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ON THE GEOMETRY OF CURVES AND FUNCTIONS

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Abstract. This survey presents some comparatively recent author's results related to large classes of curves, real and complex functions. The ideas, methods and problems come from our preceding studies related to Gamma-lines. However, they also touch other topics in pure mathematics: integral geometry, Nevanlinna-Ahlfors theories in complex analysis, and Hilbert problem 16 (part 1) in the real algebraic geometry. Also, the results in Section 3 can be considered as certain assertions in different fields of physics.

რეზიუმე. წარმოდგენილ მიმოხილვაში გადმოცემულია ავტორის ბოლოდროინდელი შედეგები წირთა, ნამდვილ და კომპლექსურ ფუნქციათა ფართო კლასების შესახებ. განვითარებული იდეები და მეთოდები სათავეს იღებს ავტორის ადრინდელ გამოკვლევებში გამა-წირების შესახებ. ამავე დროს ნაშრომში განხილულია წმინდა მათემატიკის სხვა საკითხებიც ისეთი, როგორიცაა: ინტეგრალური გეომეტრია, ნევანლინა-ალფორსის თეორია, ჰილბერტის მე-16 პრობლემა ნამდვილ ალგებრულ გეომეტრიაში. ნაშრომის მე-3 პარაგრაფში განხილულია საკითხები ფიზიკის სხვადასხვა დარგიდან.

1. Point-Domain Inequality

In what follows D is a domain with piecewise smooth boundary ∂D, $\bar{D} := D \cup \partial D$; $S(D)$ is the area of D, $l(\partial D)$ is the length of the boundary ∂D.

First, we give an inequality which can be equally attributed to geometry and analysis.

Let $z_1, z_2, \dots, z_n$ be a finite set of arbitrary complex numbers lying in a domain D, $m_1, m_2, \dots, m_n$ a set of arbitrary positive numbers. Denote by $\operatorname{dist}(z_\nu, \partial D)$ the distance between the points z_ν and ∂D.

Point-domain inequality (announced in [9]**):**

$$\sum_{\nu=1}^{n} m_\nu \operatorname{dist}(z_\nu, \partial D) \le \frac{1}{4} \iint\limits_D \left| \sum_{\nu=1}^{n} \frac{m_\nu}{z - z_\nu} \right| d\sigma. \tag{1.1}$$

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Taking $z_1 = z_2 = \cdots = z_n = 0$, $m_1 = m_2 = \cdots = m_n = 1$ and $D := \{z : |z| < 1\}$ we have

$$\sum_{\nu=1}^{n} m_\nu \operatorname{dist}(z_\nu, \partial D) = n, \quad \frac{1}{4} \iint\limits_{D} \left| \sum_{\nu=1}^{n} \frac{m_\nu}{z - z_\nu} \right| d\sigma = \frac{\pi}{2}\, n,$$

so that (1.1) gives asymptotically correct growth when $n \to \infty$.

2. Complex Functions

2.1. **An inequality for the derivatives of arbitrary meromorphic function in a given domain.** In what follows we denote by D a bounded domain with piecewise smooth boundary whose intersection with any line consists of finite number of intervals.

Theorem 2.1 ([11], section 4)**.** *For any meromorphic function f in $\bar{D}$ and any integer $k \geq 1$,*

$$\iint\limits_{D} \left| \frac{f'(z)}{f(z)} \right| d\sigma \leq \iint\limits_{D} \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| d\sigma + \frac{k\pi}{2}\, l(\partial D). \tag{2.1}$$

For any collection of pairwise different points a_ν, $\nu = 1, 2, \ldots, q$,

$$\sum_{\nu=1}^{q} \iint\limits_{D} \left| \frac{f'}{f - a_\nu} \right| dy\, dx \leq \iint\limits_{D} \left| \frac{f^{(k+1)}}{f^{(k)}} \right| dy\, dx + \frac{k\pi^2 q}{\rho} \iint\limits_{D} |f'|\, dy\, dx + \frac{k\pi}{2}\, l(\partial D), \tag{2.2}$$

where ρ is the minimal distance between the a_ν's.

Sharpness. For function $f(z) = \exp z$ in the disk $|z| < r$ we have $\iint\limits_{D} \left| \frac{f'(z)}{f(z)} \right| d\sigma = 2\pi r^2$, $\iint\limits_{D} \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| d\sigma = 2\pi r^2$ and $l(\partial D) = 2\pi r$ so that the ratio of the left and the right sides in (2.1) tends to 1 when $r \to \infty$. This means that (2.1) is asymptotically sharp when $r \to \infty$.

Clearly (2.2) is an analogue of Nevanlinna theorem. The case $k = 1$ was considered in [4].

2.2. **An equality (invariance) for arbitrary meromorphic function in a given domain. A complement to the Ahlfors theory of covering surfaces and a geometric interpretation of deficient values.** For a given meromorphic in D function $w(z)$ and a given complex values a we make use well known characteristics: $n(D, a)$ is the number of a-points of w in D (with counting multiplicities), $A(D)$ is Ahlfors-Simizu characteristic, $L(D)$ is the spherical length of $w(\partial D)$, that is $A(D) := \frac{1}{\pi} \iint_D \frac{|w'|^2}{(1+|w|^2)^2}\, d\sigma$ and $L(D) := \frac{1}{2\pi} \int_{\partial D} \frac{|w'|}{1+|w|^2}\, ds$. Also we need the following

Definition. Let $w(z)$ be a meromorphic function in $\bar{D}$ and $a \in \mathbb{C}$. Assume (for simplicity) that $w(z)$ does not have any multiple points and a-points on the boundary ∂D. Consider the part of the boundary $\{w(z) : z \in \partial D\}$, which is over the disk $|w-a| < 1$ for $a \neq \infty$ and over $|w| > 1$ for $a = \infty$. This set is a union of a collection of curves γ_a. We denote by $2\pi\nu_{\gamma_a}$ the increment of $\arg(1/(w(z)-a)$ on γ_a, in the case when $a \neq \infty$, and the increment of $\arg w(z)$ in the case when $a = \infty$. Further, we denote

$$\nu(D,a) = \nu(D,a,w) = \sum_{(\gamma_a)} [\nu_{\gamma_a}]',$$

where $[x]'$ is the entire part of x and the sum is taken over all γ_r. We will refer $\nu(D,a)$ as *number of windings* around a.

Theorem 2.2 ([4], see also [5], Section 2.2)**.** *For any meromorphic function $w(z)$ in $\bar{D}$ non admitting multiple points and a-points of w on the boundary ∂D we have*

$$\nu(D,a) + n(D,a) = A(D) + hL(D), \tag{2.3}$$

where $|h| < h(a) = const < \infty$.

