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NEUMANN BOUNDARY VALUE PROBLEMS OF SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS WITH JACOBI OPERATORS

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Abstract. In this paper, by using the critical point theory some sufficient conditions for the existence and multiplicity of solutions for the Neumann boundary value problems to second nonlinear functional difference equations with Jacobi operators are obtained. The proof is based on the variational structure and Linking Theorem.

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1. Introduction

Let **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers, respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \dots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$. k is a positive integer and * is the transpose sign for a vector.

Consider the second order functional difference equation

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}) \tag{1}$$

with Neumann boundary value conditions

$$\Delta u_0 = A, \ \ \Delta u_k = B, \tag{2}$$

where the operator L is the Jacobi operator

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n,$$

a_n and b_n are real valued for each $n \in \mathbf{Z}$, $f \in C(\mathbf{R}^4, \mathbf{R})$, A and B are constants and Δ is the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$.

Jacobi operators appear in a variety of applications, see for example [16]. They can be viewed as the discrete analogue of Sturm-Liouville operators

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and their investigation has many similarities with Sturm-Liouville theory. Whereas numerous books about Sturm-Liouville operators have been written, only few on Jacobi operators exist. In particular, there are currently fewer researches available which cover some basic topics (like positive solutions, periodic operators, boundary value problems, etc.) typically found in textbooks on Sturm-Liouville operators [9].

We may think of Eq. (1) as being a discrete analogue of the following equation

$$Su(t) = f(t, u(t+1), u(t), u(t-1)),\ t \in \mathbf{R} \tag{3}$$

which includes the following equation

$$c^2 u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)),\ t \in \mathbf{R}. \tag{4}$$

Here S is the Sturm-Liouville differential expression and $f \in C(\mathbf{R}^4, \mathbf{R})$. Eq. (4) has been studied extensively by many scholars. For example, Smets and Willem [15] have obtained the existence of solitary waves of Eq. (4).

The difference equations have widely occurred as the mathematical models describing real life situations in probability theory, matrix theory, electrical circuit analysis, combinatorial analysis, queuing theory, number theory, psychology and sociology, etc., see [1,5,7,8,10,14]. For example, the simple logistic equation

$$u_{n+1} = ru_n$$

is a formula for approximating the evolution of an animal population over time, where u_n is the number of animals this year, u_{n+1} is the number next year and r is the growth rate or fecundity. The the price-demand curve of cobweb phenomenon

$$D_n = -m_d p_n + b_d,\ m_d > 0,\ b_d > 0$$

is the economics application of difference equations, where D_n is the number of units demanded in period n, p_n is the price per unit in period n and m_d represents the sensitivity of consumers to price.

Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity and results on oscillation and other topics, see [1,2-4,6,8,11,13,17-19]. However, to our best knowledge, no similar results are obtained in the literature for the Neumann boundary value problem (BVP) (1) with (2). Since f in Eq. (1) depends on u_{n+1} and u_{n-1}, the traditional ways of establishing the functional in [2,17-19] are inapplicable to our case.

Our aim in this paper is to use the critical point theory to give some sufficient conditions for the existence and multiplicity of the BVP (1) with (2). The main idea in this paper is to transfer the existence of the BVP (1) with (2) into the existence of the critical points of some functional.

Our main results are as follows.

Theorem 1. *Assume that $A = 0$, $B = 0$ and the following hypotheses are satisfied:*

(F_1) *there exists a functional $F(n, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ with $F(0, \cdot) = 0$ such that*

$$\lim_{r \to 0} \frac{F(n, v_1, v_2)}{r^2} = 0, \; r = \sqrt{v_1^2 + v_2^2}, \; \forall n \in \boldsymbol{Z}(1, k);$$

(F_2) *there exist constants $R > 0$, $\beta > 2$ such that for any $n \in \boldsymbol{Z}(1, k)$,*

