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ON FREE AND FORCED OSCILLATIONS OF PRETWISTED LONG ORTHOTROPIC CYLINDRICAL SHELLS

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Abstract. The paper considers free and forces oscillations of long pretwisted orthotropic shells. The influence of torques, normal pressure and orthotropic parameters on lower frequencies, critical load and forms of oscillations is investigated. When considering forced oscillations, it is assumed that a shell is under the action of a disturbing arbitrary normal load which varies in time according to the harmonic law. For the solution of the problem, the use was made of the Fourier method. The results are obtained with regard for the principal boundary conditions.

რეზიუმე. ნაშრომში გამოკვლეულია მგრეხავი მომენტების, ნორმალური წნევის და ორთოტროპიული პარამეტრების გავლენა დაბალ სიხშირეზე, კრიტიკულ დატვირთვაზე და რხევის ფორმაზე. იძულებითი რხევების გამოკვლევების დროს იგულისხმება, რომ გარსზე მოქმედებს შეშფოთებული ნორმალური დატვირთვა, რომელიც იცვლება დროის მიხედვით ჰარმონიული კანონით. ამოხსნის დროს გამოყენებული იქნა ფურიეს მეთოდი. მიღებულია შესაბამისი ამონახსნები ძირითადი სასაზღვრო პირობების გათვალისწინებით.

1. In the present paper we consider free and forced oscillations of long orthotropic cylindrical shells under the preliminary action of torques, axial forces applied to the end-walls of the shell, and normal pressure distributed uniformly over the whole lateral shell surface. As a disturbing reason we consider an arbitrary normal load varying in time according to the harmonic law. Investigation is carried out on the basis of a system of equations of oscillation of prestressed orthotropic cylindrical shells of arbitrary length, with exception of very short ones, when initial stressed state cannot be assumed momentless [1]. The formulas for finding lower frequencies and critical loads are derived. The degree of influence of elastic parameters, including shear modulus of an orthotropic material, on the lower and higher frequencies is elucidated. The solution of the problem dealing with forced

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oscillations is obtained in the form of series expansion with respect to eigenfunctions of free oscillations of pretwisted shells with regard to the principal boundary conditions.

The resolving equation of oscillations of prestressed orthotropic cylindrical shells with due regard to lengthening and angles of rotation of linear elements of its midsurface, as well as for improved relations of the shell theory (regarding to the radial displacement w) is of the form [1]

$$\begin{aligned}
&\varepsilon\bigg(\Delta_1\Delta_2+\bigg\{d_1\frac{\partial^6 w}{\partial\xi^6}+d_2\frac{\partial^6 w}{\partial\xi^4\partial\varphi^2}+d_3\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4}+2d_4\frac{\partial^6 w}{\partial\varphi^6}+\\
&\quad+d_5\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2}+d_4\frac{\partial^4 w}{\partial\varphi^4}\bigg\}\bigg)+\frac{\partial^4 w}{\partial\xi^4}+d_4(t_{12}^0-t_{22}^0)\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2}-\\
&\quad-\bigg[t_{12}^0\frac{\partial^2}{\partial\xi^2}+\Big(t_{22}^0\frac{\partial^2}{\partial\varphi^2}+1\Big)+2s_2^0\frac{\partial^2}{\partial\xi\partial\varphi}\bigg]\Delta_2 w-\\
&\quad-2s_2^0\frac{\partial^2}{\partial\xi\partial\varphi}\Big(g_1\frac{\partial^2}{\partial\xi^2}+d_4\frac{\partial^2 w}{\partial\varphi^2}\Big)=\\
&\quad=\Omega_2\frac{\partial^2}{\partial t^2}\bigg\{-\Delta_2 w+p_1^{-1}(1-v_1v_2+p_2)\frac{\partial^2 w}{\partial\xi^2}+p_1p_2^{-1}\frac{\partial^2 w}{\partial\varphi^2}+\\
&\quad+(1+p_2)p_2^{-1}\Big(\frac{\partial^2}{\partial\xi^2}+d_4\frac{\partial^2}{\partial\varphi^2}\Big)\bigg[\varepsilon_*\Big(\Delta_1+\frac{\partial^2}{\partial\varphi^2}+1\Big)-\\
&\quad-\Big(\bar{t}_{12}^{\,0}\frac{\partial^2}{\partial\xi^2}+\bar{t}_{22}^{\,0}\frac{\partial^2}{\partial\varphi^2}+2\bar{s}_2^{\,0}\frac{\partial^2}{\partial\xi\partial\varphi}\Big)\bigg]w+p_2^{-1}\Omega_2^*\frac{\partial^2}{\partial t^2}\bigg[-p_1p_2^{-1}-\\
&\quad-\varepsilon_*p_1p_2^{-1}\Big(\Delta_1+2\frac{\partial^2}{\partial\varphi^2}+1\Big)+(1+p_2)\Big(\frac{\partial^2}{\partial\xi^2}+d_4\frac{\partial^2}{\partial\varphi^2}\Big)+\\
&\quad+\Big(2\bar{t}_{12}^{\,0}\frac{\partial^2}{\partial\xi^2}+\bar{t}_{22}^{\,0}\frac{\partial^2}{\partial\varphi^2}+2\bar{s}_1^{\,0}\frac{\partial^2}{\partial\xi\partial\varphi}\Big)-\Omega_1^*\frac{\partial^2}{\partial t^2}\bigg]w\bigg\},
\end{aligned}\tag{1.1}$$

$$\begin{aligned}
&\Delta_1=\delta_1\frac{\partial^4}{\partial\xi^4}+\delta_2\frac{\partial^4}{\partial\xi^2\partial\varphi^2}+\frac{\partial^4}{\partial\varphi^4}\,,\quad \Delta_2=\frac{\partial^4}{\partial\xi^4}+\delta_3\frac{\partial^4}{\partial\xi^2\partial\varphi^2}+\delta_1^{-1}\frac{\partial^4}{\partial\varphi^4}\,,\\
&\delta_1=\frac{E_1}{E_2}\,,\quad \delta_2=\frac{4G(1-v_1v_2)}{E_2}+2v_1,\quad \delta_3=\frac{E_2}{G}-2v_2,\\
&d_1=2v_1,\quad d_2=6-8v_1v_2+2v_1\Big[\frac{E_2}{G}-\frac{4G(1-v_1v_2)}{E_1}\Big],\\
&d_3=2\Big[\frac{4G(1-v_1v_2)}{E_1}+\frac{E_2}{G}-v_2\Big],\quad d_4=\frac{E_2}{E_1}\,,\\
&d_5=\frac{4G(1-v_1v_2)}{E_2}+\frac{E_2}{G}-2v_2,\\
&g_1=\frac{E_2}{G}+\frac{v_1E_2}{2G(1-v_1v_2)}\Big(1-\frac{E_2}{E_1}\Big)-(1+2v_1)\frac{E_2}{E_1}\,,\\
&\bar{t}_{ij}^{\,0}=(1-v_1v_2)t_{ij}^0,\quad \bar{s}_i^{\,0}=(1-v_1v_2)s_i^0\quad (i,j=1,2),
\end{aligned}\tag{1.2}$$

