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ON ONE PROBLEM OF FINDING AN EQUALLY STRONG CONTOUR FOR A SQUARE WHICH IS WEAKENED BY A HOLE AND BY CUTTINGS AT VERTICES

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Abstract. The problem of finding an equally strong contour for a square which is weakened by a hole and by cuttings at vertices is considered. The hole and cutting boundaries are assumed to be free from external forces, and to the remaining part of the square boundary are applied the same absolutely smooth rigid punches subjected to the action of external normal contractive forces with the given principal vectors.

Relying on the Kolosov-Muskhelishvili's formulas, the problem reduces to a mixed problem of the theory of analytic functions (the Keldysh-Sedov problem), and the solution of the latter allows us to construct complex potentials and equations of an unknown contour efficiently (in analytical form). The analysis of the obtained results is carried out and the formula of tangential normal stress is derived.

რეზიუმე. განხილულია თანაბრად მტკიცე კონტურის მოძებნის ამოცანა ხვრელითა და წვეროებში ამონაჭრებით შესუსტებული კვადრატისათვის იმ დაშვებით, რომ ხვრელისა და ამონაჭრების საზღვრები თავისუფალია გარეგანი დატვირთვებისაგან, ხოლი საზღვრის დანარჩენ ნაწილზე მოდებულია ერთნაირი აბსოლუტურად გლუვი ხისტი შტამპები, რომლებზეც მოქმედებენ მოცემული ნაკრები ვექტორის მქონე ნორმალური მკუმშავი ძალები.

კოლოსოვ-მუსხელიშვილის ფორმულების საფუძველზე განხილული ამოცანა მიყვანილია ანალიზურ ფუნქციათა თეორიის შერეულ სასაზღვრო ამოცანაზე (კელდიშ-სედოვის ამოცანა) და ამ უკანასკნელის ამოხსნის გზით საძიებელი კომპლექსური პოტენციალები და თანაბრად მტკიცე კონტურის განტოლება აგებულია ეფექტურად (ანალიზური ფორმით). ჩატარებულია მიღებული ამონახსნების გამოკვლევა და დადგენილია ტანგენციალური ნორმალური ძაბვის გამოსათვლელი ფორმულა.

In the present work we consider the problem of finding an equally strong contour for a square which is weakened by an equally strong hole and by cuttings at vertices. The boundary of the hole and cutting is in the whole

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assumed to be an equally strong contour which is free from external forces. To the remaining part of the square boundary are applied the same absolutely smooth rigid punches which are subjected to the action of external normal contractive forces with the given principal vectors.

Our problem is to find both a stressed state of the square and analytical form of the equally strong contour under the condition that the tangential normal stress on it takes constant value (the condition for the contour to be equally strong).

On the basis of the Kolosov-Muskhelishvili's formulas, the problem is reduced to the mixed problem of the theory of analytic functions, and the solution of the latter allows us to construct complex potentials and equations of an unknown contour efficiently (in analytical form).

Analogous problems of the plane theory of elasticity for domains weakened by equally strong contours have been studied in [1-8]. In the present work we improve some results (regarding parameters appearing in a solution) obtained in [4], [8].

Statement of the Problem. Let to the boundary of the square which is weakened by an interior hole and cuttings at vertices be applied the same absolutely smooth rigid punches subjected to the action of external normal contractive forces with the known principal vectors. The hole and cutting boundary is free from external forces.

Consider the problem: find an elastic equilibrium of the square and analytic form of the hole and cutting contours under the condition that tangential normal stress on them take one and the same constant value $\sigma_s = k =$ const. In these conditions, we call the assemblage of hole and cutting boundaries an equally strong contour.