Comment 2.1. A complement to the Ahlfors theory. Here we assume that the reader is familiar with the key notations and results of Nevanlinna value distribution theory [20] (1920 s) and Ahlfors theory of covering surfaces [1] (1935). Obviously (2.3) reflects an invariance (with respect to a) analogous to the first fundamental theorem of Nevanlinna, that is $m(r,a) + N(r,a) + T(r) + O(1)$. In [1] Ahlfors obtained his first fundamental theorem, that is $\mu(D,G) + n(D,G) = A(D) + hL(D)$, where G is a domain occurring instead of a. Also he discusses in [1] (see also [20], chapter 13) why his result (for the domains G) does not lead to an invariance of type (2.3) (for the value a). He mentions that when G tends to a point a the magnitude $n(D,G)$ tends to $n(D,a)$, however, the magnitude $\mu(D,G)$ become meaningless for G tending to a. Thus, one needs to introduce a new characteristic which could play in Ahlfors theory a role analogous to that of Nevanlinna approximation function $m(r,a)$. Thus we can see that $\nu(D,a)$ fulfills a role of similar characteristic and respectively (2.3) plays a role of the first theorem in Ahlfors theory for the values a.

Comment 2.2. A geometric interpretation of deficient values. In the case of meromorphic functions in the complex plane the deficiencies in Nevanlinna and Ahlfors theories are defined, respectively, as $\liminf_{r\to\infty} \frac{m(r,a)}{T(r)}$ and $\liminf_{r\to\infty} \frac{\mu(D_r,G)}{A(D_r)}$, where $D_r := \{z : |z| < r\}$. Naturally, the number of windings leads to the following *geometric deficiency*: $\bar{\delta}(a) := \liminf_{r\to\infty} \frac{\nu(D_r,G)}{A(D_r)}$. It plays quite a similar role and we can, at least in qualitative reasonings, think on deficiencies in terms of $\nu(D_r, a)$. We see that $\nu(D, a)$ is of essentially different nature than $\mu(D, G)$ or $m(r, a)$: while, say the magnitude $m(r, a)$ operates with the closeness of the values $w(\partial D_r)$ to the value a the magnitude $\nu(D_r, a)$ operates with the number of windings of the curve $w(\partial D_r)$ around this value a. Thus, this observation permits us to interpret all the results related to the deficiencies in terms of the windings.

2.3. **Gamma-lines: an inequality for arbitrary meromorphic function in a given domain.** For a given complex function $w(z)$ in a given domain D and for a given curve Γ we call Γ-lines the preimages $w^{-1}(\Gamma)$ in D.

Obviously, the concept of Gamma-lines generalizes two concepts widely studied in pure and applied mathematics: a-points of complex functions $w(z)$ (that is $w^{-1}(a)$) and level sets of real functions $u(x, y)$ (that is the set of solutions of $u(x, y) = const$), which can clearly be considered as Γ-lines of complex functions $w = u + iv$ when $\Gamma := \{w : \mathrm{Im} = 0\}$.

For Gamma-lines are valid some assertions analogous to that of the main results in Nevanlinna and Ahlfors theories, see [5]. Moreover, Gamma-lines have led to the so-called proximity property which reveals a new property related to the geometric distribution of a-ponts and, in addition, implies the key results in Nevanlinna-Ahlfors theories related to the number of a-points.

Here we present only one of these results in the spirit of the second fundamental theorem of Nevanlinna related to the length of Gamma-lines of arbitrary meromorphic function.

Denote
$$\nu(\Gamma) = Var_{z\in\Gamma}\alpha_\Gamma(z),$$
where $\alpha_\Gamma(z)$ is the angle between the tangent to Γ at the point $z \in \Gamma$ and the real axis.

Theorem 2.3 (see [5], chapter 4). *Let $w(z)$ be a function meromorphic in $\bar{D}$ and let Γ_ν, $\nu = 1, 2, \dots, q$, be a collection of some disjoint bounded Jordan curves with continuous tangent satisfying $\nu(\Gamma_\nu) < \infty$. Then*
$$\sum_{i=1}^{q} L(D, \Gamma_\nu) \le K \iint\limits_D \left|\frac{w''}{w'}\right| d\sigma + h(\Gamma_1, \dots, \Gamma_q) \iint\limits_D \frac{|w'|}{1+|w|^2} d\sigma + \\ + \sqrt{2}\, l(\partial D), \tag{2.4}$$
where K is an absolute constant and $h(\Gamma_1, \dots, \Gamma_q)$ is a constant depending only on $\Gamma_1, \dots, \Gamma_q$.

Comment 2.3. For meromorphic functions in the complex plane, the second and the third terms in the right hand side in (2.4) "are small" and we are able to estimate the first integral in terms of Ahlfors characteristics $A(D, w)$. This permits to derive (a new type) deficiency relations for

Gamma-lines which, in turn, leads to other deficiency relations related to Gelfond's magnitudes, Nevanlinna problem on the singularity of Riemann surfaces and proximity property of a-points of meromorphic functions (see [5], chapter 4).

2.4. **"Universal value distribution": for arbitrary meromorphic function in a given domain.** Some purely geometric results analogous to the second fundamental theorems in the classical Nevanlinna and Ahlfors theories are revealed. These analogs are valid for arbitrary analytic (meromorphic) functions in given domains unlike the classical studies that are valid only for some known subclasses of functions that have "equidistributions"(see [12]).

The zeros of complex functions, generally their a-points, play a pivotal role in pure and applied mathematics. They are studied in the classical Nevanlinna value distribution theory [20] and Ahlfors theory of covering surfaces [1], see also [20], chapter 13. These theories have had an essential influence on many branches of mathematics. Meantime both these theories (the one analytical the other one metric-topological) work properly only for those classes of functions that have "equidistributions": meromorphic functions in the complex plane as well as in the disks but provided that the corresponding characteristic functions grow rather strongly.

As to the most applicable case, for arbitrary meromorphic functions in a given domain, we had no theory and, moreover, no idea whether there are general value distribution type regularities in this case, see open problem's collection [6].

Surprisingly, a similar regularity has been obtained long ago (1981) in [4], but it was stated there as an auxiliary result (Lemma 1) and only for the functions in the disks. In this paper we present the general case. Also the results in [4] were not expounded as some generalities in the complex analysis. We do it in this paper.