$$t_{i1}^0 = T_i^0/E_1 h, \quad t_{i2}^0 = T_i^0/E_2 h, \quad s_i^0 = S^0/E_i h, \quad \varepsilon_* = h^2/12R^2,$$

$$p_i = \frac{G}{E_i}(1 - v_1 v_2), \;\; \Omega_i^* = \frac{\rho R^2}{E_i}(1 - v_1 v_2), \;\; \Omega_i = \frac{\rho R^2}{E_i}\,, \;\; \varepsilon = (1 - v_1 v_2)^{-1}\varepsilon_*.$$

Here T_i^0, S^0 $(i = 1, 2)$ are normal and shearing forces of the initial statical state, ξ, φ and R, ℓ, h are, respectively, radius, length and thickness of the shell; E_1, E_2, v_1, v_2 are elasticity moduli and Poisson coefficients in axial and angular directions $(E_1 v_2 = E_2 v_1)$; G is the shear modulus; t is time.

For the brevity of our writing we denote the left part by $F(w)$ and the right one by $T(w)$. Then equation (1.1) can be written as

$$F(w) = T(w).$$

In the sequel, we will mean long shells for which the condition

$$(\pi R/\ell)^2 \lesssim 12\varepsilon^{1/2}$$

is fulfilled.

Since the influence of the boundary conditions on the frequency characteristics of long shells is of no particular importance, we will take into account only principal boundary conditions

$$w(0, \varphi) = w(\ell/R, \varphi) = 0. \tag{1.3}$$

Thus, the problem of finding proper oscillations of pretwisted shell is reduced to finding nonzero solutions of equation (1.1) under the boundary conditions (1.3).

The solution for free harmonic oscillations will be sought in the form of a series

$$\begin{aligned} w &= \sum_{mn} w_{mn}(\xi, \varphi)\sin(\omega_{mn} t - \alpha), \\ w_{mn} &= \sin\lambda_m \xi\big[A_{mn}\sin n(\varphi - \gamma\xi) + B_{mn}\cos n(\varphi - \gamma\xi)\big], \end{aligned} \tag{1.4}$$

where $\lambda_m = m\pi R/\ell$. Due to the boundary conditions (1.3), the expansion (1.4) does not contain the terms of the type $A'_{mn}\cos\lambda_m\xi\sin n(\varphi - \gamma\xi)$ and $B'_{mn}\cos\lambda_m\xi\cos n(\varphi - \gamma\xi)$.

We represent the expression (1.4) as follows:

$$w = \sum_{mn}\bigg\{\frac{A_{mn}}{2}\big(\cos[n(\varphi - \gamma\xi) - \lambda_m\xi] - \cos[n(\varphi - \gamma\xi) + \lambda_m\xi]\big) +$$

$$+\frac{B_{mn}}{2}\big(\sin[n(\varphi-\gamma\xi)+\lambda_m\xi]-\sin[n(\varphi-\gamma\xi)-\lambda_m\xi]\big)\bigg\}\sin(\omega_{mn}t-\alpha). \tag{1.5}$$

Substituting (1.5) into (1.1), we get

$$\sum_{mn} = \Big\{\frac{A_{mn}}{2}\,F(n, -m)\cos\big[n(\varphi - \gamma\xi) - \lambda_m\xi\big] -$$

$$
\begin{aligned}
&- \frac{A_{mn}}{2} F(n,m) \cos\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big] + \\
&+ \frac{B_{mn}}{2} F(n,m) \sin\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big] - \\
&- \frac{B_{mn}}{2} F(n,-m) \sin\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]\Big\} \sin(\omega_{mn}t-\alpha)=0,
\end{aligned} \tag{1.6}
$$

$$
\begin{aligned}
F(n,\pm m) &= Q^{mn} - \Omega_2\omega_{mn}^2\big[d_2^{mn} - \Omega_2^*\omega_{mn}^2 p_2^{-1}(d_1^{mn} - \Omega_1^*\omega_{mn}^2)\big], \\
Q^{mn} &= d_3^{mn} - d_4^{mn},
\end{aligned} \tag{1.7}
$$

$$
\begin{aligned}
d_1^{mn} &= p_1p_2^{-1} + (1+p_2)B + p_1p_2^{-1}\varepsilon_*(\overline{\Delta}_1 - 2n^2 + 1) + 2\bar{t}_{12}^{\,0}\mu_\pm^2 + \\
&\quad + \bar{t}_{22}^{\,0}n^2 + 2\bar{s}_1^{\,0}\mu_\pm n, \\
d_2^{mn} &= \overline{\Delta}_2 + p_2^{-1}(1+p_2-v_1v_2)\mu_\pm^2 + p_1p_2^{-1}n^2 + \\
&\quad + (1+p_2)p_2^{-1}B\big[\varepsilon_*(\overline{\Delta}_1 - 2n^2+1) - (\bar{t}_{12}^{\,0}\mu_\pm + \bar{t}_{22}^{\,0}n^2 + 2\bar{s}_2^{\,0}\mu_\pm n)\big], \\
&\quad \mu_2 = -n\gamma \pm \lambda_m, \quad B = \mu_\pm^2 + d_4n^2, \\
d_3^{mn} &= \varepsilon\big(\overline{\Delta}_1\overline{\Delta}_2 - \{d_1\mu_\pm^6 + d_2\mu_\pm^4n^2 + d_3\mu_\pm^2n^4 + d_4(2n^6-n^4) - \\
&\quad - d_5\mu_\pm^2n^2\}\big) + \mu_\pm^4, \\
d_4^{mn} &= (t_{12}^0 - t_{22}^0)d_4\mu_\pm^2n^2 + t_{12}^0\mu_\pm^2\overline{\Delta}_2 + t_{22}^3(n^2-1)\overline{\Delta}_2 + \\
&\quad + 2s_2^0\mu_\pm n\big[\overline{\Delta}_2 - (g_1\mu_\pm^2 + d_4n^2)\big], \\
\overline{\Delta}_1 &= \delta_1(\mu_\pm^4 + \delta_1^{-1}\delta_2\mu_\pm^2n^2 + \delta_1^{-1}n^4), \quad \overline{\Delta}_2 = \mu_\pm^4 + \delta_3\mu_\pm^2n^2 + \delta_1^{-1}n^4.
\end{aligned} \tag{1.8}
$$