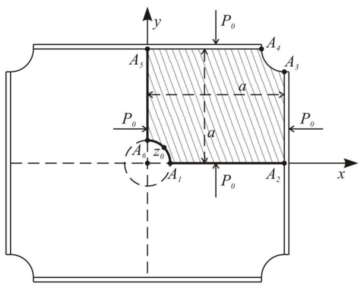

Figure 1

Solution of the problem. Owing to the symmetry of the problem, we will content ourselves with the consideration of an elastic equilibrium of a part S of the square (a shaded part in Fig.1) whose boundary consists of rectilinear segments $L_1 = \cup L_1^{(j)}$, $L_1^{(j)} = A_j A_{j+1}$ $(j = 1, 2, 4, 5)$ and unknown arcs $L_0 = L_0^{(1)} \cup L_0^{(2)}$ $L_0^{(1)} = A_3A_4$, $L_0^{(2)} = A_6A_1$.

It is not difficult to see that in this case the tangential stresses $\tau_{ns} = 0$ on the whole boundary $L = L_1 \cup L_0$ of the domain S, the normal displacement $v_n = v = \text{const}$ on $L_1^{(2)} \cup L_1^{(3)}$, and $v_n = 0$ on $L_1^{(1)} \cup L_1^{(5)}$.

On the basis of the well-known Kolosov-Muskhelishvili's formulas [9], the problem under consideration reduces to finding two functions $\varphi(z)$ and $\psi(z)$, holomorphic in the domain S, by the boundary condition on $L = L_1 \cup L_0$:

$$\operatorname{Re} e^{-i\alpha(t)}\big[\varkappa\,\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)}\big] = 2\mu\, v_n(t), \ \ t \in L_1, \tag{1}$$

$$\operatorname{Re} e^{-i\alpha(t)}\big[\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\big] = C(t), \ \ t \in L_1, \tag{2}$$

$$\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} = B_j(t), \quad t \in L_0^{(j)} \quad (j = 1, 2), \tag{3}$$

$$\operatorname{Re}\big[\varphi'(t)\big] = \frac{k}{4}\,, \quad t \in L_0, \tag{4}$$

where $\alpha(t)$ is the angle made by the outer normal to the contour L_1 and the ox-axis,

$C(t) = \operatorname{Re}\Big[i\int\limits_{A_1}^{t}\sigma_n(s_0)\exp i[\alpha(t_0) - \alpha(t)]ds_0 + \exp(-i\alpha(t))(c_1 + ic_2)\Big]$, $t \in L_1$;

$B_j(t) = i\int\limits_{A_n}^{t}\sigma_n(s_0)\exp i\alpha(t_0)ds_0 + c_1 + ic_2$, $t \in L_0^{(j)}$ $(j = 1, 2)$; c_1 and c_2 are arbitrary real constants. It is easy to notice that $c(t)$ is a piecewise constant and $B_j(t)$ is a constant function.

Summing the equalities (1) and (2), differentiating with respect to the arc abscissa s and taking into account that the functions $v_n(t)$ and $c(t)$ are piecewise constant, we obtain

$$\operatorname{Im}\varphi'(t) = 0, \quad t \in L_1. \tag{5}$$

The conditions (4) and (5) are the boundary conditions of the mixed boundary value problem

$$\operatorname{Re}\left[\varphi'(t) - \frac{k}{4}\right] = 0, \ \ t \in L_0; \quad \operatorname{Im}\left[\varphi'(t) - \frac{k}{4}\right] = 0, \ \ t \in L_1,$$

which has a unique solution $\varphi'(z) = \frac{k}{4}$, and thus we have

$$\varphi(z) = \frac{k}{4}\,z \tag{6}$$

(an arbitrary constant of integration is assumed to be equal to zero).