It is easy to notice that the mentioned classical theories have no even appropriate characteristics to solve a similar problem. To see this let us take a function $w(z)$ which is meromorphic in the closure $\bar{D}$ of a given domain D with smooth boundary ∂D. Ahlfors theory covers more large classes of functions so that it is enough to consider the question in this case. Ahlfors theory works with the spherical area $A(D)$ of $w(D)$, with the spherical length $L(D)$ of $w(\partial D)$ and with the number $n(D, a)$ of a-points of w (taken with counting multiplicities) in D. Taking now the simplest function $w = z^n/n$ in the disk $D(1) := \{z| \ |z| < 1\}$ we observe that for "enough large" n the magnitudes $A(D(1))$ and $L(D(1))$ are as small as we please and the magnitude $n(D(1), 0)$ is as large as we please. This means that that the magnitudes A, L and n can not be evaluated in terms of each other. Thus, to describe value distribution of analytic functions in the given domains, in other words, to describe $n(D,a)$ for a prescribed set of values a, we should deal with another set of characteristic functions or we should make use of an additional characteristic along with A and L.

The role of such a characteristic play the following magnitudes:

$$K(D) := \frac{1}{2\pi} \int\limits_{w(\partial D)} |k(s)| ds,$$

where $k(s)$ is the curvature of the curve $w(\partial D)$.

In this paper we show that the above problem can be solved in terms of this geometrically harmonious triple of characteristics $A(D)$, $L(D)$ and $K(D)$.

Below D stands for a simply connected domain with the boundary having continuous curvature.

We prove the following

Theorem 2.4. *Let $a_1, \ldots, a_q$, $q \geq 1$, be the set of pairwise different bounded complex values and w be a meromorphic function in $\bar{D}$ admitting no multiple points, poles and $a_1, \ldots, a_q$-points on ∂D* [1]*. Then*

$$\sum_{\nu=1}^{q} |n(D, a_\nu) - A(D))| \leq K(D) + h(a_1, \ldots, a_q) L(D), \tag{2.5}$$

where $h(a_1, \ldots, a_q)$ is a finite positive constant depending on $a_1, \ldots, a_q$.

Now we show the sharpness of (2.5) and compare this inequality with the classical results.

Sharpness. The result is asymptotically sharp for the functions in the given domains (particularly disks) and also for the functions in the complex plane. First, we consider two standard functions in the complex plane: the exponential function (entire function) and double periodic function (meromorphic function). Let us write (1) for these functions in the disks $D(r) := \{z | \, |z| < r\}$ with arbitrary $a_1, \ldots, a_q$ including 0 and ∞. It is easy to see that the ratio of the left and the right sides in (1) tends to 1 when $r \to \infty$. Thus, the result is asymptotically sharp in the class of entire and meromorphic functions in the plane. To show the sharpness in the bounded domains, we again consider the function $w = z^n$ in the disk $D(r)$. For r tending to zero we have $A(D)$ and $L(D)$, meantime $n(D(r), a_\nu) = n$ and $K(D) = n$ so that again (2.5) is asymptotically sharp for $q = 1$. For $q > 1$ we take $a_1 = 0, \ldots, a_q = q - 1$ and take the function

$$w_k = (z-0)^{km_1}(z-1)^{km_2} \cdots (z-q-1)^{km_q}$$

and observe that if D_ε is the ε-neighborhood of the interval $(0, q-1)$ on x-axis. Then for ε tending to zero and k tending to infinity we have:

[1] The last assumption is just for simplicity.

$A(D)$ tends to zero; $L(D) = o\,(km_1 + km_2 + \cdots + km_q)$, when $k \to \infty$; $n(D, a_\nu) = km_\nu$ and $K(D)$ tends asymptotically to $km_1 + km_2 + \cdots + km_q$. Consequently, the ratio of the left and the right sides in (2.5) tends to 1 when ε tends to zero and, simultaneously, k tends to infinity. Thus, the result is asymptotically sharp in this case as well.

Comparison with the classical results and some discussions. The second fundamental theorem in Ahlfors theory asserts: for any set of pairwise different complex values $a_1, \ldots, a_q$, $q \geq 3$, we have

$$\sum_{\nu=1}^{q} [A(D) - n(D, a_\nu)] \leq 2A(D) + h_1(a_1, \ldots, a_q)L(D), \tag{2.6}$$

where $h_1(a_1, \ldots, a_q)$ is a positive and finite constant depending on $a_1, \ldots, a_q$.

First, let us pay attention to the very important circumstance that in (2.5) we deal with the modules $|n(D, a_\nu) - A(D))|$, meantime in (2.6) with the difference $A(D) - n(D, a_\nu)$. Due to this circumstance, the inequality (2.5) is meaningful and describes the distribution of the a-points for any function in D. Let us compare this with (2.6). It is well known that (2.6) describes distribution of the a-points only when $L(D)$ is essentially less than $A(D)$: corresponding *exhausting surfaces* (see [1], [20], Chapter 13) are regularly exhausting. As it was mentioned above, this is so for only those classes of functions that have "equidistributions": for instance, for meromorphic functions in the complex plane as well as in the disks $D(r)$, but provided that corresponding characteristic function grows rather strongly. Ahlfors' theorem does not work when we have enough powerful set of vales $a_1, \ldots, a_q$ such that $n(D, a_\nu)$ are essentially larger than $A(D)$. But this is quite common and important in application case (remember the simplest example z^n/n) when we deal with the functions in arbitrary domains. Corresponding Riemann surfaces have a very interesting geometry. They have one or many neighborhoods of the algebraic branch points that look like some thin gimlets with many coils, so that $K(D)$ is large and can give upper bounds of $\sum_{\nu=1}^{q} |n(D, a_\nu) - A(D)|$ and consequently can give bounds of $\sum_{\nu=1}^{q} n(D, a_\nu)$ since we assumed in this case that $A(D)$ is comparatively small.

In the case when w is meromorphic in the complex plane Miles showed [18] that

$$\sum_{\nu=1}^{q} |A(D) - n(D, a_\nu)| \leq CA(D) + h_1(a_1, \ldots, a_q)L(D), \tag{2.7}$$

where C is an absolute constant. The same result followed also from our Theorem 1 in [4].

3. Real Functions

3.1. **On the geometry of level sets of real functions: sharp estimates as well as Nevanlinna type of results for the length, integral curvature, Hilbert's ovals, cardinality of the level sets of real functions.**

Introduction: the geometry of the level sets in terms of rotational variational characteristic.

In what follows we denote by D the domain whose boundary ∂D is a piecewise smooth curve of finite length. Denote by $\bar{\Omega}$ the closure of Ω.

Assume that $u(x, y) \in C^1(\bar{D})$ $(\in C^2(\bar{D}))$ and $|\text{grad}u| \neq 0$ in $\bar{D}$. Similar functions we denote by $u(x, y) \in \tilde{C}^1(\bar{D})$ $(\in \tilde{C}^2(\bar{D}))$.

The level set $\gamma(A)$ of similar functions, that is the set $\gamma(A) := \{(x, y) \in \bar{D} : \ u(x, y) = A\}$, $A \in R$ consists of smooth curves.

