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ON ONE PROPERTY OF A PERIODIC DECIMAL FRACTION, INVERSE TO A PRIME NUMBER

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Abstract. Let there exist two numbers each of which is written by means of k figures $\overline{a_1a_2\dots a_k}$ and $\overline{b_1b_2\dots b_k}$. The figures $\overline{a_1a_2\dots a_k}$ and $\overline{b_1b_2\dots b_k}$ will be called "identical on a circumference" if there exists $i \in \overline{1;k-1}$ such that $\overline{b_1b_2\dots b_k} = \overline{a_{i+1}a_{i+2}\dots a_ka_1a_2\dots a_i}$.

It is proved that: (I) if for natural $p > 2$ there exists the other than zero number $\overline{a_1a_2\dots a_{p-1}}$, $a_i \in \overline{0;9}$ which after multiplication by every $i \in \overline{1;p-1}$ results in "identical on the circumference" numbers, then p is a prime number, and $1/p = 0,(\overline{a_1a_2\dots a_{p-1}})$;

(II) Let p be a prime number, $1/p = 0,(\overline{a_1a_2\dots a_{p-1}})$ and, in addition, $\overline{a_1a_2\dots a_{p-1}}$ be not multiple of p, then for every $n \in N$, the length of writing the period of the number $1/p^{n+1}$ in the form of a periodic decimal fraction will be equal to $p^n(p-1)$; multiplying that period by any, not multiple of p number from the interval $[1;p^{n+1}-1]$, we obtain the "identical on the circumference" numbers.

რეზიუმე. ვთქვათ მოცემულია 2 რიცხვი, რომელთაგან თითოეული ჩაწერილია k ციფრით $\overline{a_1a_2\dots a_k}$ და $\overline{b_1b_2\dots b_k}$. $\overline{a_1a_2\dots a_k}$ და $\overline{b_1b_2\dots b_k}$ რიცხვებს ვუწოდოთ "ერთი და იგივე რიცხვები წრეწირზე", თუ არსებობს ისეთი $i \in \overline{1;k-1}$, რომ $\overline{b_1b_2\dots b_k} = \overline{a_{i+1}a_{i+2}\dots a_ka_1a_2\dots a_i}$.

დამტკიცებულია: (I) თუ $p > 2$-თვის არსებობს $p-1$ ციფრისგან ჩაწერილი, ნულისაგან განსხვავებული $\overline{a_1a_2\dots a_{p-1}}$ რიცხვი, რომლის გადამრავლებით ყოველ $i \in \overline{1;p-1}$-ზე მიიღება "ერთი და იგივე რიცხვები წრეწირზე", მაშინ p მარტივი რიცხვია და $1/p = 0,(\overline{a_1a_2\dots a_{p-1}})$;

(II) თუ p მარტივი რიცხვია, $1/p = 0,(\overline{a_1a_2\dots a_{p-1}})$ და ამასთან $\overline{a_1a_2\dots a_{p-1}}$ არაა p-ს ჯერადი, მაშინ ყოველი $n \in N$-თვის $1/p^{n+1}$ რიცხვის პირიოდულ ათწილადად ჩაწერისას მიღებული პერიოდის სიგრძე $p^n(p-1)$ და მიღებული პერიოდის გადამრავლებით p-ს არაჯერად ნებისმიერ მთელ რიცხვზე $[1;p^{n+1}-1]$ შუალედიდან მიიღება "ერთი და იგივე რიცხვები წრეწირზე".

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Throughout the paper, under the period of a decimal fraction will be meant the least period.

Theorem 1. *Let p and q be natural numbers, $p > 2$. Then for the representation of the number $\frac{q}{p}$ in the form of a decimal fraction a number of figures after the decimal point to the end of the period does not exceed $(p-1)$.*

Proof. We represent the number $\frac{q}{p}$ in the form of a periodic decimal fraction and divide q by p. The quotient after the division of q by p we denote by a_0 (integral part of the decimal fraction), and the reminder by b_0. To the number b_0 we add 0 and the obtained number $\overline{b_0 0}$ divide by p. The quotient after the division of $\overline{b_0 0}$ by p we denote by a_1 (decimal figure in the periodic writing of the number $\frac{q}{p}$) and the reminder by b_1. To the number b_1 we add 0 and the obtained number $\overline{b_1 0}$ we divide by p. The quotient obtained after the division of $\overline{b_1 0}$ by p we denote by a_2 (the hundredth figure in the periodic writing of the number $\frac{q}{p}$), and the reminder by b_2, and so on. In a general case for $k > 1$, the quotient after the division of $\overline{b_{k-1} 0}$ by p we denote by a_k and the reminder by b_k.

Consider separately the cases where for every k, $b_k \neq 0$, and for some k, $b_k = 0$.