Taking into account (6) and putting $c_1 = c_2 = 0$, the boundary conditions (1)–(3) yield

$$\begin{array}{llll}
\operatorname{Im}\left[\dfrac{k}{2}\,t+\psi(t)\right]=0; & \operatorname{Im}\left[\dfrac{k}{2}\,t-\psi(t)\right]=0, & t\in L_1^{(1)}; \\
\operatorname{Re}\left[\dfrac{k}{2}\,t+\psi(t)\right]=P_0; & \operatorname{Re}\left[\dfrac{k}{2}\,t-\psi(t)\right]=ka-P_0, & t\in L_1^{(2)}; \\
\operatorname{Re}\left[\dfrac{k}{2}\,t+\psi(t)\right]=P_0; & \operatorname{Im}\left[\dfrac{k}{2}\,t-\psi(t)\right]=P_0, & t\in L_0^{(1)}; \\
\operatorname{Im}\left[\dfrac{k}{2}\,t+\psi(t)\right]=ka-P_0; & \operatorname{Im}\left[\dfrac{k}{2}\,t-\psi(t)\right]=P_0, & t\in L_1^{(4)}; \\
\operatorname{Re}\left[\dfrac{k}{2}\,t+\psi(t)\right]=0; & \operatorname{Re}\left[\dfrac{k}{2}\,t-\psi(t)\right]=0, & t\in L_1^{(5)}; \\
\operatorname{Re}\left[\dfrac{k}{2}\,t+\psi(t)\right]=0; & \operatorname{Im}\left[\dfrac{k}{2}\,t-\psi(t)\right]=0, & t\in L_0^{(2)}.
\end{array} \tag{7}$$

Let the function $z = \omega(\zeta)$ map conformally the upper half-plane ($\operatorname{Im} \zeta > 0$) onto the domain S. By a_k we denote the preimages of the points A_k ($k = \overline{1,6}$) and assume that $a_3 = -1$; $a_4 = 1$; $\zeta_0 = -\infty$ (where $z_0 = \omega(\zeta_0)$ is the midpoint of the arc A_6A_1). Moreover, owing to the symmetry, we may assume that $a_5 = -a_2$; $a_6 = -a_1$.

The boundary conditions (7) with respect to the function

$$\Phi(\zeta) = -i\left[\frac{k}{2}\,\omega(\zeta) + \psi[\omega(\zeta)]\right]; \quad \Psi(\zeta) = \frac{k}{2}\,\omega(\zeta) - \psi[\omega(\zeta)] \tag{8}$$

can be written in the form

$$\begin{array}{l}
\operatorname{Im}\,\Phi(\tau) = 0, \ \ \tau \in (-\infty; a_1) \cup (-a_2; \infty); \quad \operatorname{Re}\,\Phi(\tau) = 0, \ \ \tau \in (a_1; a_2); \\
\operatorname{Im}\,\Phi(\tau) = -P_0, \ \ \tau \in (a_2; 1); \quad \operatorname{Re}\,\Phi(\tau) = ka - P_0, \ \ \tau \in (1; -a_2);
\end{array} \tag{9}$$

$$\begin{array}{l}
\operatorname{Im}\,\Psi(\tau) = 0, \ \ \tau \in (-\infty; a_2) \cup (-a_1; \infty); \\
\operatorname{Re}\,\Psi(\tau) = ka - P_0, \ \ \tau \in (a_2; -1); \\
\operatorname{Im}\,\Psi(\tau) = P_0, \ \ \tau \in (-1; -a_2); \quad \operatorname{Re}\,\Psi(\tau) = 0, \ \ \tau \in (-a_2; -a_1).
\end{array} \tag{10}$$

The problems (9) and (10) represent the Keldysh-Sedov problems [10], [11] for the half-plane $\operatorname{Im} \zeta > 0$.

Consider now the problem (9). We will seek for a bounded at infinity solution of the problem. The necessary and sufficient condition for the existence of such a solution is of the form

$$-i\,P_0 \int\limits_{a_2}^{1} \frac{d\tau}{\chi_1(\tau)} + (ka - P_0) \int\limits_{1}^{-a_2} \frac{d\tau}{\chi_1(\tau)} = 0 \tag{11}$$

