz-axis there act uniformly distributed tangential forces of constant intensity T. Let the punch under the action of these forces rigidly displace in the direction of the Oz-axis by the value δ. To solve this contact problem, we first solve the following mixed boundary value problem for the base strip $\omega_0 = \{-\infty < x < \infty,\ -H \le y \le 0\}$. Let the lower bound of that strip $y = -H$ be again rigidly fixed and on the upper bound $y = 0$ in the direction of the Oz-axis there act tangential forces of intensity $\tau(x)$. Since for the antiplane deformation the only other than zero component of displacements is u_z in the direction of the Oz-axis, which in the domain ω is a harmonic function, therefore the above-described mixed boundary value problem for the strip can be formulated mathematically in the form of the following problem:

$$\begin{cases} \Delta\varphi = \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} = 0 \quad \{-\infty < x < \infty;\ -H < y < 0\}, \\ \dfrac{\partial\varphi}{\partial y}\bigg|_{y=0} = \tau(x) \quad (-\infty < x < \infty), \quad \varphi\big|_{y=-H} = 0. \end{cases} \tag{36}$$

Here $\varphi = \varphi(x,y) = Gu_z(x,y)$ $((x,y) \in \omega)$. We also introduce a conjugate to φ function ψ which is connected with it by the Cauchy-Riemann conditions

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

and the complex potential

$$W = \varphi + i\psi \quad (z = x + iy \in \omega).$$

$\varphi(x,y) = \text{const}$ will be the lines of equal reduced displacements, and the lines $\psi(x,y) = \text{const}$ are the trajectories of tangential stresses, since

$$\psi(x,y) = \text{const} \Rightarrow \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy = 0 \Rightarrow$$
$$\Rightarrow -\frac{\partial\varphi}{\partial y}\,dx + \frac{\partial\varphi}{\partial x}\,dy = -\tau_{yz}dx + \tau_{xz}dy = 0 \Rightarrow \frac{dx}{\tau_{xz}} = \frac{dy}{\tau_{yz}}\,,$$

where τ_{xz} and τ_{yz} are the components of tangential stresses in the antiplane problem. Here the function $\psi(x,y)$ is an analogy of the flow function in hydromechanics.

By virtue of the integral Fourier transformation, for $y = 0$, a solution of the problem (36) is given by the formula

$$u_z(x,0) = \frac{1}{G}\,\varphi(x,0) = \frac{1}{\pi G}\int\limits_{-\infty}^{\infty} \ln\operatorname{cth}\left[\frac{\pi|x-s|}{4H}\right]\tau(s)\,ds \quad (-\infty < x < \infty).$$

Now, a solution of the above-described contact problem is reduced to the solution of the integral equation

$$\frac{1}{\pi G}\int\limits_{-b}^{b} \ln\operatorname{cth}\left[\frac{\pi|x-s|}{4H}\right]\tau(s)\,ds = \delta \quad (-b < x < b), \tag{37}$$

which by means of a simple transformation of variables turns into equation (26). A solution of equation (37) should satisfy the condition of equilibrium of the punch:

$$\int\limits_{-b}^{b} \tau(s)\,ds = T. \tag{38}$$

Generalizing the given contact problem, we consider the following mixed boundary value problem of elasticity under the antiplane deformation for the strip ω whose physical meaning is clear:

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} = 0 \quad \{-\infty < x < \infty;\ -H < y < 0\}, \\ \varphi(x,y)\big|_{y=0} = w_0 = \text{const} \quad (-\infty < x < -b), \\ \varphi(x,y)\big|_{y=0} = 0 \quad (b < x < \infty), \\ \dfrac{\partial\varphi}{\partial y}\Big|_{y=0} = f(x) \ \ (-b<x<b), \quad \dfrac{\partial\varphi}{\partial y}\Big|_{y=-H} = 0 \ \ (-\infty<x<\infty). \end{cases} \tag{39}$$

Using the integral Fourier transformation, the solution of the boundary value problem (39) can be reduced to the solution of the following integral

M. G. Krein's equation:

$$\int_{-b}^{b} \ln \operatorname{cth} \left[\frac{\pi|x-s|}{4H}\right] \chi(s)ds = -\pi h(x) + \pi c, \quad h(x) = \int_{-b}^{x} f(s)\, ds. \tag{40}$$