Transformation of (1.6) results in

$$
\begin{aligned}
\sum_{mn}\Big\{ &A_{mn}\big[F(n,-m) - F(n,m)\big]\cos\lambda_m\xi\cos n(\varphi-\gamma\xi) + \\
&+ A_{mn}\big[F(n,-m) + F(n,m)\big]\sin\lambda_m\xi\sin n(\varphi-\gamma\xi) + \\
&+ B_{mn}\big[F(n,m) - F(n,-m)\big]\sin\lambda_m\xi\cos n(\varphi-\gamma\xi) + \\
&+ B_{mn}\big[F(n,m) + F(n,-m)\big]\cos\lambda_m\xi\sin n(\varphi-\gamma\xi)\Big\}\sin(\omega_{mn}t-\alpha) = 0,
\end{aligned}
$$

whence

$$
\begin{aligned}
A_{mn}\big[F(n,-m) - F(n,m)\big] = 0, &\quad A_{mn}\big[F(n,-m) + F(n,m)\big] = 0, \\
B_{mn}\big[F(n,-m) - F(n,m)\big] = 0, &\quad B_{mn}\big[F(n,-m) + F(n,m)\big] = 0.
\end{aligned}
$$

Consequently, for any pair m, n there must exist the equality

$$
F(n,m) = 0, \quad F(n,-m) = 0. \tag{1.9}
$$

Thus, for a nontrivial solution of equation (1.1) under the boundary conditions (1.3), it is necessary and sufficient that there exist integers m and n satisfying the conditions (1.9).

The relation (1.9) represents the cubic equation with respect to ω^2 (the indices m and n are omitted),

$$\begin{gathered}\alpha^3 - a_1^{\pm}\alpha^2 + a_2^{\pm}\alpha - c^{\pm} = 0, \quad c^{\pm} = a_3^{\pm} - a_4^{\pm}, \quad \alpha = \Omega_2^*\omega^2, \\ a_1^{\pm} = d_1^{\pm}\frac{E_1}{E_2}, \;\; a_2^{\pm} = d_2^{\pm}p_2\frac{E_1}{E_2}, \;\; a_j^{\pm} = d_j^{\pm}p_2(1 - v_1v_2)\frac{E_1}{E_2} \;\; (j = 3, 4).\end{gathered} \tag{1.10}$$

For an isotropic load-free shell ($t_{1i}^0 = t_{2i}^0 = s_i^0$, $\gamma = 0$), the given equation is in a good agreement with the cubic equation (of hinged supported shell) given in [6].

The frequency equation allows us to define three different proper frequencies for every given pair m, n. Since the expansion coefficients of tangential displacements depend on the frequency, therefore the forms of proper oscillations for these frequencies differ.

Practically, of greatest interest are the lowest frequencies. Taking into account that $\Omega_i^* \ll 1$, $d_2^{\pm} > d_1^{\pm} > 1$ for the least root of equation (1.10) we obtain

$$\Omega_2^*\omega^2 = c^+/a_2^+, \quad \Omega_2^*\omega^2 = c^-/a_2^-. \tag{1.11}$$

In the case of axially symmetric oscillations ($n = 0$) we find that the lower frequency to within small values does not depend on prestresses.

Consider the case corresponding to “beam” type oscillations for $n = 1$. Taking into account the fact that for long shells $\mu_{\pm}^2 \ll 1$ (this inequality is always fulfilled for sufficiently long shells if m is not large), after a number of simplifications, on the basis of equality (1.11), we obtain

$$\Omega_1\omega^2 = \frac{\mu_+^2\{\mu_+^2 + (2d_4 + \delta_3\mu_+^2)t_{12}^0 - d_4t_{22}^0 + 2s_2^0\mu_+[(\delta_3 - g_1) + \mu_+^2]\}}{2(1 + \beta\mu_+^2)}, \tag{1.12}$$

$$\Omega_1\omega^2 = \frac{\mu_-^2\{\mu_-^2 + (2d_4 + \delta_3\mu_-^2)t_{12}^0 - d_4t_{22}^0 + 2s_2^0\mu_-[(\delta_3 - g_1) + \mu_-^2]\}}{2(1 + \beta\mu_-^2)}, \tag{1.13}$$

$$\mu_{\pm} = -\gamma \pm \lambda_m, \quad \Omega_1 = \Omega_2 E_2/E_1, \quad \beta = (\delta_3 - E_2G + 1)2E_2/E_1,$$

it is not difficult to see that account of tangential forces of inertia corresponds to the coefficient $1/2(1 + \beta\mu_{\pm})$, i.e., it decreases frequency by more than $1/\sqrt{2}$.

We introduce the notation

$$M = \delta_3 - g_1 = \frac{E_2}{E_1}\left(1 + \frac{v_1}{1 - v_1v_2}\,\frac{E_2 - E_1}{2G}\right). \tag{1.14}$$

and adopt the condition

$$M \gg \mu_{\pm}^2. \tag{1.15}$$

Since $\mu_{\pm}^2 \gg 1$, inequality (1.15) is, as usual, fulfilled for practically encountering shells (if $E_1 \gtrsim E_2$).