(I) If for every k, $b_k \neq 0$, then b_k may be only one of the numbers $1; 2; \dots; (p-1)$, and hence we can get maximum $(p-1)$ different values. Therefore for some k we obtain a reminder which is equal to someone we obtained earlier: $b_k = b_i$, $i < k$. Obviously, in this case, $a_{k+1} = a_{i+1}$, $a_{k+2} = a_{i+2}, \dots$ and $\frac{q}{p} = a_0, \overline{a_1 a_2 \dots a_i (a_{i+1} \dots a_k)}$.

If $i = 0$, then $\frac{q}{p} = a_0, (\overline{a_1 a_2 \dots a_k})$. Since $b_i \neq b_j$, if $i \neq j$, a number of figures in the period may not exceed a number of different reminders obtained after the division by p, not more than $(p-1)$, which was to be demonstrated.

If $i > 0$, then the reminder b_i after the decimal point is encountered twice (as a result of division of $\overline{b_{i-1} 0}$ by p, we obtain the quotient a_i, the reminder b_i, and after the division of $\overline{b_{k-1} 0}$ by p we obtain the quotient a_k and the same reminder $b_i : b_k = b_i$), but in this case the reminder may have maximum $p-2$ different values, all values except b_0. And in this case, $k \leq (p-2) + 1 = p - 1$. Thus the theorem is complete.

(II) If for some k, $b_k = 0$, then the number $\frac{q}{p}$ will be written in the form of a finite decimal fraction $\frac{q}{p} = a_k$ (an infinite periodic decimal fraction of period 0). This is quite possible if and only if there exist two nonnegative integers m and n at least one of which is the other than zero, such that $p = 2^m \cdot 5^n$, and $k = \max(m, n)$. In the decimal writing of the number $\frac{q}{p}$, a number of figures after the decimal point to the end of the period (we mean period 0) is $k+1$.

If $n \neq 0$, then $p > 2^k \geq k+1$. Consequently, $k+1 \leq p-1$, and in this case the theorem is complete.

If $n = 0$, then $p = 2^m$, $k = m$ and since $p > 2$, consequently, $m > 1$, and hence $p = 2^m > m+1 = k+1$. Thus the theorem in this case too, is complete.

Let we have two numbers, each written by means of k figures $\overline{a_1a_2\dots a_k}$ and $\overline{b_1b_2\dots b_k}$. The numbers $\overline{a_1a_2\dots a_k}$ and $\overline{b_1b_2\dots b_k}$ will be called "identical on the ciscumference" if there exists $i \in \overline{1;k-1}$ such that $\overline{b_1b_2\dots b_k} = \overline{a_{i+1}a_{i+2}\dots a_ka_1a_2\dots a_i}$. For example, the numbers 35084, 50843, 08435, 84350, 43508 are "identical on the circumference". □

Theorem 2. *Let $p > 2$ be a natural number and $\frac{1}{p} = 0,(\overline{a_1a_2\dots a_k})$. Then in the interval $[2;p-1]$ there exist $k-1$ integers such that the numbers obtained after the multiplication of $a_1a_2\dots a_k$ by the above-mentioned numbers and the number $\overline{a_1a_2\dots a_k}$ are the numbers, "identical on the circumference".*

Proof. We represent the number $\frac{1}{p}$ in the form of a decimal fraction: we divide 10 by p. The quotient after the division of 10 by p is a_1 (the decimal figure in the periodic writing of the number $\frac{1}{p}$), and the reminder we denote by b_1. To the number b_1 we add 0 and the obtained number $\overline{b_10}$ we divide by p. The quotient after the division of $\overline{b_10}$ is a_2 (the hundredth figure in the periodic writing of the number $\frac{1}{p}$), and the reminder we denote by b_2, and so on. Since $\frac{1}{p} = 0(\overline{a_1a_2\dots a_k})$, therefore the quotient after the division of $\overline{b_{k-1}0}$ by p is a_k, and the reminder is equal to 1 ($b_k = 1$).

Obviously, for every integer $i \in \overline{1;k-1}$, the reminder $b_i \in \overline{2;p-1}$, and in addition,

$$\frac{b_i}{p} = 0,(\overline{a_{i+1}\dots a_ka_1\dots a_1}) = \frac{\overline{a_{i+1}\dots a_ka_1\dots a_1}}{\underbrace{99\dots9}_{k}}.$$

On the other hand,

$$\frac{b_i}{p} = b_i\,\frac{1}{p} = b_i\cdot\frac{\overline{a_2a_2\dots a_k}}{\underbrace{99\dots9}_{k}},$$

therefore the equality

$$b_i\cdot\overline{a_1a_2\dots a_k} = \overline{a_{i+1}\dots a_ka_1\dots a_1}$$

holds.

Consequently, in the interval $[2;p-1]$ there exist $k-1$ integers (namely, reminders b_i) such that the obtained after the multiplication of $\overline{a_1a_2\dots a_k}$ by these numbers and the number $\overline{a_1a_2\dots a_k}$ are the numbers, "identical on the circumference".