Now, the solution of equation (40) with the right-hand side $h(x)$ we denote by $\chi_0(x)$, and with the right-hand side equal to 1, by $q(x,b)$. Then

$$\chi(x) = -\pi\chi_0(x) + \pi c q(x,b) \quad (-b < x < b), \tag{41}$$

where the solution $\chi_0(x)$ can be obtained by the M. G. Krein's formula [2,3], and $q(x,b)$ is given by the formula [2,3]

$$q(x,b) = \left\{ H Q_{-1/2}\left[\operatorname{ch}\left(\frac{\pi b}{H}\right)\right] \sqrt{2\left[\operatorname{ch}\left(\frac{\pi b}{H}\right) - \operatorname{ch}\left(\frac{\pi x}{H}\right)\right]} \right\}^{-1}.$$

In addition, using the calculated function $\chi(x)$ from (41), we express components of the complex potential in the given problem by the formulas ($a = \pi b/H,\ -\infty < x < \infty,\ -H \le y \le 0$)

$$\varphi(x,y) = \frac{w_0}{2} - \int_{-b}^{b} \left[\chi_0(s) - c q(s,b)\right] ds \int_{0}^{\infty} \frac{\operatorname{ch}[\lambda(y+H)]}{\lambda \operatorname{ch}(\lambda H)} \sin\left[\lambda(x-s)\right] d\lambda,$$

$$\psi(x,y) = -\int_{-b}^{b} \left[\chi_0(s) - c q(s,b)\right] ds \int_{0}^{\infty} \frac{\operatorname{sh}[\lambda(y+H)]}{\lambda \operatorname{ch}(\lambda H)} \cos\left[\lambda(x-s)\right] d\lambda,$$

$$c = (\pi H_0 - w_0)/\pi Q_0, \quad H_0 = \int_{-b}^{b} \chi_0(s)ds, \quad Q_0 = M(a) = \frac{P_{-1/2}(\operatorname{ch} a)}{Q_{-1/2}(\operatorname{ch} a)}.$$

4. In the above-considered contact problem for the strip ω under antiplane deformation we now replace the condition of rigid fastening of the strip bound by the condition of loading of that bound by tangential forces of intensity $\tau_1(x)$ acting opposing the Oz-axis. To solve this contact problem we consider first the following auxiliary boundary value problem for the strip ω ($\varphi(x,y) = G\, u_z(x,y)$):

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} = 0 \quad \{-\infty < x < \infty;\ -H < y < 0\}, \\ \left.\dfrac{\partial\varphi}{\partial y}\right|_{y=0} = \tau(x) \quad (-\infty < x < \infty), \\ \left.\dfrac{\partial\varphi}{\partial y}\right|_{y=-H} = \tau_1(x) \quad (-\infty < x < \infty). \end{cases} \tag{42}$$

Using the integral Fourier transformation with respect to the variable x and the expressions of the known integrals from [10] (p.517, formula 3.981.1 and p.530, formula 4.116.3), the solution of the boundary value problem (42), for $y = 0$, is represented by the formula

$$u_z(x,0) = \frac{1}{G}\,\varphi(x,0) = \frac{1}{\pi G}\int\limits_{-\infty}^{\infty} \ln\frac{1}{2\,\mathrm{sh}[\pi|x-s|/2H]}\,\tau(s)\,ds -$$
$$-\frac{1}{\pi G}\int\limits_{-\infty}^{\infty} \ln\frac{1}{\mathrm{ch}[\pi(x-s)/2H]}\,\tau_1(s)\,ds \quad (-\infty < x < \infty).$$

Next, realizing the condition of the punch contact with the elastic strip $u_z(x,0) = \delta$ $(-b < x < b)$, we obtain another integral M. G. Krein's equation [2,3]

$$\frac{1}{\pi G}\int\limits_{-b}^{b} \ln\frac{1}{2\,\mathrm{sh}[\pi|x-s|/2H]}\,\tau(s)\,ds = f(x) \quad (-b < x < b),$$
$$f(x) = \delta + g(x), \quad g(x) = \frac{1}{\pi G}\int\limits_{-\infty}^{\infty} \ln\frac{1}{\mathrm{ch}[\pi(x-s)/2H]}\,\tau_1(s)\,ds. \tag{43}$$

The solution of the above equation should satisfy the condition

$$\int\limits_{-b}^{b} \tau(s)\,ds = T, \tag{44}$$

which coincides with the condition of equilibrium of the punch.