Taking into account (1.15), the expressions (1.11) and (1.12) take the form

$$2\Omega_1\omega^2 = (\gamma-\lambda_m)^4 + \big[2d_4(\gamma-\lambda_m)^2 + \delta_3(\gamma-\lambda_m)^4\big]t_{12}^0 - \\ - d_4(\gamma-\lambda_m)^2 t_{22}^0 - 2Ms_2^0(\gamma-\lambda_m)^3,$$

$$2\Omega_1\omega^2 = (\gamma+\lambda_m)^4 + \big[2d_4(\gamma+\lambda_m)^2 + \delta_3(\gamma+\lambda_m)^4\big]t_{12}^0 - \\ - d_4(\gamma+\lambda_m)^2 t_{22}^0 - 2Ms_2^0(\gamma+\lambda_m)^3.$$

Adding and subtracting the above equalities, we obtain

$$\Omega_1\omega^2 = \frac{1}{2}\big[(\gamma^4 + 6\gamma^2\lambda_m^2 + \lambda_m^4) + 2d_4(\gamma^2+\lambda_m^2)t_{12}^0 + \delta_3(\gamma^4 + \\ + 6\gamma^2\lambda_m^2 + \lambda_m^4)t_{12}^0 - d_4(\gamma^2+\lambda_m^2)t_{22}^0 - 2Ms_2^0\gamma(\gamma^2+3\lambda_m^2)\big], \quad (1.16)$$

$$2\gamma(\gamma^2+\lambda^2) + 2d_4\gamma t_{12}^0 + \delta_3 2\gamma(\gamma^2+\lambda_m^2)t_{12}^0 - \gamma d_4 t_{22}^0 - \\ - Ms_2^0(3\gamma^2+\lambda_m^2) = 0. \quad (1.17)$$

It is not difficult to see that the lower frequency takes place for $m = 1$.

Formula (1.16) with regard for (1.17) yields

$$\Omega_1\omega^2 = \frac{\lambda_1^2}{2}\,\frac{(1-\overline{\gamma}^{\,2})^3 + \bar{t}_{11}(1-\overline{\gamma}^{\,2})) + \delta_3 d_4^{-1}\bar{t}_{11}(1-\overline{\gamma}^{\,2})^3\lambda_1^2 - \bar{t}_{21}(1-\overline{\gamma}^{\,2})^2}{1+3\overline{\gamma}^{\,2}}\,,$$
$$d_4^{-1}\delta_3 = \frac{E_1}{G} - 2v_1. \quad (1.18)$$

Relying on (1.17), we obtain the equation

$$2\overline{\gamma}^{\,3} - 3\widetilde{s}\overline{\gamma}^{\,2} + (2 + \bar{t}_{11} - \bar{t}_{21})\gamma - \widetilde{s} = 0, \quad (1.19)$$

where we introduced the notation

$$\overline{\gamma} = \gamma/\lambda_1, \quad \widetilde{s} = M\overline{s}, \quad \overline{s} = s_2^0/\lambda_1, \quad \bar{t}_{11} = 2t_{11}^0/\lambda_1, \quad \bar{t}_{21} = t_{21}^0/\lambda_1. \quad (1.20)$$

Neglecting in the numerator of (1.18) the third term, smaller than the second one, we have

$$\Omega_1\omega^2 = \frac{(1-\overline{\gamma}^{\,2})^3 + \bar{t}_{11}(1-\overline{\gamma}^{\,2})^2 - \bar{t}_{21}(1-\overline{\gamma}^{\,2})^2}{1+3\overline{\gamma}^{\,2}}\,. \quad (1.21)$$

For $\widetilde{s} = 0$, from formula (1.19) we arrive at $\gamma = 0$, and on the basis of formula (1.21) we have $\Omega_1\omega^2 = 0{,}5\lambda_1^4(1 + \bar{t}_{11} - \bar{t}_{21})$, whence $\omega = 0$ for $\bar{t}_{11} - \bar{t}_{21} = -1$. For $\bar{t}_{21} = 0$, we obtain $\bar{t}_{11} = -1$, whereas for $\bar{t}_{11} = 0$ we get $\bar{t}_{21} = 1$. These equalities for an isotropic shell are the well-known formulas [7].

For $\overline{\gamma} = 1$, on the basis of formula (1.21), we obtain $\Omega_1\omega^2 = 0$, and on the basis of (1.19), we obtain a critical load $\widetilde{s}_* = 1 + (\bar{t}_{11} - \bar{t}_{21})/4$. Thus, for $\bar{t}_{11} = \bar{t}_{21} = 0$, we have $\widetilde{s}_* = 1$. The obtained formula in an expanded form looks as the equality

$$\tau_*^0 = E_1\left(1 + \frac{v_1}{1 - v_1 v_2}\,\frac{E_2 - E_1}{2G}\right)^{-1}\lambda_1, \quad \tau_*^0 = S_*^0/h. \tag{1.22}$$

For an isotropic case, formula (1.22) agrees completely with Timoshenko-Greenhill's formula [5] and represents the formula for critical torques of a hinged supported beam with a ring cross-section.

Moreover, it should be noted that since for $s = 0$ we have $\gamma = 0$, the above-given solution (1.4) transforms into the solution of a hinged supported shell, and on the basis of formula (1.21), for $\bar{t}_{11} = \bar{t}_{21} = 0$ we have $\Omega_1\omega^2 = 0,5\lambda_1^4$, i.e, we obtain the well-known formula for the load-free long shell, coinciding with the formula of lower frequency of hinged fixed beam with a ring cross-section [6].

Taking into account the fact that the loading of the shell with torques can be realized only within the value no more than its critical value ($\widetilde{s} < \widetilde{s}_*$), we find that the discriminant of the cubic equation (1.19) is $D > 0$. Consequently, we have one real solution

$$\begin{gathered}\bar{\gamma} = u_1 + u_2 + 0,5\widetilde{s}, \quad u_{1,2} = (-q \pm \sqrt{D})^{1/3}, \quad D = q^2 + p^s,\\ q = -\widetilde{s}\big(\widetilde{s}^{\,2} - \bar{t}_{11} + \bar{t}_{21}\big)/2^3, \quad p = \Big[\frac{2}{3}(2 + \bar{t}_{11} - \bar{t}_{21}) - \widetilde{s}^{\,2}\Big]/2^2.\end{gathered} \tag{1.23}$$

Finding on the basis of equality (1.23) the values $\bar{\gamma}$ for the given $\widetilde{s}, \bar{t}_{11}, \bar{t}_{21}$ and substituting them into (1.21), we obtain the corresponding value ω.

In Fig. 1 we can see dimensionless values $\bar{\gamma}$ and $\bar{\omega}$ of $\widetilde{s}$ ($t_{11} = t_{21} = 0$), where $\bar{\omega} = \sqrt{\Omega_1}\,\omega(0,5\lambda_1^4)^{-1/2}$.

