Proceedings of A. Razmadze
Mathematical Institute
Vol. **152** (2010), 59–71

ON APPROXIMATE SOLVING OF SOME DYNAMIC PROBLEMS OF ELASTICITY THEORY

N. KOBLISHVILI, Z. SANIKIDZE AND M. ZAKRADZE

Abstract. An algorithm for the determination of elastico-dynamic state of homogeneous isotropic elastic body with a free surface over finite interval of time is given. In the present paper we consider the case,when the mentioned state of the elastic body is caused by the action of a simple concentrated force applied to a fixed point of the body, with the force varying in time by a non-periodical law. The realization of the algorithm is based on the method of fundamental solutions. The effectiveness of the proposed algorithm in comparison with the methods of integral transformation and Green's function is shown. For illustration of efficiency, an example has been discussed.

რეზიუმე. მოცემულია თავისუფალი ზედაპირის მქონე ერთგვაროვანი და იზოტროპული დრეკადი სხეულის დინამიკური მდგომარეობის განსაზღვრის ალგორითმი დროის სასრულ ინტერვალზე. ნაშრომში განხილულია შემთხვევა, როცა დრეკადი სხეულის აღნიშნული მდგომარეობა გამოწვეულია მის ფიქსირებულ წერტილში მოდებული მარტივი შეყურსული ძალის მოქმედებით. ალგორითმის რეალიზაცია დაფუძნებულია ფუნდამენტურ ამოხსნათა მეთოდზე. ნაჩვენებია შემოთავაზებული ალგორითმის ეფექტურობა ინტეგრალური გარდაქმნის და გრინის ფუნქციის მეთოდებთან შედარებით. ეფექტურობის საილუსტრაციოდ განხილულია მაგალითი.

1. Introduction

It is well-known that a mathematical description of dynamic physical process is much more complicated than a statistical one. Because of mentioned complicacy the three-dimensional problems of dynamic elasticity theory are not so well studied as problems of statistics. For analytical representation of classical solution of three-dimensional problems of dynamics there are mainly two well-known methods: 1) the Laplace integral [1]; 2) the Green's function [1,2].

2010 *Mathematics Subject Classification.* 74H15.
Key words and phrases. Elasticity theory, dynamic problem, method of fundamental solutions.

Complicacy which is connected with application of the first method for numerical realization is in detail described in Section 3. Mentioned complicacy makes this method ineffective from the point of view of numerical realization.

The use of the second method for numerical solution means first of all construction of Green's function, which in turn is connected with finding solution of a concrete three-dimensional boundary problem of dynamics in analytical form. Even if Green's function is known, during numerical realization the situation is the same as in the first case. Namely, approximating computation of improper integral by approximate quadrature formula, which, for its part, needs counting of volume and surface integrals for each node of quadrature formula.

Because of this we have chosen essentially other algorithm (which is based on the method of fundamental solutions (MFS)) for approximate solution of stated problem. Our choice is conditioned by its simplicity for numerical realization, and by the experience and results, which have been received while solving analogous problem by the MFS, where the time function was exponential-periodic or harmonic [3,4,5]. In these cases passing to auxiliary static problems are done, therefore in the indicated papers the algorithm of approximate solution of the stated problems is easier than the one, given in the present paper for a general case.

2. Statement of Problem

Consider a homogeneous isotropic and elastic body D with a boundary S in Euclidean space E_3. We assume, that D is convex and the surface S is smooth.

Suppose that in some fixed point $x_0(x_0^1, x_0^2, x_0^3) \in D$, at the moment $t = 0^+$ a known simple concentrated force $\Phi(x_0, t) = \Phi(x_0)f(t)$, is applied. This means that the function $f(t)$(describing a change of a concentrated force with respect to time) and the vector $\Phi(x_0) = (C_1, C_2, C_3)$ are given. Moreover, we assume, that the surface S is free (i.e., on the surface S the stress vector $F(y, t)$ is equal to zero at all moments) and a value of a deformation (caused by action of the force $\Phi(x_0, t)$ is in the limits of a infinitesimal deformation theory, i.e., in the limits of the Hook's law.

In these conditions consider the following problem for the domain D:

Find the elastico-dynamic state of the elastic body D, which is caused by the action of the concentrated time-dependent force $\Phi(x_0, t)$ in the time interval $[a, b]$.