Notice that the number (cardinality) of level sets of real functions u (that is solutions of $u = A$) plays a role quite similar to that of the number of a-points of complex functions w (that is solutions of $w = a$). We will study particularly this cardinality for large classes of functions. Remember that in the particular case, when we deal with polynomials $P(x, y)$ the cardinality was widely studied in the frame of the Hilbert problem 16 [17]: "to study the number, form and positions of connected components" of the polynomials. Thus, in fact we study this problem in a much more general setting: for large classes of real functions. Moreover, we also study the geometry of these level sets. For a given set $\Psi \subset \bar{D}$ we give bounds for: the integral $\int_{\gamma(A)\cap\Psi} K dl$; the length $L(\Psi, A, u) := \int_{\gamma(A)\cap\Psi} dl$; the absolute integral curvature $T(\Psi, A, u) = \int_{\gamma(A)\cap\Psi} |k| dl$, where $|k|$ is the curvature of $\gamma(A)$; the cardinality $C^{\text{O}}(D, A, u)$ of *Hilbert's ovals*, that is the number of maximal closed connected components of $\gamma(A) \cap D$; the cardinality $C_d(D, A, u)$ in general case, that is the number of those maximal connected components of $\gamma(A) \cap D$ (both closed and non closed) which intersect $d \subset D$.

The results are connected with the integral geometry, Gamma-lines, Nevanlinna theory, Hilbert problem 16, applied topics, admit different modifications and make use of various characteristics. Moreover, in some cases we have a long lists of preceding studies. Respectively, a usual presentation starting with the references and historical comments would not be optimal in this case. This is why we prefer to give first a summary of related concepts and results. This should show demonstrably the interrelations.

Clearly, when we consider the above concepts for a given function u and different values, say $A_1, A_2, \dots, A_q$, the outcomes can be quite different but they can also be somehow interrelated. The last situation we will refer as Nevanlinna type phenomena since his known deficiency relation is of the type: $\sum_\nu \delta(a_\nu) \leq 2$.

We will give below some sharp inequalities related to the above concepts and to a given value (level) A. Next, for each concept we will present the corresponding Nevanlinna type phenomenon dealing with $A_1, A_2, \ldots, A_q$ [2].

Inequalities for a given A in variational terms.

Let $\bar{X}(\theta)$ be the oriented straight line (axis) passing through zero and having direction θ, that is we obtain $\bar{X}(\theta)$ by turning the x-axis positively on the angle θ. We will use notation $X(\theta)$ for the coordinate on $\bar{X}(\theta)$. Denote by $J_{X(\theta)}$ the oriented straight line composing the angle $\theta+\pi/2$ with x-axis and passing through the point on $\bar{X}(\theta)$ with the coordinate $X(\theta)$. We will use notation $Y(\theta)$ for the coordinate on $J_{X(\theta)}$. Denote by $(X(\theta))$ the orthogonal projection of D on axis $\bar{X}(\theta)$.

For a given interval ω notation $\mathrm{Var}_\omega f$ stands for the variation of function f on ω. Denoting by β the angle made by the gradient vector of u and by positive direction of the x-axis, we can define now the following *rotational variational characteristic*

$$\mathrm{V_R}\,(D,K,u) := \frac{1}{\pi}\int\limits_0^\pi \int\limits_{(X(\theta))} \mathrm{Var}_{J_{X(\theta)}\cap D}(K\sin(\beta-\theta))dX(\theta)d\theta.$$

Theorem 3.1 (the integral)**.** *For any function $u(x,y) \in \tilde{C}^1(\bar{D})$, any continuous function $K(x,y)$in $\bar{D}$ and any $A \in R$ we have*

$$\int\limits_{\gamma(A)\cap D} Kdl + \frac{1}{2}\int\limits_{\gamma(A)\cap\partial D} Kdl \le \mathrm{V_R}\,(D,K,u) + \frac{2}{\pi}\int\limits_{\partial D\setminus\gamma(A)} |K|dl. \tag{3.1}$$

Theorem 3.2 (the length)**.** *For any function $u(x,y) \in \tilde{C}^1(\bar{D})$ and any $A \in R$ we have*

$$L(D,A,u) + \frac{1}{2}L(\partial D,A,u) \le \mathrm{V_R}\,(D,1,u) + \frac{2}{\pi}\int\limits_{\partial D\gamma(A)} dl. \tag{3.2}$$

Theorem 3.3 (the absolute integral curvature)**.** *For any function $u(x,y) \in \tilde{C}^2(\bar{D})$ and any $A \in R$ we have*

$$T(D,A,u) + \frac{1}{2}T(\partial D,A,u) \le \mathrm{V_R}\,(D,|k|,u) + \frac{2}{\pi}\int\limits_{\partial D\setminus\gamma(A)} |k|dl, \tag{3.3}$$

where $k(x,y)$ stands for the curvature of $\gamma(A^*)$, $A^* = u(x,y)$.

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[2]To study the mentioned problems we have introduced four types of characteristics for a given real function. Some results can respectively be given in different forms depending on a given characteristic. We present here only one version utilizing the so-called *rotational variational characteristic.*

Theorem 3.4 (the cardinality of the Hilbert's ovals)**.** *For any function $u(x,y) \in \tilde{C}^2(\bar{D})$ and any $A \in R$ we have*

$$C^{\mathrm{O}}(D,A,u) \leq \frac{1}{2\pi}\,\mathrm{V_R}\left(D,|k|,u\right) + \frac{1}{2\pi^2}\int\limits_{\partial D\setminus\gamma(A)} |k|dl. \tag{3.4}$$

Theorem 3.5 (the cardinality in a general case)**.** *For any function $u(x,y) \in \tilde{C}^2(\bar{D})$ any domain $d \subset D$ with $\Delta := \mathrm{dist}(\partial d, \partial D) > 0$and any $A \in R$ we have*

$$\begin{aligned} C_d(D,A,u) \leq & \frac{1}{2\pi}\,\mathrm{V_R}\left(D,|k|,u\right) + \frac{1}{\Delta}\,\mathrm{V_R}\left(D,1,u\right) + \\ & + \frac{2}{\pi}\int\limits_{\partial D}\left(\frac{|k|}{2\pi} + \frac{1}{\Delta}\right)dl. \end{aligned} \tag{3.5}$$

Nevanlinna type phenomenon for $A_1, A_2, \ldots, A_q$ in variational terms.

The following results are analogs of the second main theorem for Gamma-lines (which in turn is an analog of the Nevanlinna second fundamental theorem).