Moreover, we notice that the parameter $\bar{\gamma}$ for $0 \le \widetilde{s} \le \widetilde{s}_*$ varies in the interval $0 \le \bar{\gamma} \le 1$, and hence the relation $\mu_\pm^2 \ll 1$ is completely valid.

Consider now the case ($t_{11} = 0$, $t_{21} \neq 0$, $s_2 \neq 0$), where

$$n \ge 2,$$

that is we consider bending oscillations when cross-sections of the shell is of star-shaped form under oblique wave formation.

Taking into account that for long shells

$$n^2 \gg \mu_\pm^2 \tag{1.24}$$

the relation (1.11) looks as follows:

$$\Omega_2\omega^2 = \frac{\varepsilon n^4(n^2-1)^2 + d_4^{-1}\mu_+^4 + t_{22}^0 n^4(n^2-1) + 2s_2^0\mu_+ n^3(n^2-1)}{n^2(n^2+1)},$$

$$\Omega_2\omega^2 = \frac{\varepsilon n^4(n^2-1)^2 + d_4^{-1}\mu_-^4 + t_{22}^0 n^4(n^2-1) + 2s_2^0\mu_- n^3(n^2-1)}{n^2(n^2+1)},$$

whence we have

$$\Omega_2\omega^2 = \frac{\varepsilon n^2(n^2-1)^2}{n^2+1} + \frac{(n\gamma)^4 + 6(n\gamma)^2\lambda_m^2 + \lambda_m^4}{d_4 n^2(n^2+1)} - \\ - \frac{2s_2^0 n\gamma n(n^2-1)}{n^2+1} + \frac{t_{22}^0 n^2(n^2-1)}{n^2+1}\,, \tag{1.25}$$

$$(n\gamma)^3 + \lambda_m^2 n\gamma - \frac{1}{2}\,s_1^0 n^3(n^2-1) = 0. \tag{1.26}$$

By virtue of formula (1.25), it is not difficult to see that the lower frequency depending on m is realized for $m = 1$.

We simplify the expression (1.25), take into account equality (1.26) and obtain

$$\Omega_1\omega^2 = \frac{d_4\varepsilon n^4(n^2-1)^2 + \lambda_1^4 + 2\lambda_1^2(n\gamma)^2 - 3(n\gamma)^4 + d_4 t_{22}^2 n^4(n^2-1)}{n^2(n^2+1)}\,. \tag{1.27}$$

Introduce the parameters α_1 and α_2,

$$E_1 = \alpha_1 E, \quad E_2 = \alpha_2 E \tag{1.28}$$

and denote

$$s^0 = S^0/Eh, \quad k_1 = s^0/s_*^0, \quad k_2 = t^0/t_*, \quad t^0 = T_2^0/Eh,$$

where

$$s_* = 12\varepsilon^{3/4}/3\sqrt{2}, \quad t_* = 3\varepsilon, \quad \Omega\omega_\infty^2 = 36\varepsilon/5, \quad \Omega = \rho R^2/E. \tag{1.29}$$

Then $s_1^0 = \alpha_1^{-1}k_1 s_*$, $d_4 t_{22}^0 = \alpha_1^{-1}k_2 t_*$, and the relation (1.27) takes the form

$$\frac{\omega^2}{\omega_\infty^2} = \\ = \frac{5\varepsilon^{-1}}{36}\left\{\frac{\alpha_2\varepsilon n^4(n^2-1)^2 + \alpha_1[\lambda_1^4 + 2\lambda_1^2(n\gamma)^2 - 3(n\gamma)^4] + k_2 t_* n^4(n^2-1)}{n^2(n^2+1)}\right\}. \tag{1.30}$$

Denoting $n\gamma = x$, (1.26) takes the form

$$x^3 + 3px + 2q = 0, \quad p = \lambda_1^2/3, \quad q = -\alpha_1^{-1}k_1 n^3(n^2-1)s_*/4. \tag{1.31}$$

Since the discriminant of equation (1.31) is $D > 0$, we have one real solution

$$x = u_1 + u_2, \quad u_{1,2} = \big(-q \pm \sqrt{q^2+p^3}\big)^{1/3}. \tag{1.32}$$

Relying on the above solution, for the given $\alpha_1, \alpha_2, v_1, k_1, k_2, n$ we obtain corresponding values $n\gamma$. Substituting the values $n\gamma$ into formula (1.30), we obtain ω.

For $n = 2$, formula (1.30) takes the form

$$\frac{\omega^2}{\omega_\infty^2} = \alpha_2 + k_2 + \frac{\alpha_1\varepsilon^{-1}}{144}\big[\lambda_1^4 + 2\lambda_1^2(n\gamma)^2 - 3(n\gamma)^4\big]. \tag{1.33}$$

In particular, for $s^0 = 0$, on the basis of (1.31) and (1.33), we get

$$\gamma = 0, \quad \omega^2/\omega_\infty^2 = \alpha_2 + k_2 + (\alpha_1 \lambda_1^4 \varepsilon^{-1}/144).$$

Introduce a geometric parameter β,

$$\left(\frac{\pi R}{\ell}\right)^2 = \beta \varepsilon^{1/2}$$

and consider the shells for different β. Note that formula (1.33) takes the form

$$\omega^2/\omega_\infty^2 = \alpha_2 + k_2 + \frac{\alpha_1}{144}\left[\beta^2 + 2\beta\varepsilon^{-1/2}(n\gamma)^2 - 3\varepsilon^{-1}(n\gamma)^4\right].$$

Thus, it is not difficult to see that the influence of the parameter α_2 on the frequency ω does not depend on pretwisting. Investigate the influence of the parameter α_1. Figs. 2 and 3 show the curves $n\varphi$ and ω of variation depending on the prestress $k_1 = s^0/s_*$, $k_2 = 0$, when $\beta = 0,5; 1; 2$, for two cases $\alpha_1 = \alpha_2 = 1$ and $\alpha_1 = 2$, $\alpha_2 = 1$ (broken curve). On the basis of these charts, we can easily see that if for small torque values k_1, the influence of the elastic parameter in the axial direction α_1 on the lower frequency ω is comparatively small, then as k_1 increases the influence of the parameter α_1 on ω increases significantly (for $s \leq 1$). Moreover, for comparison, for these cases in Fig. 3 are drawn the curves A [2] (without regard for the boundary condition). Along the Oy-axis, we have dimensionless frequency ω^2/ω_∞^2 ($\omega_\infty^2 = 36\Omega^{-1}\varepsilon/5$) and along the Ox-axis there is a dimensionless value s^0/s_*. It is easy to see that as β decreases these curves approach the corresponding curves A, that is, the boundary conditions for sufficiently long shells ($\beta < 0.5$) are practically of no importance for the lower frequency. Moreover, it should be noted that these curves for $\alpha_1 > 1$ (as elastic parameter E_1 increases) stray away from the corresponding curve A. Critical loads look as the points of intersection of curves with the Ox-axis, showing dependence of frequency on the load. Note that for sufficiently long isotropic shells (for $\beta \leq 0,5$, $t_{22} = 0$) by virtue of equalities (1.25) and (1.26) it is not difficult to show that $\omega = 0$ for $s_* = 4\varepsilon^{3/4}/\sqrt{2}$, i.e., we obtain the well-known Timoshenko's formula [5].