It should be noted that if $D \equiv E_3$, then under action of the force $\Phi(x_0, t)$ only two types of spreading fronts of elastic oscillations are possible: longitudinal and transverse. In this case the displacement of a point

$x \in E_3$ $(x \neq x_0)$ caused by the force $\Phi(x_0,t)$ has the form [1,2]

$$U_0(x,t) = \sum_{j=1}^{3} C_j \Psi^j(x,x_0,t) \, ,$$

where $\Phi^j(x,x_0,t)$ is j-th column (row) of the matrix $\Psi(x,x_0,t)$ of fundamental solutions (for the function $f(t)$) of the operator $A\big(\frac{\partial}{\partial x},t\big)\big)$,

$$\Psi(x,x_0,t) \equiv \big(\Psi^1,\Psi^2,\Psi^3\big) = \big\|\Psi_{kj}(x,x_0,t)\big\|_{3\times 3}, \quad \Psi_{kj} = \Psi_{jk},$$
$$\Psi^j(x,x_0,t) = \big(\Psi_{1j},\Psi_{2j},\Psi_{3J}\big) \quad (k,j=1,2,3).$$

In notation $\Psi_{kj}(x,x_0,t)$ the index k denotes a component of displacement vector of the point x. The index j indicates the cause of displacement, namely indicates on the unit concentrated force $E^j(x_0,t)$, directed along the axis $Ox^j (j=1,2,3)$. x denotes a moving point, and x_0 denotes the point in which a point force is applied.

In a bounded body with reflecting boundary a number of displaced fronts increases with time, and the stress state of a body is a result of superposition of incident and reflected waves. In comparison with the static state or steady oscillation in this case new mathematical difficulties arise, which correspond to a complicated physical picture of the dynamic state.

It is known [1,6] that in the indicated conditions the elastico-dynamic state of the body D is defined by solution of the direct dynamic problem of the elasticity theory.

$$A\Big(\frac{\partial}{\partial x},t\Big)U(x,t) + \Phi(x_0)f(t)\delta(x-x_0) = \Theta \, , \quad x\in D \, , \quad t\in[0,\infty), \quad (2.1)$$

$$T\Big(\frac{\partial}{\partial y},n\Big)U(y,t) = \Theta, \quad y\in S, \quad t\in[a,b], \quad (2.2)$$

$$U(x,t)\big|_{t=0} = \Theta, \quad \frac{\partial U(x,t)}{\partial t}\Big|_{t=0} = \Theta, \quad x\in\overline{D}, \quad (2.3)$$

where $A\big(\frac{\partial}{\partial x},t\big)$ and $T\big(\frac{\partial}{\partial y},n\big)$ are the matrix differential operators [1]:

$$A\Big(\frac{\partial}{\partial x},t\Big) = \Big\|A_{kj}\Big(\frac{\partial}{\partial x},t\Big)\Big\|_{3\times 3}, \quad (k,j=1,2,3), \quad (2.4)$$

$$A_{kj}\Big(\frac{\partial}{\partial x},t\Big) = \delta_{kj}\Big[\mu\Delta\Big(\frac{\partial}{\partial x}\Big) - \rho\frac{\partial^2}{\partial t^2}\Big] + (\lambda+\mu)\frac{\partial^2}{\partial x^k\partial x^j},$$

$$T\Big(\frac{\partial}{\partial y},n\Big) = \Big\|T_{kj}\Big(\frac{\partial}{\partial y},n\Big)\Big\|_{3\times 3}, \quad (2.5)$$

$$T_{kj}\Big(\frac{\partial}{\partial y},n\Big) = \lambda n_k\frac{\partial}{\partial y^j} + \mu n_j\frac{\partial}{\partial y^k} + \mu\delta_{kj}\frac{\partial}{\partial n} \, ;$$

$U(x,t)$ is a displacement vector of a point x at the moment t; δ is the Dirac function (delta-function); $\Theta=(0,0,0)$ is zero vector; $a=\frac{R_1}{c_1}$, $R_1=$

$\min|y-x_0|$, $y\in S$; c_1 is a velocity of a longitudinal wave in the body D; Δ is the Laplace operator; δ_{kj} is the Kronecker symbol; λ and μ are the Lame elastic constants; ρ is the density of the body D.