Corollary. *Let $p>2$ be a natural number such that $\frac{1}{p}=0,(\overline{a_1a_2\dots a_{p-1}})$, then for any $i\in\overline{2;p-1}$, the numbers obtained by the multiplication of $\overline{a_1a_2\dots a_{p-1}}$ by i and the number $\overline{a_1a_2\dots a_{p-1}}$ are the numbers, "identical on the circumference".*

For example, $\frac{1}{7}=0,(142857)$, therefore by the corollary of Theorem 2, multiplying 14275 by 2 and 3 and so on up to 6, we obtain the numbers, "identical on the circumference". Indeed, $2\cdot 142857=28574$, $3\cdot 142857=428571$, $4\cdot 142857=571428$, $5\cdot 142857=714285$, $6\cdot 142857=857142$.

Here we present several examples of such numbers:
0588235294117647 (period of the number $\frac{1}{17}$);
052631578947368421 (period of the number $\frac{1}{19}$);
0434782608699652173913 (period of the number $\frac{1}{23}$);
0344827586206896551724137931 (period of the number $\frac{1}{29}$);
0212765957446808510638297872340425531914893617 (period of the number $\frac{1}{47}$). $\square$

Theorem 3. *Let $p>2$ be a natural number such that $\frac{1}{p}=0,(\overline{a_1a_2\dots a_{p-1}})$, then p is a prime number.*

Proof. Assume to the contrary that $\frac{1}{p}=0,(\overline{a_1a_2\dots a_{p-1}})$ and p is a composite number: $p=q\cdot r$, $1<q<p$, $1<r<p$. Since the length of the period of writing the number $\frac{1}{p}$ in the form of periodic decimal fraction is equal to $p-1$, therefore for the writing of the number $\frac{1}{p}$ in the form of periodic decimal fraction we obtain as reminders any integers from the interval $[1;p-1]$. In particular, for some i the reminder is $q: b_i=q$. Therefore we have

$$\frac{1}{r}=\frac{q}{p}=\frac{b_i}{p}=\frac{\overline{a_{i+1}a_{i+2}\dots a_{p-1}a_1a_2\dots a_i}}{\underbrace{99\dots 9}_{p-1}}=0,(\overline{a_{i+1}a_{i+2}\dots a_{p-1}a_1a_2\dots a_i}).$$

Thus we have found that for the writing of the number $\frac{1}{r}$ in the form of a periodic decimal fraction the period consists of $p-1$ figures.

On the other hand, if we write the number $\frac{1}{r}$ in the form of a periodic decimal fraction, then the length of the period is, by Theorem 1, no more than $(r-1)$. Since $r-1<p-1$, we obtain the contradiction which proves our theorem. $\square$

Lemma. *Let p and q be mutually prime numbers, and the number p be not multiple of 2 and 5. If the period length of writing the number $\frac{1}{p}$ in the form of a periodic decimal fraction is equal to k, then for the writing of the number $\frac{q}{p}$ in the form of a periodic decimal fraction the period length will likewise be equal to k.*

Proof. Without loss of generality, we assume that $q < p$. Since p is not multiple of 2 and 5, the number $\frac{1}{p}$ will be written in the form of a pure periodic decimal number

$$\frac{1}{p} = 0,(\overline{a_1 a_2 \dots a_k}) = \frac{\overline{a_1 a_2 \dots a_k}}{\underbrace{99\dots9}_{k}}. \tag{1}$$

By virtue of (1),

$$\frac{q}{p} = \frac{q \cdot \overline{a_1 a_2 \dots a_k}}{\underbrace{99\dots9}_{k}}.$$

This implies that for the writing of the number $\frac{q}{p}$ in the form of a periodic decimal fraction the period length does not exceed k. Let the period length be equal to m:

$$\frac{q}{p} = 0,(\overline{b_1 b_2 \dots b_m}) = \frac{\overline{b_1 b_2 \dots b_m}}{\underbrace{99\dots9}_{m}}, \quad m \le k,$$

$$\frac{q \cdot \overbrace{99\dots9}^{m}}{p} = \overline{b_1 b_2 \dots b_m}.$$

Since $\frac{q \cdot \overbrace{99\dots9}^{m}}{p}$ is integer, and p and q are mutually prime numbers, therefore $\frac{\overbrace{99\dots9}^{m}}{p}$ is integer. According to (1), k is the least number for which $\overbrace{99\dots9}^{m}$ is divisible by p without remainder, i.e., $m \ge k$. On the other hand, $m \le k$, consequently, $m = k$, which was to be demonstrated. □

Theorem 4. *Let $p \ge 7$ be a prime number. If for the writing of the number $\frac{q}{p}$ in the form of a periodic decimal fraction the length of the period is equal to k, then k is the divisor of $p - 1$.*

Proof. By Theorem 1, $k \le p - 1$. If $k = p - 1$, the theorem is complete.