For the shells $\beta < 0,5$. $4,5\alpha_1^{-2}k_1^2 n^6(n^2-1)^2 \gg 1$ for $n = 2$, we obtain $q^2 \gg p^3$. Then, on the basis of (1.32), we have

$$n\gamma = \left[0,5\alpha_1^{-1}k_1 s_* n^3(n^2-1)\right]^{1/3} \tag{1.34}$$

whence for $n = 2$ we find that $n\gamma = 2,632\alpha_1^{-1/3}k_1^{1/3}\varepsilon^{1/4}$. Analogously, on the basis of (1.35), it is easy to find the values $n\gamma$ for $n = 3, 4, \ldots$,

$$n\gamma = 5,475\alpha_1^{-1/3}k_1^{1/3}s^{1/4}\ (n = 3), \quad n\gamma = 9,002\alpha_1^{-1/3}k_1^{1/3}\varepsilon^{1/4}\ (n = 4).$$

Substituting these values n and $n\gamma$ for fixed $k_1, k_2, \alpha_1, \alpha_2, v_1$ into (1.30), we obtain the corresponding value for the frequency ω.

For the cases under consideration, in Fig. 4 we can see the curves of frequency variation depending on s/s_* for $n = 3, 4, 5$. On the basis of these curves it is not difficult to notice that the influence of an elastic parameter in the axial direction for the cases $n = 3, 4$ (for $s \leq 1$) is comparatively not great, whereas for $n = 5, 6, \dots$ it is practically inessential, and the curves for $\beta = 0.5, 1, 2$ merge practically with the corresponding curves without regard for the boundary conditions [2], i.e., unlike the lower frequency, the influence of the boundary conditions here is, as was to be expected, much lesser.

Note that our investigation covers likewise the shells of middle length when the condition (1.24) is fulfilled, because the above-given solution for the tending to zero torques ($s \to 0$) transforms into that corresponding to a hinged supported shell, while in the other limiting state of greatest influence of torques on the lower frequency as $\omega \to 0$, we obtain critical torques which are very good approximation for critical torques of hinged supported shells (in particular, for an isotropic case see [4]).

Thus, we have investigated the influence of elastic orthotropic parameters both on the lower and on the higher frequencies for pretwisted long cylindrical shells and also showed essential influence of pretorques on the lower frequencies of long orthotropic shells and comparatively weak influence of the higher frequencies.

2. Forced Oscillations. We now investigate the problem of forced oscillations of a long orthotropic pretwisted cylindrical shell. As a disturbing motive, we consider an arbitrary normal load varying in time according to the harmonic law, with a ring frequency k,

$$P^*(\xi, \varphi, t) = P(\xi, \varphi) \sin kt. \tag{2.1}$$

The corresponding equation is of the form

$$F(w) = T(w) + p_2^{-1} L(p), \quad p = P^* R^2 / E_2 h, \tag{2.2}$$

$$p_2^{-1} L = \Delta_2 - p_2^{-1} \Omega_1 \frac{\partial^2}{\partial t^2} \Big[\Big(\ell_1 \frac{\partial^2}{\partial \xi^2} + \ell_2 \frac{\partial^2}{\partial \varphi^2} \Big) - \Omega_2 \frac{\partial^2}{\partial t^2} \Big],$$

$$\ell_1 = \frac{E_1}{E_2} + p_2, \quad \ell_2 = 1 + p_2, \quad p_2 = (1 - v_1 v_2) \frac{G}{E_2}\,.$$

It is assumed that $p(\xi, \varphi)$ can be expanded in Fourier series

$$p(\xi, \varphi) = \sum_{mn} p_{mn}(\xi, \varphi), \tag{2.3}$$

$$p_{mn} = \sin \lambda_m \big[C_{mn} \sin n(\varphi - \gamma \xi) + D_{mn} \cos n(\varphi - \gamma \xi) \big].$$

A solution of equation (2.2) will be sought in the form

$$w = \sin kt \sum w_{mn}(\xi, \varphi), \tag{2.4}$$

where w_{mn} is of the form (1.4).

In connection with the forthcoming calculations, we represent $p(\xi,\varphi)$ as follows:

$$p(\xi,\varphi)=\sum_{mn}\frac{C_{mn}}{2}\Big(\cos\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]-\cos\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]+ \\ +\frac{D_{mn}}{2}\Big(\sin\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]-\sin\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]\Big).$$

Substituting (2.1), (2.3) and (2.4) into equation (2.2) and contracting by $\sin kt$, we obtain

$$\sum_{mn}\Big\{A_{mn}F(n,-m)\cos\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]- \\ -A_{mn}F(n,m)\cos\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]+ \\ +B_{mn}F(n,m)\sin\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]- \\ -B_{mn}F(n,-m)\sin\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]\Big\}= \tag{2.5}$$

$$=\sum_{mn}\Big\{C_{mn}f(n,-m)\cos\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]- \\ -C_{mn}f(n,m)\cos\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]+ \\ +D_{mn}f(n,m)\sin\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]- \\ -D_{mn}f(n,m)\sin\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]\Big\}, \tag{2.6}$$

$$F(n\pm m)=Q^{mn}-\Omega_2k^2\big[d_2^{mn}-\Omega_2^*k^2p_2^{-1}(d_1^{mn}-\Omega_2k^2)\big],\ \Omega_i=\rho R^2/E_i, \tag{2.7}$$

$$f(n\pm m)=\overline{\Delta}_2+p_2^{-1}\Omega_1k^2\big[(\ell_1\mu_\pm^2+\ell_2n^2)-\Omega_2^*k^2\big],\ \Omega_i^*=\Omega_i(1-v_1v_2), \tag{2.8}$$

where d_j^{mn} $(j=1,\dots,4)$ represents the above-given expression (1.8).