In the stress operator $T\big(\frac{\partial}{\partial y},n\big)$ the term $n(y)=(n_1(y),n_2(y),n_3(y))$ is the outward unit normal vector; $\frac{\partial}{\partial n}=\sum\limits_{j=1}^{3}n_j(y)\frac{\partial}{\partial y^j}$. Let $F^{(n)}(y,t)$ be a stress vector which acts on the surface element of S at point y along the normal $n(y)$. Then the stress vector $F^{(n)}(y,t)$ is expressed by the displacement vector $U(x,t)=(U_1,U_2,U_3)$ using the formula [1]

$$F^{(n)}(y,t)=\lim_{D\ni x\to y}T\Big(\frac{\partial}{\partial x},n(x)\Big)U(x,t)=T\Big(\frac{\partial}{\partial y},n(y)\Big)U(y,t)\ ,$$

where the operator $T\big(\frac{\partial}{\partial x},n(x)\big)$ has the form (2.5).

Taking into account, that j – th column (row) $\Psi^j(x,x_0,t)\equiv(\Psi_{1j},\Psi_{2j},\Psi_{3j})$ $(j=1,2,3)$ of the matrix $\Psi(x,x_0,t)$ satisfies the equation

$$A\Big(\frac{\partial}{\partial x},t\Big)\Psi^j(x,x_0,t)+(\delta_{j1},\delta_{j2},\delta_{j3})f(t)\delta(x-x_0)=\Theta,\quad x\in E_3,\quad t\in(-\infty,\infty)$$

and the initial conditions (2.3), the solution of the problem (2.1),(2.2),(2.3) can be presented in the form :

$$U(x,t)=\sum_{j=1}^{3}C_j\Psi^j(x,x_0,t)+V(x,t),\tag{2.6}$$

where $V(x,t)$ is the solution of the following dynamic problem:

$$A\Big(\frac{\partial}{\partial x},t\Big)V(x,t)=\Theta,\quad x\in D,\quad t\in[0,\infty),\tag{2.7}$$

$$T\Big(\frac{\partial}{\partial y},n\Big)V(y,t)=g^1(y,t),\quad y\in S,\quad t\in[a,b],\tag{2.8}$$

$$V(x,t)\big|_{t=0}=\Theta,\quad \frac{\partial V(x,t)}{\partial t}\Big|_{t=0}=\Theta,\quad x\in\overline{D},\tag{2.9}$$

$$g^1(y,t)=-T\Big(\frac{\partial}{\partial y},n\Big)\sum_{j=1}^{3}C_j\Psi^j(y,x_0,t)\equiv-\sum_{j=1}^{3}C_j\Psi^j(y,x_0,n,t).$$

Thus, the problem (2.1), (2.2), (2.3) is reduced to the problem (2.7), (2.8), (2.9), in which the right-hand side of the equation (2.7) and the initial data (2.9) are zero vectors, and the boundary function $g^1(y,t)$ is special. In particular

$$g^1(y,t)=\Theta\quad \mathit{for}\quad y\in S\quad \text{and}\quad t<\frac{R_1}{c_1},\tag{2.10}$$

as the concentrated force (at the point x_0) comes into action $t=0$ moment.

The vector function $g^1(y,t)$ has partial derivatives(with respect to t) of arbitrary order and on the basis of (2.10) for all integer numbers m ($m \geq 0$)

$$\frac{\partial^m g^1(y,t)}{\partial t^m}\Big|_{t=0} = \Theta. \tag{2.11}$$

On the basis of (2.9) and (2.11) the compatibility conditions are automatically fulfilled, which is necessary for existence of the classical solution of the problem (2.7),(2.8),(2.9), i.e., $V(x,t) \in C^1\{\overline{D} \times [0,\infty)\} \bigcap C^2\{D \times [0,\infty)\}$. Besides, if the boundary S is sufficiently smooth ($e.g.$, $S \in L_7(0)$), then the solution is unique [1].