Theorem 3.6 (the integral)**.** *For any function $u(x,y) \in \tilde{C}^1(\bar{D})$, any continuous function $K(x,y)$in $\bar{D}$, any continuous function $\omega(t) > 0$on R and any real values $A_1 < A_2 < \cdots < A_q$, $1 < q < \infty$ we have*

$$\sum_{\nu=1}^{q}\int\limits_{\gamma(A_\nu)\cap D} Kdl + \frac{1}{2}\sum_{\nu=1}^{q}\int\limits_{\gamma(A_\nu)\cap\partial D} Kdl \leq \mathrm{V_R^*}\left(D,K,u\right), \tag{3.6}$$

where

$$\mathrm{V_R^*}\left(D,K,u\right) := \mathrm{V_R}\left(D,K,u\right) + \frac{2}{\rho}\iint\limits_{D} |K|\,|\mathrm{grad}\; u|\,\omega(u)d\sigma + \frac{2}{\pi}\int\limits_{\partial D\setminus\gamma(A)} |K|dl,$$

$\rho = \min_\nu \int_{A_\nu}^{A_{\nu+1}} \omega(t)dt$.

Theorem 3.7 (the length)**.** *Under the same assumptions we have*

$$\sum_{\nu=1}^{q} L(D,A_\nu,u) + \frac{1}{2}\sum_{\nu=1}^{q} L(\partial D,A_\nu,u) \leq \mathrm{V_R^*}\left(D,1,u\right). \tag{3.7}$$

Theorem 3.8 (the absolute integral curvature)**.** *Assuming in addition that $u(x,y) \in \tilde{C}^2(\bar{D})$, we have*

$$\sum_{\nu=1}^{q} T(D,A_\nu,u) + \frac{1}{2}\sum_{\nu=1}^{q} T(\partial D,A_\nu,u) \leq \mathrm{V_R^*}\left(D,|k|,u\right). \tag{3.8}$$

Theorem 3.9 (the cardinality of the Hilbert's ovals)**.** *Under the same assumptions we have*

$$\sum_{\nu=1}^{q} C^{\mathrm{O}}(D, A_\nu, u) \leq \frac{1}{2\pi}\, \mathrm{V}_{\mathrm{R}}^{*}\,(D, |k|, u)\,. \tag{3.9}$$

Theorem 3.10 (the cardinality in a general case)**.** *Under the same assumptions we have*

$$\sum_{\nu=1}^{q} C_d(D, A_\nu, u) \leq \frac{1}{2\pi}\, \mathrm{V}_{\mathrm{R}}^{*}\,(D, |k|, u) + \frac{1}{2\pi\Delta}\, \mathrm{V}_{\mathrm{R}}^{*}\,(D, 1, u)\,. \tag{3.10}$$

Comments on connections with different fields of physics. In this subsection we deal with the level sets of large classes of real functions which admit several interpretations in different fields of physics: the sets where velocity, temperature, elasticity, tension, pressure etc. are equal to a given constant. All the above-given results can be considered as some assertions in different fields of physics. Indeed, on one hand, the magnitudes ($\int K dl$, L, T, C_d) we study admit interpretation (as they describe the geometry of the above mentioned physical sets). On the other hand, since the above-mentioned magnitude we estimate in terms of the gradients (occurring in V_{R} and $\mathrm{V}_{\mathrm{R}}^{*}$) which also admit corresponding physical interpretation. For instance, applying the results to the velocity function $u(x, y)$ of the given process, we describe the behavior of the streaming lines of the process in terms of the gradient grad$u$. Both components here (the streaming lines and the gradient) play a crucial role in physics. Thus similar mathematical result are immediately converted into some assertion in physics.

As one can easily see, Theorem 3.7 here is an analog of Theorem 2.4 related to Gamma-lines. Similar transfer of Gamma-lines type of results into real analysis can be traces in [7]**,** [10] and [13]**.**

3.2. **Rolle type results for real functions of two variables.** In what follows D stands for a domain in the plane (x, y) with piecewise smooth boundary ∂D of finite length.

In the essential part of this book we study the level sets of rather large classes of functions $u(x, y)$ in $\bar{D} = D \cup \partial D$ that is study the sets $\lambda(\bar{D}, u) := \{(x, y) :\ u(x, y) = 0\} \cap \bar{D}$.

The set $\lambda(\bar{D}, u)$ we refer as *proper level set* if $\lambda(\bar{D}, u)$ implies a set $\lambda_0(\bar{D}, u)$ such that $\lambda(\bar{D}, u) \backslash \lambda_0(\bar{D}, u)$ is a union of countably many smooth curves without singular points and $\lambda_0(\bar{D}, u)$ is a countable set of points in $\bar{D}$ which do not admit limit points in D.

First, we consider a narrow class of *proper functions in* $\bar{D}$, that is function $u \in C^1(\bar{D})$ for which the set $\lambda(\bar{D}, u)$ is a proper level set of u and the sets

$\lambda(\bar{D}, u'_s)$, where u'_s are the directional derivatives taken for any direction s, are the proper level set of u'_s [3].

Denote by $L(D, u)$ the total length of all curves belonging to $\lambda(\bar{D}, u)$; by $L(D, u'_s)$ the total length of curves belonging to $\lambda(\bar{D}, u'_s)$; by $l(\Gamma)$ the length of a given curve Γ.

Let $\bar{X}(\theta)$ be the oriented straight line passing through zero which composes the angle θ with the positive direction of axis x. Notation $s(\theta)$ stands for the direction $\theta + \pi/2$.

Inequality A. *For any proper function $u(x, y)$ in D*

$$L(D, u) \leq \frac{1}{2} \int_0^\pi L\left(D, u'_{s(\theta)}\right) d\theta + \frac{1}{2}\, l(\partial D) \leq$$

$$\leq \frac{\pi}{2} \sup_{0 \leq \theta < \pi} L(D, u'_{s(\theta)}) + \frac{1}{2}\, l(\partial D). \tag{3.11}$$

Here and in what follows, under the integrals will be meant the Lebesgue integrals.

Sharpness. Consider $u = \sqrt{1 - x^2 - y^2}$ in the unit disk. The set of solutions of $u = 0$ coincides with the boundary of the disk and the set of solutions of $u'_{s(\theta)} = 0$ coincides with the diameter of the disk having direction θ. We have therefore $L(D, u) = 2\pi$, $L(D, u'_s) = 2$ for any θ (consequently $\sup_{0 \leq \theta < \pi} L(D, u'_s) = 2$) and $l(D) = 2\pi$. Thus we have equality in the double inequality (3.11) for this function.

Some analogous, but more particular, inequalities were announced without proofs in [9], [11].

Comment 3.1. Inequality A as a Rolle type result.

One of the key results in real analysis, the Rolle's theorem, asserts: between arbitrary two zeros of a real differentiable function of one variable $f(x)$ there is a zero of $f'(x)$.