We reduce the relation (2.6) to the form

$$\sum_{mn}\Big\{A_{mn}\big[F(n,-m)-F(n,m)\big]\cos\lambda_m\xi\cos n(\varphi-\gamma\xi)+ \\ +A_{mn}\big[F(n,-m)+F(n,m)\big]\sin\lambda_m\xi\sin n(\varphi-\gamma\xi)+ \\ +B_{mn}\big[F(n,m)-F(n,-m)\big]\sin\lambda_m\xi\cos n(\varphi-\gamma\xi)+ \\ +B_{mn}\big[F(n,m)+F(n,-m)\big]\cos\lambda_m\xi\sin n(\varphi-\gamma\xi)\Big\}= \\ =\sum_{mn}\Big\{C_{mn}\big[f(n,-m)-f(n,m)\big]\cos\lambda_m\xi\cos n(\varphi-\gamma\xi)+ \\ +C_{mn}\big[f(n,-m)+f(n,m)\big]\sin\lambda_m\xi\sin n(\varphi-\gamma\xi)+ \\ +D_{mn}\big[f(n,m)-f(n,-m)\big]\sin\lambda_m\xi\cos n(\varphi-\gamma\xi)+ \\ +D_{mn}\big[f(n,m)+f(n,-m)\big]\cos\lambda_m\xi\sin n(\varphi-\gamma\xi)\Big\}.'$$

whence it follows that for all m and n,

$$\begin{aligned}
&A_{mn}\big[F(n,-m)-F(n,m)\big]-C_{mn}\big[f(n,-m)-f(n,m)\big]=0,\\
&A_{mn}\big[F(n,-m)+F(n,m)\big]-C_{mn}\big[f(n,-m)+f(n,m)\big]=0,\\
&B_{mn}\big[F(n,m)-F(n,-m)\big]-D_{mn}\big[f(n,m)-f(n,-m)\big]=0,\\
&B_{mn}\big[F(n,m)+F(n,-m)\big]-D_{mn}\big[f(n,m)+f(n,-m)\big]=0.
\end{aligned} \tag{2.9}$$

The first two equations (2.9) yield

$$A_{mn}F(n,-m)-C_{mn}f(n,-m)=0,$$
$$A_{mn}F(n,m)-C_{mn}f(n,m)=0,$$

whence

$$A_{mn}=C_{mn}\,\frac{f(n,m)}{F(n,m)}\,,\quad A_{mn}=C_{mn}\,\frac{f(n,-m)}{F(n,-m)}\,.$$

Adding and subtracting the above equalities, we get

$$\begin{aligned}
A_{mn}&=\frac{1}{2}\,C_{mn}\left[\frac{f(n,m)}{F(n,m)}+\frac{f(n,-m)}{F(n,-m)}\right],\\
C_{mn}&\left[\frac{f(n,m)}{F(n,m)}-\frac{f(n,-m)}{F(n,-m)}\right]=0,\quad C_{mn}\neq 0.
\end{aligned} \tag{2.10}$$

from which it follows that

$$\frac{f(n,m)}{F(n,m)}=\frac{f(n,-m)}{F(n,-m)}\,,\quad A_{mn}=C_{mn}\,\frac{f(n,m)}{F(n,m)}\,. \tag{2.10$'$}$$

Analogously, the third and fourth equations (2.9) result in

$$B_{mn}=D_{mn}\,\frac{f(n,m)}{F(n,m)}\,. \tag{2.11}$$

Substituting (2.10′) and (2.11) into (2.4), we obtain the particular solution of the given inhomogeneous equation

$$\begin{aligned}
w=\sin kt\sum_{mn}\frac{f(n,m)}{F(n,m)}\,\sin\lambda_m\xi\big[&C_{mn}\sin n(\varphi-\gamma\xi)+\\
&+D_{mn}\cos n(\varphi-\gamma\xi)\big].
\end{aligned} \tag{2.12}$$

For $k=\omega_{mn}$, we have $F(n,\pm m)=0$. Consequently, $w\to\infty$, i.e., there takes place the resonance phenomenon when frequency of forced oscillations k coincides with one of the proper frequencies ω_{mn} of a prestressed shell.

In the expression (2.12), C_{mn} and D_{mn} are the Fourier coefficients for the series (2.3) expansion; they are of the form

$$\begin{aligned}
C_{mn}=\frac{2}{\pi\bar\ell}\int\limits_0^{\bar\ell}\int\limits_0^{2\pi}&p(\xi,\varphi)\sin\lambda_m\xi(\cos n\gamma\xi\sin n\varphi-\\
&-\sin n\gamma\xi\cos n\varphi)d\xi\,d\varphi,
\end{aligned}$$

$$D_{mn} = \frac{2}{\pi\overline{\ell}} \int\limits_0^{\overline{\ell}} \int\limits_0^{2\pi} p(\xi,\varphi) \cos\lambda_m\xi (\cos n\gamma\xi \sin n\varphi + \\ + \sin n\gamma\xi \sin n\varphi) d\xi\, d\varphi, \quad \overline{\ell} = \ell/R.$$

The function $p(\xi,\varphi)$ is assumed to be regular enough for the series (2.12) and for those obtained by differentiation to be uniformly convergent.

As an example, we consider the case where the shell is loaded with the concentrated force P applied to an arbitrary point (ξ_1,φ_1). Around the point (ξ_1,φ_1) we single out an element of the cylindrical surface with the vertex at the points $(\xi_1-\varepsilon,\varphi_1+\eta)$, $(\xi_1+\varepsilon,\varphi_1+\eta)$, $(\xi_1-\varepsilon,\varphi_1-\eta)$, $(\xi_1+\varepsilon,\varphi_1-\eta)$. We replace the force P by a continuous load q distributed over the above-mentioned elementary surface, then $q = \frac{P}{4R^2\varepsilon\eta}$ and

$$C_{mn} = \frac{2}{\pi\overline{\ell}} \int\limits_{\xi_1-\varepsilon}^{\xi_1+\varepsilon} \int\limits_{\varphi_1-\eta}^{\varphi_1+\eta} \overline{q} \sin\lambda_m\xi (\cos n\gamma\xi \sin n\varphi - \\ - \sin n\gamma\xi \cos n\varphi) d\xi\, d\varphi, \quad \overline{q} = qR^2/E_2h.$$