3. On Representation by the Laplace Integral

It should be noted that solution algorithms of the direct boundary problems for steady states are significantly less laborious than solution algorithms for general dynamic problems. Actually, under calculation of steady states, a elliptic boundary problem is solved for some characteristics of oscillation and in addition it is not necessary to care for satisfaction of the initial conditions. This important and clear facilitating circumstance for computing may generate a tendency that practical solution of dynamics problems will be reduced to multiple solution of boundary problems for steady states by some integral transformation [1,2]. We would like to notify representatives of computational mathematics that application of the method of integral transformation for approximate solving of the problem (2.7),(2.8),(2.9) is a hopeless problem at present. Numerical experiments have shown: in order to get by the method of integral transformation the total dynamic picture of the elastico-dynamic state of the elastic body for a force with a finite time interval of action catastrophically large time is needed (e.g., some ten thousand times larger time than is needed for solution of a problem for the steady state [7]).

Indeed, with the help of the Laplace integral transformation

$$\widetilde{U}(x,\tau) = \int_0^\infty e^{-\tau t} U(x,t)dt,$$

the solution of the problem (2.7), (2.8), (2.9) is represented by the Laplace integral [1].

$$U(x,t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\tau t} \widetilde{U}(x,\tau) d\tau. \tag{3.1}$$

In (3.1): $\tau = \sigma + i\omega$ is a complex variable ($\sigma = const; \sigma > 0$); The function $\widetilde{U}(x,\tau)$ represents the solution of the boundary problem of a pseudo-oscillation

$$\begin{aligned} A\Big(\frac{\partial}{\partial x},\tau\Big)\widetilde{U}(x,\tau) &= \Theta, \quad x \in D, \\ T\Big(\frac{\partial}{\partial y},n\Big)\widetilde{U}(y,\tau) &= \widetilde{g}^1(y,\tau), \quad y \in S, \end{aligned} \tag{3.2}$$

where the matric differential operator $A\big(\frac{\partial}{\partial x},\tau\big)$ is given by the formula

$$A\Big(\frac{\partial}{\partial x},\tau\Big) = \Big\|A_{kj}\Big(\frac{\partial}{\partial x},\tau\Big)\Big\|_{3\times 3} \quad (k,j = 1,2,3),$$

$$A_{kj}\Big(\frac{\partial}{\partial x},\tau\Big) = \delta_{kj}\Big[\mu\Delta\Big(\frac{\partial}{\partial x}\Big) - \rho\tau^2\Big] + (\lambda+\mu)\frac{\partial^2}{\partial x^k \partial x^j},$$

and

$$\widetilde{g}^1(y,\tau) = \int_0^\infty e^{-\tau t} g^1(y,t)dt.$$

In [1] it is shown, that the solution of the problem (3.2) is expressed by the series

$$\widetilde{U}(x,\tau) = \sum_{k=1}^{\infty} H_k(\tau)\varphi^{(k)}(x,\tau), \quad x \in D,$$

which for fixed τ uniformly converges in the domain D, where $H_k(\tau)$ and $\varphi^{(k)}(x,\tau)$ are expressed in quadratures with the help of the given functions. In particular

$$H_k(\tau) = \int_S \varphi^{(k)}(y,\tau)\widetilde{g}^1(y,\tau)d_yS,$$

and $\varphi^{(k)}(x,\tau)$ are some linear combinations of the vectors $T\Psi^1(x,z_j,n,\tau)$, $T\Psi^2(x,z_j,n,\tau)$, $T\Psi^3(x,z_j,n,\tau)$ ($j = 1,2,\dots,k$), where "auxiliary points" z_j are situated outside of the domain $\overline{D}$.

On the basis of a definition of a improper integral for approximate calculation of the function $U(x,t)$ in the time interval $t \geq 0$ for $x \in D$ we can suppose that

$$U(x,t) \approx U^*(x,t) = \frac{1}{2\pi i}\int_{\sigma-iA}^{\sigma+iA} e^{\tau t}\widetilde{U}(x,\tau)d\tau, \tag{3.3}$$

where A is a sufficiently large positive number [8].

Evidently, if $A \to \infty$ then $U^*(x,t) \to U(x,t)$. As concerns calculation of the integral (3.3), it we can calculated by well-known methods.

Thus, after application of a quadrature formula the approximate expression of the function $U(x,t)$ has the form

$$U(x,t) \approx \frac{1}{2\pi i} \sum_{k=1}^{N} A_k e^{\tau_k t} \widetilde{U}(x,\tau_k),$$

where A_k are the coefficients of the quadrature formula, τ_k are its nodes, and $\widetilde{U}(x,\tau_k)$ is the solution of the boundary problem (3.2) for $\tau = \tau_k$.