Let $k < p$ and q be any integer from the interval $[1; p - 1]$. Since p is a prime number, the numbers p and q are mutually prime numbers, and due to the lemma ($p \ge 7$ is a prime number, not multiple of 2 and 5), the length of the period for the writing of the number $\frac{q}{p}$ in the form of a periodic decimal fraction is equal to k:

$$\frac{q}{p} = \frac{\overline{a_1^{(q)} a_2^{(q)} \dots a_k^{(q)}}}{\underbrace{99\dots9}_{k}}.$$

Recall once more the rule of representation of the number $\frac{q}{p}$ in the form of a periodic decimal fraction: we divide $\overline{q0}$ by p. The remainder obtained after the division of $\overline{q0}$ by p is $a_1^{(q)}$ (decimal figure in the periodic writing of the number $\frac{q}{p}$), and we denote the remainder by $b_1^{(q)}$. To the number $b_1^{(q)}$ we add 0 and the obtained number $\overline{b_1^{(q)}0}$ we divide by p. The reminder obtained after the division of $\overline{b_1^{(q)}0}$ by p is $a_2^{(q)}$ (the hundredth figure in the periodic writing of the number $\frac{q}{p}$), we denote the reminder by $b_2^{(q)}$, and so on. Since $\frac{q}{p} = \frac{\overline{a_1^{(q)}a_2^{(q)}\dots a_k^{(q)}}}{\underbrace{99\dots9}_{k}}$, the quotient obtained after the division of $\overline{b_{k-1}^{(q)}0}$ by p is $a_k^{(q)}$, and the remainder is equal to $q : b_k^{q)} = q$. Obviously, for every i, $1 \le i < k$, the equality

$$\frac{b_i^{(q)}}{p} = 0,(\overline{a_{i+1}^{(q)}a_{i+2}^{(q)}\dots a_k^{(q)}a_1^{(q)}a_2^{(q)}\dots a_i^{(q)}})$$

holds.

That is, for every i, $1 \le i \le k$, the period of the numbers $\frac{b_i^{(q)}}{p}$ are the numbers, "identical on the circumference".

Let q' be the integer from the interval $[1; p-1]$ such that $q' \notin \{b_i^{(q)},\ i \in \overline{1;k}\}$. By the lemma, the period length of the writing of the number $\frac{q'}{p}$ in the form of a periodic decimal fraction is equal to k. Let us show that

$$\{b_i^{(q)},\ i \in \overline{1;k}\} \cap \{b_j^{(q')},\ j \in \overline{1;k}\} = \varnothing..$$

If in the interval $[1; k]$ there are the integers i and j such that $b_i^{(q)} = b_j^{(q')}$, then we find that $b_{i+1}^{(q)} = b_{j+1}^{(q')}$, $b_{i+2}^{(q)} = b_{j+2}^{(q')}$, and so on, i.e., every element from the set $\{b_i^{(q)},\ i \in \overline{1;k}\}$ coincides with that of the set $\{b_j^{(q')},\ j \in \overline{1;k}\}$, in particular, we find that $b_k^{(q')}$ is the element of the set $\{b_i^{(q)},\ i \in \overline{1;k}\}$. But this is impossible because $b_k^{(q')} = q'$ and $q' \notin \{b_i^{(q)},\ i \in \overline{1;k}\}$.

Consequently, the set $\overline{1; p-1}$ decomposes into nonintersecting sets, each consisting of k elements. This is quite possible if and only if $p-1$ is multiple of k, which was to be demonstrated. $\square$

Corollary. *If $p \ge 7$ is a prime number and $\frac{1}{p} = 0,(\overline{a_1a_2\dots a_k})$, then the interval $\overline{1; p-1}$ decomposes into nonintersecting sets consisting of k elements, and in addition, multiplying elements of each set by $\overline{a_1a_2\dots a_k}$, we obtain the numbers which are "identical on the circumference".*

Of significance is the requirement of the theorem for the number p to be prime. For example,

$$\frac{1}{49} = 0,(020408163265306122448979591836734693877551),$$

the length of the period is 42 which is not a divisor of the number 48.

Theorem 5. *If for $p > 2$ there exists the other than zero number $\overline{a_1 a_2 \dots a_{p-1}}$ after multiplication of which by every $i \in \overline{1; p-1}$ we obtain "identical on the circumference" numbers, then p is a prime number, and $\frac{1}{p} = 0, (\overline{a_1 a_2 \dots a_{p-1}})$.*

Proof. It is not difficult to verify that if there exists the number $\overline{a_1 a_2 \dots a_{p-1}}$, then $p > 5$. Here we prove the validity of the theorem for $p > 5$.