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),$$

$$\frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \le \beta F(n, v_1, v_2) < 0, \tag{5}$$

$$\forall \sqrt{v_1^2 + v_2^2} \ge R;$$

(F_3) *for any $n \in \boldsymbol{Z}(1, k-1)$, $a_n > 0$; for any $n \in \boldsymbol{Z}(1, k)$, $b_n + a_{n-1} + a_n \equiv 0$. Then the BVP* (1) *with* (2) *possesses at least three solutions.*

Remark 1. (5) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$F(n, v_1, v_2) \le -a_1 \left(\sqrt{v_1^2 + v_2^2} \right)^{\beta} + a_2, \; \forall n \in \mathbf{Z}(1, k). \tag{6}$$

Corollary 1. *Assume that $A = 0$, $B = 0$ and $(F_1) - (F_3)$ are satisfied. Then the BVP* (1) *with* (2) *possesses at least two nontrivial solutions.*

The rest of the paper is organized as follows. In Section 2 we shall establish the variational framework for the BVP (1) with (2) in order to apply the critical point method and give some useful lemmas. In Section 3 we shall complete the proof of the main results and give an example to illustrate the main result.

2. Variational Structure and Some Lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (1) with (2) and give some basic notations and useful lemmas.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v \rangle = \sum_{j=1}^{k} u_j v_j, \;\; \forall u, v \in \mathbf{R}^k, \tag{7}$$

by which the norm $\| \cdot \|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2 \right)^{\frac{1}{2}}, \;\; \forall u \in \mathbf{R}^k. \tag{8}$$

On the other hand, we define the norm $\|\cdot\|_r$ on $\mathbf{R}^k$ as follows:

$$\|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r\right)^{\frac{1}{r}}, \tag{9}$$

for all $u \in \mathbf{R}^k$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_r \leq c_2\|u\|_2,\ \forall u \in \mathbf{R}^k. \tag{10}$$

Clearly, $\|u\| = \|u\|_2$. For the BVP (1) with (2), when $k > 1$, consider the functional J on $\mathbf{R}^k$ as follows:

$$\begin{aligned} J(u) = &\frac{1}{2}\sum_{n=1}^{k-1} a_n(\Delta u_n)^2 - \frac{1}{2}\sum_{n=1}^{k}(b_n + a_{n-1} + a_n)u_n^2 + \\ &+\sum_{n=1}^{k} F(n, u_{n+1}, u_n) - a_k B u_k + a_0 A u_1, \\ &\forall u = (u_1, u_2, \ldots, u_k)^* \in \mathbf{R}^k,\ \ \Delta u_0 = A,\ \ \Delta u_k = B. \end{aligned} \tag{11}$$

Clearly, $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $u = \{u_n\}_{n\in\mathbf{Z}(1,k)} \in \mathbf{R}^k$, by using $\Delta u_0 = A,\ \Delta u_k = B$, we can compute the partial derivative as

$$\begin{aligned} \frac{\partial J}{\partial u_n} &= -\,a_n\Delta u_n + a_{n-1}\Delta u_{n-1} - (b_n + a_{n-1} + a_n)u_n + f(n, u_{n+1}, u_n, u_{n-1}) \\ &= -\,Lu_n + f(n, u_{n+1}, u_n, u_{n-1}),\ \ n \in \mathbf{Z}(1,k). \end{aligned}$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}),\ \forall n \in \mathbf{Z}(1,k).$$

We reduce the existence of the BVP (1) with (2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1) with (2).

Remark 2. In the case $k = 1$ is trivial, and we omit its proof.

Denote

$$W = \{(u_1, u_2, \ldots, u_k)^* \in \mathbf{R}^k | u_n \equiv v,\ v \in \mathbf{R},\ n \in \mathbf{Z}(1,k)\}$$

and Y be the direct orthogonal complement of $\mathbf{R}^k$ to W, i.e., $\mathbf{R}^k = Y \oplus W$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to be satisfying the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(k)}\right\} \subset E$ for which $\left\{J\left(u^{(k)}\right)\right\}$ is bounded and $J'\left(u^{(k)}\right) \to 0 (k \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 1 (Linking Theorem [12,20])**.** *Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, \boldsymbol{R})$ satisfies the P.S. condition and*

(J_1) there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;

(J_2) there exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{re | 0 < r < R_0\}$.

Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$, where id denotes the identity operator.

Let

$$p_{\max} = \max\{a_n : n \in \mathbf{Z}(1, k-1)\},\ p_{\min} = \min\{a_n : n \in \mathbf{Z}(1, k-1)\},$$
$$q_{\max} = \max\{b_n + a_{n-1} + a_n : n \in \mathbf{Z}(1, k)\},$$
$$q_{\min} = \min\{b_n + a_{n-1} + a_n : n \in \mathbf{Z}(1, k)\},$$
$$p = \max\{|a_n| : n \in \mathbf{Z}(0, k)\},\ q = \max\{|b_n + a_{n-1} + a_n| : n \in \mathbf{Z}(1, k)\}.$$

Lemma 2. *Assume that $A = 0$, $B = 0$ and $(F_1) - (F_3)$ are satisfied. Then the functional J is bounded from above in $\boldsymbol{R}^k$*

Proof. By (8), for any $u \in \mathbf{R}^k$,

$$\begin{aligned} J(u) =& \frac{1}{2} \sum_{n=1}^{k-1} a_n (\Delta u_n)^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \leq \\ \leq& p_{\max} \sum_{n=1}^{k-1} (u_{n+1}^2 + u_n^2) - a_1 \sum_{n=1}^{k} \left(\sqrt{u_{n+1}^2 + u_n^2}\right)^\beta + a_2 k \leq \\ \leq& 2p_{\max} \sum_{n=1}^{k} u_n^2 - a_1 \sum_{n=1}^{k} |u_n|^\beta + a_2 k \leq \\ \leq& 2p_{\max} \|u\|^2 - a_1 c_1^\beta \|u\|^\beta + a_2 k. \end{aligned} \tag{12}$$

Since $\beta > 2$, there exists a constant $M_1 > 0$ such that $J(u) \leq M_1,\ \forall u \in \mathbf{R}^k$. The proof of Lemma 2 is complete. □

Lemma 3. *Assume that $A = 0$, $B = 0$ and $(F_1) - (F_3)$ are satisfied. Then the functional J satisfies the P.S. condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k,\ l \in \mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded. Then there exists a positive constant M_2 such that

$$-M_2 \leq J\left(u^{(l)}\right) \leq M_2,\ \forall l \in \mathbf{N}.$$

By the proof of Lemma 2, it is easy to see that

$$-M_2 \leq J\left(u^{(l)}\right) \leq 2p_{\max}\left\|u^{(l)}\right\|^2 - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + a_2 k.$$

That is,

$$a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta - 2p_{\max}\left\|u^{(l)}\right\|^2 \leq M_2 + a_2 k.$$

Since $\beta > 2$, there exists a constant $M_3 > 0$ such that

$$\left\|u^{(l)}\right\| \leq M_3, \ \forall l \in \mathbf{N}.$$

Therefore, $\left\{u^{(l)}\right\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\left\{u^{(l)}\right\}$ possesses a convergence subsequence in $\mathbf{R}^k$. And thus the P.S. condition is verified. □

3. Proof of the Main Results

Proof of Theorem 1. Assumptions (F_1) and (F_2) imply that $F(n,0) = 0$ and $f(n,0) = 0$ for $n \in \mathbf{Z}(1,k)$. Then $u = 0$ is a trivial solution of the BVP (1) with (2).

By Lemma 2, J is bounded from the upper on $\mathbf{R}^k$. We define $c_0 = \sup_{u\in\mathbf{R}^k} J(u)$. The proof of Lemma 2 implies $\lim_{\|u\|\to+\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in \mathbf{R}^k$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J.

We claim that $c_0 > 0$. Indeed, by (F_1), for any $\epsilon = \frac{1}{12}p_{\min}\lambda_2$($\lambda_2$ can be referred to (13)), there exists $\rho > 0$, such that

$$|F(n,v_1,v_2)| \leq \frac{1}{12}p_{\min}\lambda_2(v_1^2+v_2^2), \forall n \in \mathbf{Z}(1,k),$$

for $\sqrt{v_1^2+v_2^2} \leq \sqrt{2}\rho$.