4. The Algorithm Based on the Method of Fundamental Solutions

The basic idea of solution of the dynamic problem (2.7),(2.8),(2.9) by the method of fundamental solutions does not differ from solution of a statical boundary problem (see e.g.[1.9.10.11.12]), but calculating process of solution is much more complicated because of addition the time variable t.

At approximate solution of the problem (2.7),(2.8),(2.9) by the method of fundamental solutions the conditions (2.7) and (2.9) are fulfilled automatically and it is necessary only to have an approximation of the boundary function $g^1(y,t)$ by this method on the set $S \times [a,b]$.

For approximation of the boundary function $g^1(y,t)$ on the set $S \times [a,b]$ we apply the following algorithm.

We approximate in successively the boundary function $g^1(y,t)$ at moments t_l $(l = 1,2,\dots,m)$, $(a = t_1 < t_2 < \dots < t_m = b)$ by the system of functions

$$\Big\{ T\Big(\frac{\partial}{\partial y}, n\Big) \Psi^i(y, z_{l,k}, t - t_{l,k}) \Big\}_{k=1}^{N_l} \equiv \big\{ \Psi^i(y, z_{l,k}, n, t - t_{l,k}) \big\}_{k=1}^{N_l}, \qquad (4.1)$$
$$y \in S \quad (i = 1,2,3),$$

where $z_{l,k}$ are the points of the auxiliary surface S_l $(S_l \bigcap S = \oslash)$; $t_{l,k} = t_l - \frac{r_{l,k}}{c_1}$, where $r_{l,k} = min|y - z_{l,k}|$, $y \in S$; N_l — the number of the auxiliary points (sources) on the S_l. In (4.1) the argument $t - t_{l,k}$ shows that the auxiliary point forces

$$P_{l,k} = \sum_{i=1}^{3} a_{i,k}^l E^i(z_{l,k}, t),$$

where $E^i(z_{l,k},t) = (\delta_{1i}, \delta_{2i}, \delta_{3i}) f(t)$, are the unit concentrated forces, which are applied at the points $z_{l,k}$, begin their action at the moment $t = t_{l,k}$.

The boundary function $g^1(y,t) = (g_1^1, g_2^1, g_3^1)$ for the moment $t = t_1$ we approximate by the sum

$$\sum_{i=1}^{3} \sum_{k=1}^{N_1} a_{i,k}^1 \Psi^i(y, z_{1,k}, n, t_1 - t_{1,k}). \qquad (4.2)$$

In (4.2) the $z_{1,k}$ are uniformly situated on those parts of the surface S_1, which lie outside the parts of the surface S where the wave (emitted from the point x_0) reach at moment $t = t_1$. As concerns the real coefficients $a^1_{i,k}$, for their definition we apply the collocation method, i.e., we solve the linear algebraic equations system of order $3N_1$:

$$\sum_{i=1}^{3}\sum_{k=1}^{N_1} a^1_{i,k}\Psi_{ri}(y_j, z_{1,k}, n, t_1 - t_{1,k}) = g^1_r(y_j, t_1)$$
$$(r = 1, 2, 3; \quad j = 1, 2, \dots, N_1),$$

where $y_j \in S$ are the collocation points. Consequently, the approximate solution of the problem (2.7),(2.8),(2.9) for the moment $t = t_1$ with a certain accuracy will be

$$V^1(x,t) = \sum_{i=1}^{3}\sum_{k=1}^{N_1} a^1_{i,k}\Psi^i(x, z_{1,k}, t - t_{1,k}).$$

We construct the approximate solution of problem (2.7),(2.8),(2.9) for the time interval $[t_1, t_2]$ in the form $V^2(x,t) = V^1(x,t) + W^2(x,t)$, where $W^2(x,t)$ is the solution of problem (2.7),(2.8),(2.9) with boundary function

$$g^2(y, t_2) = g^1(y, t_2) - T\Big(\frac{\partial}{\partial y}, n\Big)V^1(y, t_2)$$

for the moment $t = t_2$.