Assume that

$$\frac{\overline{a_1 a_2 \dots a_{p-1}}}{\underbrace{99 \dots 9}_{n-1}} = \frac{n}{m},$$

and the fraction $\frac{n}{m}$ is irreducible. Let us show that $n = 1$, $m = p$.

The remainder obtained after the division of the number $10 \cdot n$ by m we denote by b_1. Let for every $i \in \overline{2; p-1}$, b_i denote the reminder after the division of $\overline{b_{i-1} 0}$ by m. Obviously, $b_{p-1} = n$. From the condition of the theorem, for every $i \in \overline{2; p-1}$ there exists $i_1 \in \overline{i; p-2}$ such that

$$i \cdot \overline{a_1 a_2 \dots a_{p-1}} = \overline{a_{i_1+1} a_{i_1+2} \dots a_{p-1} a_1 a_2 \dots a_{i_1}},$$

hence

$$\frac{i \cdot n}{m} = \frac{i \cdot \overline{a_1 a_2 \dots a_{p-1}}}{\underbrace{99 \dots p}_{n-1}} = \frac{\overline{a_{i_1+1} a_{i_1+2} \dots a_{p-1} a_1 a_2 \dots a_{i_1}}}{\underbrace{99 \dots p}_{n-1}},$$

and this implies that $b_{i_1} = i \cdot n - b_{i_1}$ is multiple of n.

Obviously, if i takes all integral values from the set $\overline{2; p-1}$, then all corresponding b_{i_1} take values from the set $\{b_1; b_2; \dots; b_{p-1}\}$. Thus all remainders are multiple of n, and the equality of the sets

$$\{b_1; b_2; \dots; b_{p-1}\} = \{n; 2n; 3n; \dots; (p-1)n\}$$

holds.

Thus we have obtained the following result: for every $i \in \overline{1; p-1}$ there exists $i_1 \in \overline{0; p-2}$ (in particular, $i_1 = 0$ for $i = 1$) such that

$$10 \cdot i \cdot n = a_{i_1+1} \cdot m + b_{i_1+1},$$

and in addition, there exists $r_i \in \overline{1; p-1}$ such that $b_{i_1+1} = r_i \cdot n$, i.e., the equalities

$$10 \cdot i \cdot n = a_{i_1+1} \cdot m + r_i \cdot n, \tag{2}$$

$$10 \cdot i - r_i = \frac{a_{i_1+1} \cdot m}{n}$$

hold.

Since $10 \cdot i - r_i$ is integer, the right-hand side of the equality (2) is integer. Owing to the fact that the fraction $\frac{n}{m}$ is irreducible, n and m are mutually prime numbers, hence $\frac{a_{i_1+1} \cdot m}{n}$ will be an integer if and only if a_{i_1+1} is a number, multiple of n, i.e., every element of the set $\{a_1; a_2; \dots; a_{p-1}\}$ is an integer which is multiple of n.

Since $0 \le a_i \le 9$ for every $i \in \overline{1; p-1}$ and every element of the set $\{a_1; a_2; \dots; a_{p-1}\}$ is an integer which is multiple of n, and hence $1 \le n \le 9$.

Let us show that the number a_{p-1} is odd. Assume to the contrary that the number a_{p-1} is even. Then for every $i \in \overline{1; p-1}$, the product $i \cdot \overline{a_1 a_2 \dots a_{p-1}} = \overline{a_{i_1+1} a_{i_1+2} \dots a_{p-1} a_1 a_2 \dots a_{i_1}}$ is even and hence every a_{i_1} is even. Obviously, if i takes all values from the set $\overline{1; p-1}$, then the corresponding a_{i_1} take all values from the set $\{a_1, a_2, \dots, a_{p-1}\}$; this implies that all elements of the set $\{a_1, a_2, \dots, a_{p-1}\}$ are even. If $a_{p-1} = 0$, then $a_i = 0$ for every $i \in \overline{1; p-1}$, and correspondingly, $\overline{a_1 a_2 \dots a_{p-1}} = 0$. But this contradicts the conditions of the theorem. Consequently, $a_{p-1} \ne 0$.

Obviously, for every other than zero even number a_{p-1} there always exists $i \in \overline{1; p-1}$ $(p > 5)$ such that a number of tens in the product $i \cdot a_{p-1}$ will be equal to 1. Since $i \cdot \overline{a_1 a_2 \dots a_{p-1}} = \overline{a_{i_1+1} a_{i_1+2} \dots a_{p-1} a_1 a_2 \dots a_{i_1}}$, therefore a_{i_1} will be the last figure in the product $i \cdot a_{p-1}$, and a_{i_1-1} will be that of $(i \cdot a_{p-2} + 1)$ and hence odd, which contradicts the condition of evenness of all a_i. Consequently, a_{p-1} cannot be even.

n cannot be even, since in this case a_{p-1} is likewise even, which is, as we obtained able, impossible.