For any $u = (u_1,u_2,\ldots,u_k)^* \in Y$ and $\|u\| \leq \rho$, we have $|u_n| \leq \rho, \ n \in \mathbf{Z}(1,k)$.

When $k \geq 2$,

$$\begin{aligned} J(u) =& \frac{1}{2}\sum_{n=1}^{k-1} a_n(u_{n+1}-u_n)^2 + \sum_{n=1}^{k} F(n,u_{n+1},u_n) \geq \\ \geq& \frac{1}{2}p_{\min}\sum_{n=1}^{k-1}(u_{n+1}-u_n)^2 - \frac{1}{12}p_{\min}\lambda_2\sum_{n=1}^{k}(u_{n+1}^2+u_n^2) \geq \\ \geq& \frac{1}{2}p_{\min}(u^*Du) - \frac{1}{4}p_{\min}\lambda_2\|u\|^2, \end{aligned}$$

where

$$D = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}_{k\times k}.$$

Clearly, $\lambda_1 = 0$ is an eigenvalue of D and $\xi = (v, v, \dots, v) \in \mathbf{R}^k (v \neq 0,\ v \in \mathbf{R})$ is an eigenvector of D corresponding to 0. Let $\lambda_2, \lambda_3, \dots, \lambda_k$ be the other eigenvalues of D. Applying matrix theory, we know $\lambda_j > 0$, $j = 2, 3, \dots, k$. Without loss of generality, we may assume that

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k, \tag{13}$$

then for any $u \in Y$, we have

$$J(u) \geq \left(\frac{1}{2} p_{\min}\lambda_2 - \frac{1}{4} p_{\min}\lambda_2\right) \|u\|^2 = \frac{1}{4} p_{\min}\lambda_2 \|u\|^2.$$

Take

$$a = \frac{1}{4} p_{\min}\lambda_2 \|\rho\|^2.$$

Therefore,

$$J(u) \geq a > 0,\ \ \forall u \in Y \cap \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{Y\cap\partial B_\rho} \geq a$. That is to say, J satisfies the condition (J_1) of the Linking Theorem.

In order to exploit the Linking Theorem in critical point theory, we need to verify other conditions of the Linking Theorem. By Lemma 3, J satisfies the P.S. condition. So it suffices to verify the condition (J_2).

Take $e \in \partial B_1 \cap Y$, for any $w \in W$ and $r \in \mathbf{R}$, let $u = re + w$. Then

$$\begin{aligned} J(u) =& \frac{1}{2} \sum_{n=1}^{k-1} a_n (re_{n+1} + \omega_{n+1} - re_n - \omega_n)^2 + \\ &+ \sum_{n=1}^{k} F(n, re_{n+1} + \omega_{n+1}, re_n + \omega_n) \leq \\ &\leq \frac{p_{\max} r^2}{2} \sum_{n=1}^{k-1} (e_{n+1} - e_n)^2 - \\ &- a_1 \sum_{n=1}^{k} \left[\sqrt{(re_{n+1} + \omega_{n+1})^2 + (re_n + \omega_n)^2}\right]^\beta + a_2 k \leq \\ &\leq 2(k-1) p_{\max} r^2 - a_1 c_1^\beta \left(\sum_{n=1}^{k} |re_n + \omega_n|^2\right)^{\frac{\beta}{2}} + a_2 k = \end{aligned}$$

$$=2(k-1)p_{\max}r^2 - a_1c_1^{\beta}(r^2+\|\omega\|^2)^{\frac{\beta}{2}} + a_2k \le$$
$$\le 2(k-1)p_{\max}r^2 - a_1c_1^{\beta}r^{\beta} - a_1c_1^{\beta}\|\omega\|^{\beta} + a_2k.$$