By analogy with $V^1(x,t)$, the approximate expression of the function $W^2(x,t)$ will have the form

$$W^2(x,t) = \sum_{i=1}^{3}\sum_{k=1}^{N_2} a^2_{i,k}\Psi^i(x, z_{2,k}, t - t_{2,k}). \tag{4.3}$$

In (4.3) the points $z_{2,k}$ are uniformly situated on those parts of the surface S_2, which lie outside parts of the surface S, where the wave (emitted from the points x_0 and $z_{1,k}$ $(k = 1, 2, \dots, N_1)$ reach at the moment $t = t_2$. From the expression of $W^2(x,t)$ we have that $W^2(y, t_1) = \Theta$ $(t_1 < t_2)$, therefore

$$T\Big(\frac{\partial}{\partial y}, n\Big)V^2(y, t_1) = g^1(y, t_1)\ , \quad y \in S$$

and

$$T\Big(\frac{\partial}{\partial y}, n\Big)V^2(y, t_2) = g^1(y, t_2)\ , \quad y \in S.$$

If the time interval $[t_1, t_2]$ is sufficiently small, then on the basis of the behavior of the function $g^1(y,t)$ on the interval [a,b] we can assume, that for the constructed function $V^2(x,t)$ the boundary condition (2.8) is fulfilled on the interval $[t_1, t_2]$. Consequently, the function $V^2(x,t)$ will be the approximate solution of the problem (2.7), (2.8), (2.9) on the interval $[t_1, t_2]$.

If we continue successively this process, then for the moment $t = t_m$ we obtain

$$V^m(x,t) = V^{m-1}(x,t) + W^m(x,t) = V^1(x,t) + \sum_{l=2}^{m} W^l(x,t),$$

where $W^l(x,t)$ is the solution of the problem (2.7),(2.8),(2.9) with the boundary function

$$g^l(y,t_l) = g^1(y,t_l) - T\Big(\frac{\partial}{\partial y}, n\Big)V^{l-1}(y,t_l), \quad y \in S,$$

for the moment $t = t_l$. The approximate expression of the function $W^l(x,t)$ will be

$$W^l(x,t) = \sum_{i=1}^{3}\sum_{k=1}^{N_l} a^l_{i,k}\Psi^i(x, z_{l,k}, t - t_{l,k}),$$

where $z_{l,k} \in S_l$ are situated according to the same rule, as in the previous cases.

On the basis of construction of the functions $W^l(x,t)$ $(l = 2,3,\dots,m)$, it is evident that

$$T\Big(\frac{\partial}{\partial y}, n\Big)W^l(y,t_j) = \Theta \ \text{ for } \ t_j < t_l$$

and consequently

$$T\Big(\frac{\partial}{\partial y}, n\Big)V^m(y,t_l) = g^1(y,t_l).$$

Thus, using the considered algorithm we constructed the function $V^m(x,t)$, which satisfies the conditions (2.7) and (2.9), and for small time intervals $[t_i, t_{i+1}]$ $(i = 1,2,\dots,m-1)$ is approximating the function $g^1(y,t)$ on the set $S \times [a,b]$. That is, with certain accuracy we can consider it as the approximate solution of the problem (2.7), (2.8), (2.9). Consequently, on the basis of (2.6) the approximate solution of the dynamic problem (2.1), (2.2), (2.3) will be

$$U(x,t) = \sum_{j=1}^{3} C_j\Psi^j(x,x_0,t) + V^m(x,t) =$$

$$= \sum_{j=1}^{3} C_j\Psi^j(x,x_0,t) + \sum_{l=1}^{m}\sum_{i=1}^{3}\sum_{k=1}^{N_l} a^l_{i,k}\Psi^i(x,z_{l,k},t-t_{l,k}),$$

$$x \in \overline{D}, \quad t \in [a,b].$$

As was mentioned, during construction of the vector-function $V^m(x,t)$, for definition of the coefficients $a^l_{i,k}$ $(i = 1,2,3;\ k = 1,2,\dots,N_l;\ l =$

$1, 2, \ldots, m$) we solve the linear algebraic equations systems of order $3N_l$:

$$\sum_{i=1}^{3} \sum_{k=1}^{N_l} a_{i,k}^l \Psi_{ri}(y_j, z_{l,k}, n, t_l - t_{l,k}) = g_r^l(y_j, t_l) \tag{4.4}$$
$$(r = 1, 2, 3; \quad j = 1, 2, \ldots, N_l),$$

where an expression of the function g_r^l is defined during of approximation process.