If $n = 9$, then every a_i, $i \in \overline{1; p-1}$ will be 9 or 0, in particular, $a_{p-1} = 9$. In this case, the last figure in the product $2 \cdot \overline{a_1 a_2 \dots a_{p-1}}$ will be 8, not 9 or 0. Consequently, $n \ne 9$.

Analogously can be proved that $n \ne 7$.

If $n = 5$, then every a_i, $i \in \overline{1; p-1}$ will be 5 or 0, in particular, $a_{p-1} = 5$. In this case, the next to the last figure in the product $2 \cdot \overline{a_1 a_2 \dots a_{p-1}}$ will be either 1, or 6, and not 5 or 0. Consequently, $n \ne 5$.

If $n = 3$, then every a_i, $i \in \overline{1; p-1}$ will be either 0 or 3, or 6, or 9. If $a_{p-1} = 3$, then the last figure in the product $4 \cdot \overline{a_1 a_2 \dots a_{p-1}}$ will be 2, and not 0, or 3, or 6, or 9. If $a_{p-1} = 9$, then the last figure in the product $2 \cdot \overline{a_1 a_2 \dots a_{p-1}}$ will be 8, and not 0, or 3, or 6, or 9. Consequently, $n \ne 3$.

Thus we have found that $n = 1$, i.e., the equality

$$\frac{1}{m} = \frac{\overline{a_1 a_2 \dots a_{p-1}}}{\underbrace{99 \dots 9}_{p-1}}$$

holds.

Assume that $m > 20$ (for $m \le 20$, the validity of the theorem is verified immediately).

Putting $n = 1$ in equality (2), we obtain $10 \cdot i = a_{i_1+1} \cdot m + r_i$, $1 \le i \le p-1$, $1 \le r_i \le p-1$, is the remainder obtained after the division of $10 \cdot i$ by m which is no more than $(p-1)$.

Let a number of tens $(p-1)$ be $(j-1)$:

$$10(j-1) \le p-1 < 10 \cdot j.$$

Since $p > 2$, hence $j < p-1$. m cannot be more than $10 \cdot j$, because in this case the remainder obtained after the division of $10 \cdot g$ by m will be $10 \cdot j$ which is more than $(p-1)$.

$m \ne 10 \cdot j$, since otherwise $\frac{1}{m}$ cannot be written in the form of a pure periodic fraction, we have

$$10(j-1) \le p-1 < m < 10 \cdot j.$$

In this case,

$$m-(p-1) < 10. \tag{3}$$

If m is a prime number, then by Theorem 4, $p-1$ is the divisor of $(m-1)$: and either (I) $p-1 = m-1$, $p = m$, or (II) $p-1 \le \frac{m-1}{2}$. In this case (since $m > 20$),

$$m-(p-1) \ge m - \frac{m-1}{2} = \frac{m+1}{2} > \frac{m}{2} > 10,$$

which contradicts the condition (3). Therefore if m is a prime number, then necessarily $p = m$. Thus the theorem is complete.

Let m be a composite number, $m = q \cdot r$. It is obvious that

$$q \le \frac{m}{2} < p-1, \quad r \le \frac{m}{2} < p-1.$$

Since the numbers $r \cdot \overline{a_1 a_2 \ldots a_{p-1}}$ and $\overline{a_1 a_2 \ldots a_{p-1}}$ are "identical on the circumference", the period length of decimal writing of the number $\frac{r}{m}$ is equal to $(p-1)$. On the other hand, $\frac{r}{m} = \frac{1}{q}$, and hence by virtue of Theorem 1, the period lenght of decimal writing of the number $\frac{r}{m}$ is no more than $q-1$, and consequently, is less than $(p-1)$. Thus we have obtained the contradiction which proves that the number m is prime.

Consequently, m is a prime, equal to p, number, which was to be demonstrated. □

Theorem 6. *Let $p > 2$ be a prime number,*

$$\frac{1}{p} = 0,(\overline{a_1 a_2 \ldots a_m}),$$

and in addition, the number $\overline{a_1 a_2 \ldots a_m}$ be not multiple of p. Then for any $n \in N$, the period length of writing of the number $\frac{1}{p^{n+1}}$ in the form of a periodic decimal fraction is equal to $p^n \cdot m$.

Proof. To prove the theorem it suffices to prove that if $s \in N$,

$$\frac{1}{p^s} = 0,(\overline{c_1 c_2 \ldots c_k}) = \frac{\overline{c_1 c_2 \ldots c_k}}{\underbrace{99\ldots9}_{k}} \tag{4}$$

and in addition, the number $\overline{c_1 c_2 \ldots c_k}$ is not multiple of p. Then:

(I) for the writing of the number $\frac{1}{p^{s+1}}$ in the form of periodic decimal fraction the length of the period will be equal to $p \cdot k$:

$$\frac{1}{p^{s+1}} = 0,(\overline{b_1 b_2 \dots b_{p\cdot k}});$$

(II) $\overline{b_1 b_2 \dots b_{p\cdot k}}$ is not multiple of p.