Equation (43) in dimensionless values

$$t = \pi x/H, \quad u = \pi s/H, \quad a = \pi b/H, \quad \overline{\tau}(u) = G^{-1}\tau(Hu/\pi),$$
$$\overline{\tau}_1(u) = G^{-1}\tau(Hu/\pi), \quad \overline{f}(t) = \frac{\pi^2}{H}\,f(Ht/\pi), \quad \overline{f}(t) = \overline{\delta} + \overline{g}(t),$$
$$\overline{\delta} = \frac{\pi^2}{H}\,\delta, \quad \overline{g}(t) = \int\limits_{-\infty}^{\infty} \ln\frac{1}{\mathrm{ch}((t-u)/2)}\,\overline{\tau}_1(u)\,du \quad (-a < t < a)$$

takes the form

$$\int\limits_{-\infty}^{\infty} \ln\frac{1}{2\,\mathrm{sh}(|t-u|/2)}\,\overline{\tau}(u)\,du = \overline{f}(t). \tag{45}$$

and the condition (44) transforms into the condition

$$\int_{-a}^{a} \overline{\tau}(u)\,du = \overline{T} \quad \Big(\overline{T} = \frac{\pi T}{HG}\Big). \tag{46}$$

A solution of the integral equation (45) can be obtained by the well-known from [2,3] formula, and the condition (46) establishes the dependence between the values $\overline{\delta}$ and $\overline{T}$.

In particular, for the concentrated force $\tau_1(x) = T\delta(x)$, where $\delta(x)$ is the known Dirac delta-function, the integral equation (45) turns into the equation

$$\int_{-\infty}^{\infty} \ln \frac{1}{2\,\mathrm{sh}(|t-u|/2)}\, \overline{\tau}(u)du = \overline{\delta} - \overline{T} \ln \big[\, \mathrm{ch}(t/2)\big].$$

Finally, it should be noted that the method of solving the integral M.G. Krein's equations is widely applied for the solution of contact problems of the theory of elasticity [13].

References

1. N. I. Muskhelishvili, Some basic problems of the mathematical theory of elasticity. (Russian). *Nauka, Moscow*, 1966.
2. M. G. Krein, On a new method of solution of linear integral equations of the first and second kind. (Russian) *Dokl. Akad. Nauk SSSR* **100** (1955), 413–416.
3. I. Ts. Gokhberg and M. G. Krein, The theory of Volterra operators in Hilbert space and its applications. (Russian) *Nauka, Moscow*, 1967.
4. V. V. Novozhilov, Theory of elasticity. (Russian) *Sudpromgiz, Leningrad*, 1958.
5. I. I. Privalov, Introduction to the theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1977.
6. M. A. Lavrent'ev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1973.
7. V. I. Aravin and S. N. Numerov, Theory of liquid and gas motion in undeformable porous medium. (Russian) *Gostechteorizdat, Moscow, Leningrad*, 1953.
8. P. F. Fil'chakov, Theory of filtration under hydrotechnical constructions. (Russian). *Izd. ANUSSR, Kiev*, 1959.
9. G. Bateman, A. Erdelyi, et al., Tables of integral transformations, I. (Russian) *Nauka, Moscow*, 1969.
10. I. S. Gradstein and I. M. Ryzhik, Tables of integrals, sums, and products. (Russian) *Nauka, Moscow*, 1971.
11. S. M. Mkhitaryan, Eigenfunctions of an integral operator generated by a logarithmic kernel on two intervals and their application to contact problems. (Russian) *Izv. Akad. Nauk Armyan. SSR Ser. Mekh.* **35** (1982), No. 6, 3–18.
12. G. Bateman, A. Erdelyi, et al., Tables of integral transformations, II. (Russian) *Nauka, Moscow*, 1970.

13. S. M. Mkhitaryan, On the application of the M. G. Krein method for the solution of integral equations in contact problems in elasticity theory. *Modern analysis and applications. The Mark Krein Centenary Conference* **2** (2009), 155–171, *Oper. Theory Adv. Appl.* 191, *Birkhauser Verlag, Basel*, 2009.

(Received 20.12.2010)

Author's address:

Institute of Mechanics
24 b, Marshal Bagramyan Avenue
Yerevan, 0019
Armenia
E-mail: smkhitaryan39@rambler.ru