A determination and physical sense of the vector functions $\Psi^i(x, z, t)$ $(i = 1, 2, 3)$ as well as successful selection of the auxiliary points $z_{l,k}$ give us a possibility to define the coefficients $a_{i,k}^l$ separately for the to each point $z_{l,k}$. Indeed, if we dispose the points $z_{l,k}$ $(l = 1, 2, \ldots, m; \quad k = 1, 2, \ldots, N_l)$, respectively, on the normals to the surface S, passing through the point y_k, then $|y_j - z_{l,k}| = r_{l,k}$ for $k = j$ and $|y_j - z_{l,k}| > r_{l,k}$ for $k \neq j$, therefore $\Psi_{ri}(y_j, z_{l,k}, n, t_l - t_{l,k}) = 0$ for $k \neq j$.

Consequently, from (4.4) for fixed k and l we obtain the linear algebraic equations system of order 3:

$$\sum_{i=1}^{3} a_{i,k}^l \Psi_{ri}(y_k, z_{l,k}, n, t_l - t_{l,k}) = g_r^l(y_k, t_l) \tag{4.5}$$
$$(r = 1, 2, 3; \quad k = 1, 2, \ldots, N_l; \quad l = 1, 2, \ldots, m).$$

5. Numerical Example

During solution of the problem (2.1),(2.2),(2.3) by the described algorithm, in numerical experiment the body D and surface S were taken as a sphere with the radius $R = 6400$ km and with the center at the origin. In the role of x_0 was taken the point $x_0 = (0, 0, 6390 \text{ km})$. The Lame constants λ, μ and the density ρ of the body, respectively, were chosen as $\lambda = 29 * 10^9 \frac{n}{m^2}$, $\mu = 34 * 10^9 \frac{n}{m^2}$, $\rho = 2.72 \frac{gr}{cm^3}$.

The functions $\Phi(x_0)$ and $f(t)$ were taken as:

$$\Phi(x_0) = (1, 0, 0), \quad f(t) = \begin{cases} \dfrac{\alpha^2 h}{\alpha^2 + (t - \beta)^2} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \leq 0, \end{cases} \tag{5.1}$$

where $\beta > 0$, $h > 0$, and α is a real number.

The law of action (5.1) of the force $\Phi(x_0, t)$ is interesting for practice as we often meet fields of displacements (seismic data), which are caused by point forces of the type (5.1). On the basis of the formula [2]

$$\Psi_{kj}(x,z,t)=\frac{1}{4\pi\rho c_2^2}\Big\{\frac{\delta_{kj}}{r}f\Big(t-\frac{r}{c_2}\Big)+$$
$$+c_2^2\frac{\partial^2}{\partial x^k\partial x^j}\int\limits_0^t\frac{1}{r}\Big[\eta\Big(t'-\frac{r}{c_1}\Big)-\eta\Big(t'-\frac{r}{c_2}\Big)\Big]f(t-t')dt'\Big\} \tag{5.2}$$

the matrix of fundamental solutions for the function (5.1) is constructed and investigated in [13].

In (5.2) constants $c_1,\ c_2$ are velocities of longitudinal and transverse waves in the elastic body D, respectively; $c_1^2=(\lambda+2\mu)/\rho$; $c_2^2=\mu/\rho$; $r=\big\{\sum\limits_{k=1}^{3}(x^k-z^k)^2\big\}^{1/2}$ is a distance between the points x and z; The function $\eta(a)$ is defined in the following form

$$\eta(a)=\begin{cases}0 & \text{for}\quad a\le 0\ ,\\ a & \text{for}\quad a>0.\end{cases}$$

The numerical experiment show that the accuracy of approximation of boundary function (2.8) (or the accuracy of a solution of the problem (2.1), (2.2),(2.3)) depends: 1). On a number and location of collocation points on the surface S; 2). On a number and choice of discrete times in the interval $[a,b]$. In numerical experiment the coefficients $a_{i,k}^l$ we found from system (4.5).