First let us prove the validity of (I).

Let the remainder of the division of $\overline{c_1 c_2 \dots c_k}$ by p be equal to r and let the length of the period of the writing of number $\frac{1}{p^{s+1}}$ in the form of a periodic decimal fraction be equal to t:

$$\begin{gathered}\frac{1}{p^{s+1}} = \frac{\overline{b_1 b_2 \dots b_t}}{\underbrace{99\dots9}_{t}}, \\ \frac{1}{p^{s}} = \frac{p}{p^{s+1}} = \frac{p\cdot\overline{b_1 b_2 \dots b_t}}{\underbrace{99\dots9}_{t}} = \frac{\overline{c_1 c_2 \dots c_k}}{\underbrace{99\dots9}_{k}}.\end{gathered} \tag{5}$$

By virtue of (4), k is the least number for which the number $\overbrace{99\dots p}^{k}$ divides by p^s. Therefore equality (6) is valid only for t, multiple of $k : t = i{\cdot}k$. Owing to (5), we obtain

$$\frac{p\cdot\overline{b_1 b_2 \dots b_{i\cdot k}}}{\underbrace{99\dots9}_{i\cdot k}} = \frac{\overline{c_1 c_2 \dots c_k}}{\underbrace{99\dots9}_{k}} = \frac{\overline{\overbrace{(c_1c_2\dots c_k)(c_1c_2\dots c_k)\dots(c_1c_2\dots c_k)}^{i}}}{\underbrace{99\dots9}_{i\cdot k}}. \tag{6}$$

(6) yields

$$\overline{b_1 b_2 \dots b_{i\cdot k}} = \frac{\overline{\overbrace{(c_1c_2\dots c_k)(c_1c_2\dots c_k)\dots(c_1c_2\dots c_k)}^{i}}}{p}.$$

The left-hand side of the obtained equality contains integer, therefore we have to choose i such that $\overline{\overbrace{(c_1c_2\dots c_k)(c_1c_2\dots c_k)\dots(c_1c_2\dots c_k)}^{i}}$ is divisible by p without remainder.

$$\begin{aligned}&\overline{\overbrace{(c_1c_2\dots c_k)(c_1c_2\dots c_k)\dots(c_1c_2\dots c_k)}^{i}} = \\ &= \overline{c_1c_2\dots c_k}\cdot(10^{k(i-1)} + 10^{k(i-2)} + \dots + 1) = \\ &= \overline{c_1c_2\dots c_k}\cdot\left[\Big(1+\overbrace{99\dots9}^{k}\Big)^{i-1} + \Big(1+\overbrace{99\dots9}^{k}\Big)^{i-2} + \dots + 1\right] =\end{aligned}$$

$$
\begin{aligned}
&=\overline{c_1c_2\ldots c_k}\cdot\Big[\big(1+\overline{c_1c_2\ldots c_k}\cdot p^s\big)^{i-1}+\big(1+\overline{c_1c_2\ldots c_k}\cdot p^s\big)^{i-2}+\ldots+1\Big]=\\
&=\overline{c_1c_2\ldots c_k}\cdot\Big[\big(1+C_{i-1}^1\overline{c_1c_2\ldots c_k}\cdot p^s\big)+\big(1+C_{i-2}^1\overline{c_1c_2\ldots c_k}\cdot p^s\big)+\ldots+\\
&+\big(1+C_2^1\overline{c_1c_2\ldots c_k}\cdot p^s\big)+f_{ik}\cdot p^{2s}+\big(1+C_1^1\overline{c_1c_2\ldots c_k}\cdot p^s\big)+1\Big]=\\
&=\overline{c_1c_2\ldots c_k}\cdot\Big[i+\big(C_{i-1}^1+C_{i-2}^1+\ldots+C_1^1\big)\cdot\overline{c_1c_2\ldots c_k}\cdot\cdot p^s+f_{pk}\cdot p^{2s}\Big]=\\
&=\overline{c_1c_2\ldots c_k}\cdot\Big[i+\frac{i\cdot(i-1)}{2}\cdot\overline{c_1c_2\ldots c_k}\cdot p^s+f_{pk}\cdot p^{2s}\Big]. \qquad (7)
\end{aligned}
$$

The remainder obtained after the division by p of the first multiplier in the product is equal to r, while that obtained after the division of the second multiplier by p is equal to i. Therefore the remainder obtained after the division of $\overbrace{\overline{c_1c_2\ldots c_kc_1c_2\ldots c_k\ldots c_1c_2\ldots c_k}}^{i}$ by p is $i\cdot r$. This implies that $\overbrace{\overline{c_1c_2\ldots c_kc_1c_2\ldots c_k\ldots c_1c_2\ldots c_k}}^{i}$ divides by p without reminder if and only if i is the number, multiple of p. The least multiple for p is p itself, therefore $i=p$ and $t=p\cdot k$. Thus we have

$$
\frac{1}{p^{s+1}}=\frac{\overline{b_1b_2\ldots b_{p\cdot k}}}{\underbrace{99\ldots 9}_{p\cdot k}}.
$$

This proves point (I).