and a solution itself is given by the formula

$$\Phi(\zeta) = \frac{\chi_1(\zeta)}{\pi i}\left[-i\,P_0\int\limits_{a_2}^{1}\frac{d\tau}{\chi_1(\tau)(\tau-\zeta)} + (ka-P_0)\int\limits_{1}^{-a_2}\frac{d\tau}{\chi_1(\tau)(\tau-\zeta)}\right], \quad (12)$$

where $\chi_1(\zeta) = \sqrt{(\zeta-a_1)(\zeta-a_2)(\zeta-1)(\zeta+a_2)}$ (under the radical sign we mean a branch whose decomposition near the point at infinity has the form $\sqrt{(\zeta-a_1)(\zeta-a_2)(\zeta-1)(\zeta+a_2)} = \zeta^2+\alpha_1\zeta+\alpha_2+\cdots$. The radicals appearing in the sequel will be meant analogously).

Similarly, the necessary and sufficient condition for the existence of a bounded at infinity solution of the problem (10) is of the form

$$(ka-P_0)\int\limits_{a_2}^{-1}\frac{d\tau}{\chi_2(\tau)} + i\,P_0\int\limits_{-1}^{-a_2}\frac{d\tau}{\chi_2(\tau)} = 0 \quad (13)$$

and such a solution is represented by the formula

$$\Psi(\zeta) = \frac{\chi_2(\zeta)}{\pi i}\left[(ka-P_0)\int\limits_{a_2}^{-1}\frac{d\tau}{\chi_2(\tau)(\tau-\zeta)} + i\,P_0\int\limits_{-1}^{-a_2}\frac{d\tau}{\chi_2(\tau)(\tau-\zeta)}\right], \quad (14)$$

where $\chi_2(\zeta) = \sqrt{(\zeta-a_2)(\zeta+1)(\zeta+a_2)(\zeta+a_1)}$.

Having found the functions $\Phi(\zeta)$ and $\Psi(\zeta)$, by virtue of (8), we can define the functions $\omega(\zeta)$ and $\psi[\omega(\zeta)]$ by the formulas

$$\omega(\zeta) = \frac{1}{k}\big[i\,\Phi(\zeta)+\Psi(\zeta)\big]; \quad \psi[\omega(\zeta)] = \frac{1}{2}\big[i\,\Phi(\zeta)-\Psi(\zeta)\big]. \quad (15)$$

Let us now pass to finding analytical form of the unknown equally strong contour. Equations for the parts $L_0^{(1)}$ and $L_0^{(2)}$ of the unknown contour can be obtained from the image of the function $\omega(\zeta)$ for $\zeta=\xi\in(-1;1)$ and $\zeta=\xi\in(-\infty;a_1)\cup(-a_1;\infty)$, respectively.

Taking into account the fact that $|\chi_2(-\xi)| = |\chi_1(\xi)|$, formulas (12), (14) and (15) result in

$$\omega(\zeta) = \frac{1}{k}\big[P_0 + A(-\xi) + i(P_0 + A(\xi))\big], \quad (16)$$

where

$$A(\xi) = \frac{|\chi_1(\xi)|}{\pi}\left[-P_0\int\limits_{a_2}^{1}\frac{d\tau}{|\chi_1(\tau)|(\tau-\xi)} + (ka-P_0)\int\limits_{1}^{-a_2}\frac{d\tau}{|\chi_1(\tau)|(\tau-\xi)}\right]. \quad (17)$$

Analogously, for $\xi\in(-\infty;a_1)\cup(-a_1;\infty)$, $\omega(\xi)$ has the form

$$\omega(\xi) = \frac{1}{k}\big[A(-\xi) + i\,A(\xi)\big]. \quad (18)$$

The conditions (11) and (12) are the same and hence to find k, we obtain the formula

$$k = \frac{P}{a}\left[1 + \frac{F_1}{F_2}\right], \tag{19}$$

where

$$F_1 = \int_{a_2}^{1} \frac{d\tau}{|\chi_1(\tau)|}\,; \quad F_2 = \int_{1}^{-a_2} \frac{d\tau}{|\chi_1(\tau)|}\,. \tag{20}$$

It should be noted that the integrals appearing in (17) and (20) are expressed in terms of elliptic integrals of the first and third kind [12].