At the early stage of mathematical education we learn that the Rolle's theorem is not valid for the functions of several variables. We ask, how could the possible multivariate analogs of Rolle's theorem look like? We meet the essential difference: zeros of functions of several variables are curves, surfaces etc. and these functions have directional derivatives (unlike functions of one variable whose zeros are points in general case and we deal with only one derivative).

[3]Notice that the class of proper functions in $\bar{D}$ is much larger than those we meet in the majority of preceding studies related to the level sets. For instance, the harmonic functions in $\bar{D}$ studied in the real and complex analysis or the polynomials studied in algebraic geometry belong to $C^\infty(\bar{D})$ and admit only a finite number of similar curves and points.

Let us consider the Rolle's theorem in a bit enlarged version: if f has n zeros on $[x_1, x_2]$ then f' should have at least $n-1$ zeros on $[x_1, x_2]$. The last statement we can express qualitatively as follows: if f has a "powerful" set of zeros of f on $[x_1, x_2]$, then f' should also have "powerful" set of zeros.

The power of zeros can also be defined in a multivariate case. A natural measure for the "power" of zeros of functions $u(x, y)$ $(u'_s(x, y))$ of two variables in $\bar{D}$ is the length $L(D, u)$ $(L(D, u'_s))$.

Inequality A has exactly the same meaning: when $u(x, y)$ has a "powerful" set of zeros of u in $\bar{D}$ (that is if $L(D, u)$ is essentially larger than $l(D)$) we should have also a "powerful" set of zeros of u'_s for at least one value θ (that is $L(D, u'_{s(\theta)})$ should be large as well). Respectively, this inequality can be considered as an analogue of Rolle's theorem for functions of two variables.

3.3. **Nevanlinna type results for real functions of two variables.** In this subsection we present an extended version of the above Rolle type theorem which is an analog of the second main theorem in Gamma-lines theory which, in turn, is an analog of the second fundamental theorem of Nevanlinna theory.

In what follows, we will utilize the notation and definitions of Section 1 and the given constants $A_1 < A_2 < \cdots < A_q$, $q \geq 1$, we denote by $\bar{C}^1(\bar{D})$ the class of functions u satisfying $u - A_\nu \in \tilde{C}^1(\bar{D})$ for any $\nu = 1, 2, \ldots, q$.

The results describe the length $\mathcal{L}(D, u - A_\nu)$ of the solutions (zeros) of $u(x, y) = A_\nu$. Notice that in the length $\mathcal{L}(D, u - A_\nu)$ we also take into account the length of the boundaries of degenerated domains where $u \equiv A_\nu$.

In what follows, we will utilize the notation and definitions of Section 1 and denote by $\bar{C}^1(\bar{D})$ the class of functions u satisfying $u - A_\nu \in \tilde{C}^1(\bar{D})$ for A_ν, $\nu = 1, 2, \ldots, q \geq 1$.

Theorem 3.11. *For arbitrary real numbers $A_1 < A_2 < \cdots < A_q$, $q \geq 1$, and arbitrary $u(x, y) \in \bar{C}^1(\bar{D})$ we have*

$$\sum_{\nu=1}^{q} \mathcal{L}(D, u - A_\nu) \leq \frac{1}{2} \int_0^\pi \mathcal{L}_\theta \left(D, u'_{s(\theta)}\right) d\theta + \\ + C \iint_D |\text{grad } u| d\sigma + \frac{1}{2} l(\partial D), \tag{3.12}$$

where $C = 0$ for $q = 1$ and $C = 1/\rho$ for $q > 1$, $\rho := \min_{\nu \neq \mu}[A_\nu, A_\mu] > 0$.

This implies more simple (and more rough) inequalities

$$\sum_{\nu=1}^{q} \mathcal{L}(D, u - A_\nu) \leq \frac{1}{2} \int_0^\pi \mathcal{L} \left(D, u'_{s(\theta)}\right) d\theta + C \iint_D |\text{grad } u| d\sigma + \frac{1}{2} l(\partial D)$$

and

$$\sum_{\nu=1}^{q} \mathcal{L}(D, u - A_\nu) \leq \frac{\pi}{2} \sup_{0 \leq \theta < \pi} \mathcal{L}(D, u'_{s(\theta)}) + C \iint\limits_D |\text{grad } u| d\sigma + \frac{1}{2}\, l(\partial D).$$

Sharpness. The last two inequalities imply inequality (3.11) (for $q = 1$) which is sharp.

4. Curves

4.1. **A Nevanlinna type inequality and deficiency relation for the plane curves.** Let γ be a curve in the plane (x, y). We can consider it as a curve in the complex plane $x + iy$ respectively $\gamma := f(t) := f_1(t) + i f_2(t)$, $t \in [0, 1]$, where $f(t)$ is a complex function of the real argument t. Denoting by $f_1^{(j)}(t)$ and $f_2^{(j)}(t)$ the derivatives we say $\gamma := f(t) \in F(k)$, k is an integer ≥ 1, if tangent of the curve $\gamma^{(j)} := f^{(j)}(t) := f_1^{(j)}(t) + i f_2^{(j)}(t)$ exists and is continuous in $[0, 1]$ for any j, $1 \leq j \leq k + 1$, and for any j, $0 \leq j \leq k + 1$ and any $t \in [0, 1]$ we have $f^{(j)}(t) \neq 0$.

Notice that $\arg f(t_0)$ means the angle between the x-axis and the vector connecting 0 and $f(t_0)$, while $\arg f'(t)$ means the angle between tangent to γ at the point $f(t_0)$ and the x-axis. Thus, if a is a point on the plane (x, y) then

$$R(a, \gamma) := \int_0^1 \left|(\arg(f(t) - a))'\right| dt$$

is the total rotation of γ around this point a. Then we consider the curve $\gamma^{(k)} := f_1^{(k)}(t) + i f_2^{(k)}(t)$ and denote by

$$T(\gamma^{(k)}) := R(0, \gamma^{(k)}) := \int_0^1 \left| \left(\arg f^{(k)}(t)\right)' \right| dt$$

the total integral curvature of $\gamma^{(k-1)}$ (or the total rotation of the curve $\gamma^{(k)}$ around $a = 0$).

Denote by $l(\gamma)$ the length of γ.

Theorem 4.1 (principle of angles and lengths)**.** *For any $\gamma := f(t) \in F(k)$, any integer $k \geq 1$, any point a*

$$R\,(a, \gamma) \leq T\left(\gamma^{(k)}\right) + k\pi. \tag{4.1}$$

For any collection of pairwise different points $a_\nu, \nu = 1, 2, \ldots, q$

$$\sum_{\nu=1}^{q} R\,(a_\nu, \gamma) \leq T\left(\gamma^{(k)}\right) + \frac{2k\pi q}{\rho}\, l(\gamma) + k\pi, \tag{4.2}$$

where ρ is the minimal distance between the points a_ν.