Prove now (II). Replacing in equality (7) i by p, we obtain

$$
\begin{aligned}
\overline{b_1b_2\ldots b_{p\cdot k}}&=\frac{\overline{\overbrace{(c_1c_2\ldots c_k)(c_1c_2\ldots c_k)(c_1c_2\ldots c_k)}^{p}}}{p}=\\
&=\frac{\overline{c_1c_2\ldots c_k}\Big[p+\frac{p\cdot(p-1)}{2}\left(\overline{c_1c_2\ldots c_k}\right)\cdot p^s+f_{pk}\cdot p^{2s}\Big]}{p}=\\
&=\overline{c_1c_2\ldots c_k}\left(1+\frac{p-1}{2}\,\overline{c_1c_2\ldots c_k}\cdot p^s+f_{pk}\cdot p^{2s-1}\right).
\end{aligned}
$$

The remainder obtained after the division by p of the first multiplier is equal to r, while that obtained after the division by p of the second multiplier is equal to 1. Hence the remainder obtained after the division of $\overline{b_1b_2\ldots b_{p\cdot k}}$ by p is r. Thus we have proved not only that $\overline{b_1b_2\ldots b_{p\cdot k}}$ is not multiple of p, but more, that the reminder obtained after the division of $\overline{b_1b_2\ldots b_{p\cdot k}}$ by p is equal to r. Thus the theorem is complete. □

Corollary 1. *Let $p>2$ be a prime number,*

$$
\frac{1}{p}=0,(\overline{a_1a_2\ldots a_{p-1}}),
$$

and in addition, $\overline{a_1a_2\dots a_{p-1}}$ be not multiple of p. Then the length of the period of writing of the number $\frac{1}{p^{n+1}}$ in the form of periodic decimal fraction is equal to $p^n(p-1)$. Multiplying this period by any, not multiple of p, number from the interval $[1;p^{n+1}-1]$, we obtain the numbers, "identical on the circumference".

Proof. By Theorem 6, if $\frac{1}{p}=0,(\overline{a_1a_2\dots a_{p-1}})$, then the length of the period of writing of the number $\frac{1}{p^{n+1}}$ in the form of a periodic decimal fraction is equal to $p^b(p-1)$. By Theorem 3, in the interval $[2;p^{n+1}-1]$ there exist $p^n(p-1)-1$ integers which are, by multiplication by the period of the number $\frac{1}{p^{n+1}}$ and the period of the number $\frac{1}{p^{n+1}}$, are the numbers, "identical on the circumference". There arises the question what are those integers from the interval $[2;p^{n+1}-1]$ whose product by the period of the number $\frac{1}{p^{n+1}}$ and the period of the number $\frac{1}{p^{n+1}}$ are not the numbers, "identical on the circumference". Let us first count an amount of such numbers. In the interval $[2;p^{n+1}-1]$ this amount is equal to $p^{n+1}-2$, therefore the amount of those integers in the interval $[2;p^{n+1}-1]$ whose product by the period of the number $\frac{1}{p^{n+1}}$ and that of the number $\frac{1}{p^{n+1}}$ are not the numbers, "identical on the circumference" is $p^{n+1}-2-(p^n(p-1)-1)=p^n-1$.

In the interval $[2;p^{n+1}-1]$, the amount of numbers multiple of p is equal to p^n-1 (these numbers are $p;2p;\dots;p^{n+1}-p=(p^n-1)p$). Show that their product by the period of the number $\frac{1}{p^{n+1}}$ and by that of the number $\frac{1}{p^{n+1}}$ are not the numbers, "identical on the circumference".

Let $s\in\overline{2;p^{n+1}-1}$ be multiple of $p:s=i\cdot p$. Since $\frac{s}{p^{n+1}}=\frac{i\cdot p}{p^{n+1}}=\frac{1}{p^n}$, the length of the period of the number $\frac{s}{p^{n+1}}$ is no more than (p^n-1). Since $p^n-1<p^n\cdot(p-1)$, therefore the period of the number $\frac{1}{p^{n+1}}$ (the length of the period is equal to $p^n(p-1)$) and that of the number $\frac{s}{p^{n+1}}$ (the length of the period is no more than (p^n-1)) are not the numbers, "identical on the circumference". Thus the corollary is proved. □

Corollary 2. *For any $n\in N$, there exist $k>n$ and the number $\overline{c_1c_2\dots c_k}$ such that multiplying $\overline{c_1c_2\dots c_k}$ by some k numbers, we obtain the numbers, "identical on the circumference".*

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