Let

$$g_1(r) = (2k-1)p_{\max}r^2 - a_1c_1^{\beta}r^{\beta},\ g_2(t) = -a_1c_1^{\beta}t^{\beta} + p_{\max}t^2 + a_2k.$$

Then

$$\lim_{r\to+\infty} g_1(r) = -\infty,\ \lim_{t\to+\infty} g_2(t) = -\infty,$$

$g_1(r)$ and $g_2(t)$ are bounded from above. It is easy to see that there exists a positive constant $R_1 > \rho$ such that for any $u \in \partial Q$, $J(u) \le 0$, where $Q = (\bar{B}_{R_1} \cap W) \oplus \{re|0 < r < R_1\}$. By the Linking Theorem, J possesses a critical value $c \ge a > 0$, where

$$c = \inf_{h\in\Gamma} \sup_{u\in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, \mathbf{R}^k) \mid h|_{\partial Q} = id\}$.

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. If $\tilde{u} \ne \bar{u}$, then the conclusion of Theorem 1 holds. Otherwise, $\tilde{u} = \bar{u}$. Then $c_0 = J(\bar{u}) = J(\tilde{u}) = c$, that is $\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma}\sup_{u\in Q} J(h(u))$. Choosing $h = id$, we have $\sup_{u\in Q} J(u) = c_0$. Since the choice of $e \in \partial B_1 \cap Y$ is arbitrary, we can take $-e \in \partial B_1 \cap Y$. Similarly, there exists a positive number $R_2 > \rho$, for any $u \in \partial Q_1$, $J(u) \le 0$, where $Q_1 = (\bar{B}_{R_2} \cap W) \oplus \{-re|0 < r < R_2\}$.

Again, by the Linking Theorem, J possesses a critical value $c' \ge a > 0$, where

$$c' = \inf_{h\in\Gamma_1} \sup_{u\in Q_1} J(h(u)),$$

and $\Gamma_1 = \{h \in C(\bar{Q}_1, \mathbf{R}^k) \mid h|_{\partial Q_1} = id\}$.

If $c' \ne c_0$, then the proof is finished. If $c' = c_0$, then $\sup_{u\in Q_1} J(u) = c_0$. Due to the fact $J|_{\partial Q} \le 0$ and $J|_{\partial Q_1} \le 0$, J attains its maximum at some points in the interior of sets Q and Q_1. However, $Q \cap Q_1 \subset W$ and $J(u) \le 0$ for any $u \in W$. Therefore, there must be a point $u' \in \mathbf{R}^k$, $u' \ne \tilde{u}$ and $J(u') = c' = c_0$. The above argument implies that the BVP (1) with (2) possesses at least two nontrivial solutions when $k \ge 2$.

In the case $k = 1$, it is easy to complete the proof of Theorem 1.

The proof of Theorem 1 is complete. □

Remark 3. Due to Theorem 1, the conclusion of Corollary 1 is obviously true.

Remark 4. As an application of Theorem 1, finally, we give an example to illustrate our main result.

Example 1. For all $n \in \mathbf{Z}(1,k)$, assume that

$$u_{n+1} + u_{n-1} - 2u_n =$$
$$= -\beta u_n \left[\varphi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\beta}{2}-1}\right] \qquad (14)$$

with Neumann boundary value conditions

$$\Delta u_0 = 0, \;\; \Delta u_k = 0, \qquad (15)$$

where $\beta > 2$, φ is continuously differentiable and $\varphi(n) > 0,\; n \in \mathbf{Z}(1,k)$ with $\varphi(0) = 0$.

We have

$$a_n = a_{n-1} \equiv 1, \;\; b_n \equiv -2,$$
$$f(n, v_1, v_2, v_3) = -\beta v_2 \left[\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\beta}{2}-1}\right]$$

and

$$F(n, v_1, v_2) = -\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}}.$$

Then

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} =$$
$$= -\beta v_2 \left[\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\beta}{2}-1}\right].$$

It is easy to verify all the assumptions of Theorem 1 are satisfied and then the BVP (14) with (15) possesses at least three solutions.

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