The inequalities have simple geometric meaning: the total rotation of γ around a does not exceed total integral curvature of γ plus π. For $k > 1$, $T(\gamma^{(k)})$ equals total rotation of the curve $\gamma^{(k)}$ so that both (4.1) and (4,.2) admit corresponding interpretations.

The reader familiar with Nevanlinna's value distribution theory and (or) with Ahlfors theory of covering surfaces will see an analogy between (4.2) and the second fundamental theorem in Nevanlinna's theory; we will discuss this below.

Sharpness. We consider the case where $k = 1$. Let $\gamma := f(t)$, $t \in [0, 1]$ be the segment connecting the points $(-1, \varepsilon)$ and $(1, \varepsilon)$ in the plane. Then $R(0, \gamma)$ is as close to π as we please when we take ε sufficiently small, meantime $T(\gamma^{(1)})$ is equal to zero. Thus, (4.1) cannot be improved.

Assume that our curve γ approaches a circumference by a spiral. Then the part of γ having N "coils" contributes to both the first and the second integrals asymptotically as N when N tends to infinity. Thus, the ratio of the left and the right magnitudes in (1.1) tends to 1 when N tends to infinity.

The inequality (4.2) is sharp as well. Let γ be the graph of the function $f_\varepsilon := \sqrt{\varepsilon}\sin\frac{1}{t+\varepsilon}$, $t \in [0, 1]$, where $0 < \varepsilon < \frac{1}{2}$ and take $a_\nu = 2(\nu - 1)$, $\nu = 1, 2, \ldots, q$. When ε tends to zero then both the left and the right sides of (4.2) tend to infinity but their ratio tends to 1.

Let $\gamma_i \in F(1)$ be a sequence of curves, each satisfying the conditions of Theorem 1.1, $\gamma_i \subset \gamma_{i+1}$, for which $T\left(\gamma^{(k)}\right) \to \infty$ when $i \to \infty$ and

$$\frac{l(\gamma_i)}{T\left(\gamma^{(k)}\right)} \to 0, \quad i \to \infty. \tag{4.3}$$

These are *bee sequences* (frequent excursions on small distances in different directions) respectively T is comparatively large and l is comparatively small. The rotations of γ_i around a_ν determines the following magnitude

$$\Delta(a_\nu) := \liminf_{i\to\infty} \frac{R\left(a_\nu, \gamma_i\right)}{T\left(\gamma^{(k)}\right)}$$

which we refer as *deficiency.* Inequality (1.2) implies the following

Deficiency relation (for the curves). For any bee sequence of curves and an arbitrary collection of pairwise different points a_ν, $\nu = 1, 2, \ldots, q$,

$$\sum_{\nu=1}^{q} \Delta(a_\nu) \leq 1. \tag{4.4}$$

4.2. **A particular case: in terms of a real functions of one variable.** Inequalities (4.1) and (4.2) for the curves imply corresponding corollaries for the real smooth functions of one variable.

The graph of a real function of one variable $\varphi(x) \in C^1[0,1]$ is the curve $\gamma := f(t) := x + i\varphi(x) \in F(1)$, $x \in [0,1]$. Thus, we can apply the above theorem to this curve. We give here only application of the inequality (4.1).With the notation $S(u) := u'/\left(1+u^2\right)$ (the spherical derivative of u) and $a_\nu = (x_\nu, y_\nu)$ we have

Theorem 4.2. *For any $\varphi(x) \in C^1[0,1]$,*

$$\int_0^1 S\left(\frac{\varphi(x)}{x}\right) dx \leq \int_0^1 S\left(\varphi'(x)\right) dx + \pi. \tag{4.5}$$

4.3. **An identity generalizing the key identity in integral geometry.** To study level sets of general classes of real functions we had to establish three mutually connected identities for the integrals $\int_\Gamma K dl$ for "good" curves Γ and function K on Γ. The identities are very simple but we did not see them anywhere else. One of them generalizes the key identity in integral geometry, the Crofton's formula, which we obtain taking $K \equiv 1$. This identity will be presented below.

Let $\bar{X}(\theta)$ be the oriented straight line passing through zero and having direction θ, that is $\bar{X}(\theta) := \{(x,y) : \ \theta := \arctan(y/x)\}$. We will use notation $X(\theta)$ for the coordinate on $\bar{X}(\theta)$. Denote by $J_{X(\theta)}$ the straight line composing the angle $\theta + \pi/2$ with x-axis and passing through the point on $\bar{X}(\theta)$ with the coordinate $X(\theta)$. We will use notation $Y(\theta)$ for the coordinate on $J_{X(\theta)}$.

In what follows, a curve means an oriented plane curve. Let Γ be a curve with continuous tangent and finite length. Thus, we can define the acute angle δ that composes the tangent to Γ with x-axis. Denote by $N(\Gamma \cap J_{X(\theta)})$ the number of elements of the set $\Gamma \cap J_{X(\theta)}$. Observe that either these elements are points (denote them by $(X(\theta), Y_i(\theta))$, or they are common parts of Γ and $J_{X(\theta)}$, which should be then some intervals on $\Gamma \cap J_{X(\theta)}$).

Due to the key identity in integral geometry (Crofton's formula), for the above defined curve we have

$$l(\Gamma) = \frac{1}{2} \int_0^\pi \int_{\Gamma \perp \bar{X}(\theta)} N(\Gamma \cap J_{X(\theta)}) dX(\theta) d\theta, \tag{4.6}$$

see [23], formula (3.17).

Now we prescribe to the elements of the set $\Gamma \cap J_{X(\theta)}$ the weight $\omega_i^*(X(\theta)) = K(X(\theta), Y_i(\theta))|$ when these elements are points and prescribe the weight $\omega_i^*(X(\theta)) = 0$ if these elements are common parts of Γ and $J_{X(\theta)}$. We define a new weight function $W^*_{\Gamma,K}(X(\theta))$ of the variable $X(\theta)$ defined on $\Gamma \perp \bar{X}(\theta)$:

$$W^*_{\Gamma,K}(X(\theta)) := \left\{ \sum_{i=1}^{N(\Gamma\cap J_{X(\theta)})} \omega^*_i(X(\theta)) \right\}.$$

Theorem 4.3. *For any Γ with continuous tangent and finite length and any continuous function K on Γ we have*

$$\int_{\Gamma} K dl = \frac{1}{2} \int_0^{\pi} \int_{\Gamma\perp\bar{X}(\theta)} W^*_{\Gamma,K}(X(\theta)) dX(\theta) d\theta. \tag{4.7}$$

Notice that for $K \equiv 1$ (4.7) implies (4.3).