Of special importance is the definition of parameters k, a_1 and a_2 involved in the above formulas.

Refer now to formulas (19) and (20). The values F_1 and F_2 are the complete elliptic integrals of the first kind ([12]), namely,

$$F_1 = M^{-1} \cdot F\Big(\frac{\pi}{2}\,/m_1\Big); \quad F_2 = M^{-1} \cdot F\Big(\frac{\pi}{2}\,/m_2\Big),$$

where

$$M = \sqrt{2} \cdot \big[a_2(a_1 - 1)\big]^{-\frac{1}{2}}, \quad F\Big(\frac{\pi}{2}\,/m\Big) = \int_0^{\pi/2} \big(1 - m\sin^2\vartheta\big)^{-\frac{1}{2}} d\vartheta,$$

$$m_1 = \frac{(a_2 - 1)(a_2 + a_1)}{2a_2(a_1 - 1)}\,, \quad m_2 = \frac{(a_2 + 1)(a_1 - a_2)}{2a_2(a_1 - 1)}$$

(of interest is the fact that $m_1 + m_2 = 1$ and $m_1 > m_2$).

Fixing the value of the parameter m_1 (and hence of the parameter $m_2 = 1 - m_1$), for finding a_1 and a_2, we obtain the equality

$$a_2^2 + (1 - 2m_1)(a_1 - 1)a_2 - a_1 = 0 \ \ \text{(under the condition } a_1 < a_2 < -1). \tag{21}$$

The discriminant of the above equation (with respect to a_2) is of the form

$$D = (1 - 2m_1)^2(a_1 - 1)^2 + 4a_1.$$

Introducing the notation $\sqrt{-a_1} = x$, from the condition $D \geq 0$, $x > 1$ we get

$$x \geq \frac{1 + 2\sqrt{m_1(1 - m_1)}}{2m_1 - 1} = A.$$

If we assume that $D > 0$, then to every value $x > A$, and hence $a_1 < -A^2$, according to (21), there correspond two values a_2, both satisfying the condition $a_2 < -1$, but this contradicts the condition of the uniqueness of the conformally mapping function $z = \omega(\zeta)$, and hence we should have $D = 0$ from which it follows that

$$a_1 = -\left[\frac{1 + 2\sqrt{m_1(1 - m_1)}}{2m_1 - 1}\right]^2; \quad a_2 = \frac{(2m_1 - 1)(a_1 - 1)}{2}\,. \tag{22}$$

Summing the obtained results, we conclude that for the fixed m_1 in the domain $\left(\frac{1}{2};1\right)$, from the table of complete elliptic integrals we can find F_1 and F_2, and using formulas (19) and (22), we define parameters k, a_1, a_2 and the conformally mapping function $z=\omega(\zeta)$ (formulas (16) and (18)) which establishes analytical form of the unknown equally strong contour.

Direct calculations show that as m_1 increases, the length of the contour $L_0^{(1)}$ decreases, $L_0^{(2)}$ increases, and k increases.

In a particular case, for $m_1=0,75$, we have approximately ([13])

$$
\begin{aligned}
&F_1=2,156;\;\; F_2=1,686;\;\; k=\frac{P_0}{a}\cdot 2,28;\;\; a_1=-13,7;\;\; a_2=-3,7;\\
&A(0)=0,743\,P_0;\;\; w(0)=(0,764\,a;\,0,764\,a);\\
&A(-1)=0,386\,P_0;\;\; \omega(-1)=(a;\,0,608\,a);\\
&A(\infty)=A(-\infty)=1,08\,P_0;\;\; \omega(\infty)=\omega(-\infty)=(0,474\,a;\,0,474\,a);\\
&A(-a_1)=1,451\,P_0;\;\; w(a_1)=(0,636\,a;\,0).
\end{aligned}
$$

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