Turning this element to the point (ξ_1,φ_1), we have

$$\lim_{\varepsilon,\eta\to 0} C_{mn} = \frac{P}{2\pi\ell R} \lim_{\varepsilon,\eta\to 0} \left(\int\limits_{\varphi_1-\eta}^{\varphi_1+\eta} \frac{\sin n\varphi\, d\varphi}{\eta} \int\limits_{\xi_1-\varepsilon}^{\xi_1+\varepsilon} \frac{\sin\lambda_m\xi \cos n\gamma\xi\, d\xi}{\varepsilon} - \\ - \int\limits_{\varphi_1-\eta}^{\varphi_1+\eta} \frac{\cos n\varphi\, d\varphi}{\eta} \int\limits_{\xi_1-\varepsilon}^{\xi_1+\varepsilon} \frac{\sin\lambda_m\xi \sin n\gamma\xi\, d\xi}{\varepsilon} \right),$$

$$\lim_{\varepsilon\to 0} \int\limits_{\varphi_1-\varepsilon}^{\varphi_1+\varepsilon} \frac{\sin n\varphi\, d\varphi}{\eta} \int\limits_{\xi_1-\varepsilon}^{\xi_1+\varepsilon} \frac{\sin\lambda_m\xi \cos n\gamma\xi\, d\xi}{\varepsilon} = \\ = -\frac{1}{2}\left[\frac{\cos\Lambda^-(\xi_1+\varepsilon) - \cos\Lambda^+(\xi_1-\varepsilon)}{\Lambda^-\varepsilon} + \frac{\cos\Lambda^+(\xi_1+\varepsilon) - \cos\Lambda^+(\xi_1-\varepsilon)}{\Lambda^-\varepsilon} \right] = \\ = \sin\Lambda^-\xi_1 + \sin\Lambda^+\xi_1, \quad \Lambda^- = \lambda_m - n\gamma, \quad \Lambda^+ = \lambda_m + n\gamma,$$

$$\lim_{\eta\to 0} \int\limits_{\varphi_1-\eta}^{\varphi_1+\eta} \frac{\sin n\varphi\, d\varphi}{\eta} = 2\sin n\varphi_1 \lim_{\eta\to 0} \frac{\sin\eta n}{\eta n} = 2\sin n\varphi_1.$$

The remaining integrals are calculated in a similar way. As a result, we obtain

$$C_{mn} = \frac{2P}{\pi\ell R} \big(\sin\Lambda^-\xi_1 + \sin\Lambda^+\xi_1\big) \sin n\varphi_1 - \big(\sin\Lambda^-\xi_1 - \sin\Lambda^+\xi_1\big) \cos n\varphi_1 =$$

$$= \frac{2P}{\pi \ell R} \big(\sin \lambda_m \xi_1 \cos n\gamma \xi_1 \sin n\varphi_1 + \cos \lambda_m \xi_1 \sin n\gamma \xi_1 \cos n\varphi_1 \big).$$

Just analogously, we have

$$D_{mn} = \frac{2P}{\pi \ell R} \big(\sin \lambda_m \xi_1 \cos n\gamma \xi_1 \cos n\varphi_1 - \cos \lambda_m \xi_1 \sin n\gamma \xi_1 \sin n\varphi_1 \big).$$

In particular, for $\gamma = 0$,

$$C_{mn} = \frac{2P}{\pi \ell R} \sin \lambda_m \xi_1 \sin n\varphi_1, \quad D_{mn} = \frac{2P}{\pi \ell R} \sin \lambda_m \xi_1 \cos n\varphi_1.$$

Thus, we have considered the action of steady disturbing load without regard for free oscillations (free oscillations were assumed to be damping).

For the joint action of free and forced oscillations, it is necessary to combine the corresponding solutions.

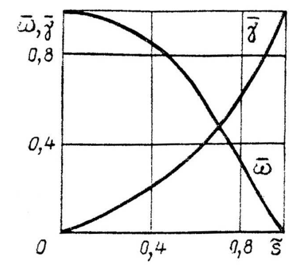

Figure 1

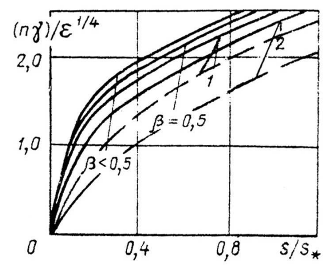

Figure 2

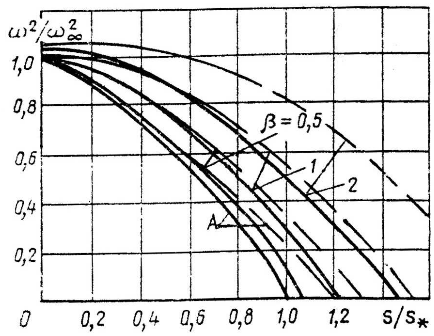

Figure 3

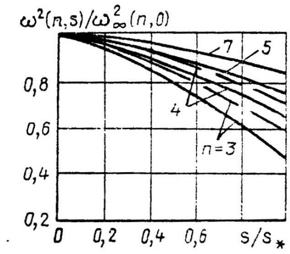

Figure 4

References

1. S. Kukudzhanov, On proper oscillations of prestressed orthotropic cylindrical shells. *Proc. A. Razmadze Math. Inst.* **149** (2009), 17–33.
2. S. N. Kukudzhanov, Oscillations of long cylindrical shells under the action of torques. (Russian) *Trudy VI Vsesoyuzn. Konf. po Teorii Obolochek i Plastin. Moscow, Nauka*, 1966, 522–527.
3. A. V. Pogorelov, Cylindrical shells under supercritical deformations. Twisting. (Russian) *Kharkov Univ.* 1962.
4. P. E. Tovstik, To the question on the stability of a cylindrical shell under twisting. (Russian) *Applied Mechanics* **16** (1980), No 9, 132–134.
5. S. P. Timoshenko, Stability of elastic systems. (Russian) *Moscow-Leningrad, Gostekhizdat* 1946.
6. V. Flugge, Statics and dynamics of shells. (Russian) *Stroiizdat, Moscow*, 1961.
7. E. I. Grigoljuk and V. V. Kabanov, Stability of shells. (Russian) *Nauka, Moscow*, 1978.

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