4.4. **An improvement and a complement of the Fáry's inequality.** We assume that $\Gamma \in C^2$ is an oriented curve so that we can define the tangential angle β, respectively curvature $k(l)$ at any point $l \in \Gamma$, and can define the magnitude

$$C(\Gamma) := \int_{\Gamma} |k(l)|\, dl$$

called usually *absolute integral curvature* of Γ. The magnitude $C(\Gamma)$ plays a crucial role in many pure and applied studies.

The problems we consider are closely connected with the known Fáry's inequality (see [15] also book [23], formula (3.26)) which asserts that *for any closed curve* $\Gamma^* \in C^2$

$$l(\Gamma^*) \leq \frac{1}{2} \operatorname{diam} \Gamma^* C(\Gamma^*). \tag{4.8}$$

This inequality has led to a lot of generalizations for very different objects in geometry (the Internet shows 11000 citations). The inequality is sharp, but only when Γ is circumference.

In this section we consider the following problems. Are there similar inequalities for closed curves that are sharp for rather large classes of curves? What can be said about non closed case?

Define the following *rotational length* $\Delta(\Gamma)$ of Γ, that is

$$\Delta(\Gamma) := \frac{1}{2} \int_0^{\pi} l(\Gamma \perp \bar{X}(\theta)) d\theta.$$

Inequality A. *For any closed curve* $\Gamma^* \in C^2$ *we have*

$$l(\Gamma^*) \leq \frac{1}{2} \operatorname{diam} \Gamma^* C(\Gamma^*) + 2\Delta(\Gamma^*) - \pi \operatorname{diam} \Gamma^*. \tag{4.9}$$

This inequality implies the Fáry's inequality since $2\Delta(\Gamma^*) - \pi \mathrm{diam}\Gamma^* \leq 0$.

Sharpness. Due to the Identity A below, (2.2) is sharp for any closed convex curve. This shows that (2.2) improves (2.1) essentially.

Inequality B. *For any non closed curve* $\Gamma \in C^2$ *we have*

$$l(\Gamma) \leq \frac{1}{2} \operatorname{diam} \Gamma C(\Gamma) + \Delta(\Gamma). \tag{4.10}$$

Sharpness. This is a rather rough inequality which meantime is sharp for any segment.

4.5. **A new identity for closed convex curves.** Now we consider seemingly a much more interesting problem: are there identities involving simultaneously the length and the curvature of a given curve?

Clearly, for a given curve we cannot have any identity determined merely by the length and curvature: some additional notions are needed for that. It turns out that in the case if we deal with convex curves, similar notion is $\Delta(\Gamma)$.

Saying convex curve Γ we mean that the intersection of Γ with any straight line consists of at most two elements which can be the points and the segments, as well.

Identity A. *For any convex closed curve* $\Gamma^* \in C^2$ *we have*

$$l(\Gamma^*) = \frac{1}{2} \operatorname{diam} \Gamma^* \left[C(\Gamma^*) - 2\pi\right] + 2\Delta(\Gamma^*). \tag{4.11}$$

5. The Cardinality and Integral Curvature for the Polynomials

Let $P(x, y)$ be a polynomial. The cardinality of the level set $\gamma(R^2, A, P) := \{(x, y) \in R^2 \backslash \infty : P(x, y) = A\}$ is of special interest since it was widely studied in the frame of the Hilbert problem 16 [17]: "to study the number, form and positions of connected components" of the polynomials.

Related studies have a long history. Traditionally, the above-mentioned number "cardinality" was studied in terms of Euler's characteristics and Betty's numbers for the polynomials $P(x, y)$; see the initial, key result by Petrovskii [21], and then by Petrovskii and Oleynik [22], Thom [24], Milnor [19].

Much later the cardinality started to play an important role in computations (complexity theory by Smale). Smale and his co-authors Blum, Cucker and Shub return in [14] to the "natural or deterministic" definition of the cardinality, more convenient (visible) than the Euler's characteristics and Betty's numbers. Notice that for a given regular value A (means $|\operatorname{grad} u(x, y)| \neq 0$ on $\gamma(R^2, A, P)$) the set $\gamma(R^2, A, P)$ consists of isolated smooth curves (without intersection points in $R^2 \backslash \infty$) so that we can determine the maximal connected components $\gamma_i \in \gamma(R^2, A, P)$,

$i = 1, 2, \ldots, C(A, P) \leq \infty$ and similarly we can determine maximal closed connected components (Hilbert's ovals) o_j, $j = 1, 2, \ldots, C^{\mathrm{O}}(A, P)$. In these terms they prove for any regular value A the following key inequality ([14], p.303)

$$C^{\mathrm{O}}(A, P) \leq \frac{1}{2}\, n(P)(n(P) - 1), \tag{*}$$

where $n(P)$ is the degree of P, and hence we derive for arbitrary A ([14], p.307 [4])

$$C(A, P) \leq n(P)(2n(P) - 1). \tag{**}$$

Notice that (*) generalizes known Harnak's inequality which is a bit stronger but relates to the irreducible polynomials merely.

Inequality 5.1. *For any $P(x, y)$ and any regular value A we have*

$$C^{\mathrm{O}}(A, P) \leq \frac{1}{2}\, n(P) n'(P).$$

where $n'(P) := \min[n(P'_x), n(P'_y)]$.

Inequality 5.2. *For any $P(x, y)$ and any regular value A we have*

$$C(A, P) \leq 2n(P)n'(P) + \pi n(P).$$

Clearly Inequality 5.1 implies (*). Moreover, when $n'(P)$ is essentially smaller than $n(P)$ Inequality 5.1 is sharper than (*) and Inequality 5.2 is sharper than (**).

One can easily see that inequality 3.5 gives far going generalization of (*) (and consequently of the preceding studies by Petrovskii-Oleynik-Thom-Milnor) and, in addition, inequality 3.10 connects these studies with Nevanlinna type results and deficiencies.

Clearly the absolute integral curvature $T(A, P)$ of $\gamma_{R^2}(A, P)$ is a concept closely connected with this ring of problems. Surprisingly this concept has not been studied for the polynomials. We prove

Inequality 5.3. *For any $P(x, y)$ and any regular value A we have*

$$T(A, P) \leq \frac{\pi}{2}\, \left[5n^2(P) - (6 - \pi)n(P)\right].$$

[4]On p. 315 in [14] the authors assert that (*) is due to Petrovskii-Oleynik-Thom-Milnor and of course this is so substantially. However, it should be stressed that (*) does not cover all the particularities we find in the mentioned works. On the other hand, it seems that some details in the proofs of (*) are due to the authors of the book. Thus (*) should not be considered simply as an exposition.

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