In the Table 1 (for illustration) the results of approximation of the boundary function $g^1(y,t)$ are given in a near zone of an epicenter. In this experiment the following values were taken: $\beta=0.02$; $\alpha=0.001$; $h=10$. The discrete moments were taken (in seconds) at: $a=t_1=1.658303$; $t_2=1.658634$; $t_3=1.659629$; $t_4=1.661284$; $t_5=1.663601$; $t_6=b=1.666574$. $t_i^*\in(t_i,t_{i+1})$, $(i=1,2,3,4,5)$ are arbitrary chosen moments. $y_1=(0,0,6400)$ is the collocation point, $y_2=(0.1,0.173205,6400)$ and $y_3=(0.166,0.364,6400)$ are the intermediate points on the surface S.

For simplicity in the Table 1 the following notations are introduced:

$$g^1=(g_1^1,g_2^1,g_3^1)=-T\Big(\frac{\partial}{\partial y},n\Big)\Psi^1(y,x_0,t)10^{15}\ n/m^2,$$
$$\widetilde{g}=(\widetilde{g}_1,\widetilde{g}_2,\widetilde{g}_3)=T\Big(\frac{\partial}{\partial y},n\Big)V^6(y,t)10^{15}\ n/m^2.$$

It should be noted that despite the fact that we realized the offered algorithm for a sphere it can be used for much more difficult bodies (for half-space among them). In this case we need just the choosing of respectively auxiliary points. In another words a concrete body requires the concrete choice of auxiliary points, which is characteristic to the MFS itself.

Table 1

t^*	y	$\widetilde{g}_1$	g_1^1	$\widetilde{g}_2$	g_2^1	$\widetilde{g}_3$	g_3^1
	y_1	-142.5514	-142.5514	0	0	0	0
$t_1^* =$	y_2	-141.5952	-141.5954	0.5549938	0.5549947	40.65319	40.65309
1.658403	y_3	0	0	0	0	0	0
	y_1	-147.1061	-147.1063	0	0	0	0
$t_2^* =$	y_2	-146.1065	-146.1067	0.5762093	0.5762202	42.23503	42.23507
1.659542	y_3	-140.7529	-140.7528	2.380855	2.380846	139.7760	139.7764
	y_1	-151.2278	-151.2282	0	0	0.00027	0
$t_3^* =$	y_2	-150.1962	-150.1967	0.5956822	0.5956726	43.68618	43.38620
1.660034	y_3	-144.6681	-144.6682	2.460748	2.460848	144.5555	144.5555
	y_1	-156.7342	-156.7361	0	0	-0.00049	0
$t_4^* =$	y_2	-155.6516	-155.6529	0.6218766	0.6210447	45.61742	45.64812
1.66251	y_3	-149.8801	-149.8803	2.568811	2.568840	151.0135	151.0143
	y_1	-163.7861	-163.7860	0	0	0	0
$t_5^* =$	y_2	-162.6449	-162.6466	0.6563141	0.6561398	48.20595	48.20596
1.664861	y_3	-156.5589	-156.5587	2.7009385	2.7009360	159.4287	159.4290

6. Concluding Remarks

Finally, it should be noted that the stated problem in the paper actually is the direct dynamic problem of theoretical seismology, when the seismic source is pointwise. The results of the numerical experiment have shown that the algorithm presented and consequently, the method of fundamental solutions must be accepted as quite adequate for approximate solution of complex mathematical physics and geophysical problems because it has obvious physical and is readily realizable with the aid of a computer.

Acknowledgment

This work was supported by the grant GNSF/ST08/3-390.

References

1. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Russian) *Nauka, Moscow*, 1976.
2. W. Nowacki, Theory of elasticity. (Russian) *Mir, Moscow*, 1975.
3. M. Zakradze, A case of the action of forces in a seismic center. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR*, **100** (1980), No. 1, 85–88.
4. M. Aleksidze and M. Zakradze, On a stability determination parameters of force in a seismic source. *Dokl. Acad. Nauk SSSR*, **259** (1981), No. 3, 559–562.
5. M. Zakradze, On the accuracy of the solution of certain direct and inverse problems of seismology. *Bull. Acad. Sci. Georg. SSR*, **108** (1982), No. 1, 65–68.

6. N. I. Muskhelishvili, Some basic problems of mathematical theory of elasticity. (Russian) *Nauka, Moscow*, 1966.
7. M. Aleksidze, The solution of some direct and inverse problems of seismology. (Russian) *Metsniereba, Tbilisi*, 1990.
8. G. M. Fikhtengolts, Course of differential and integral calculus, part II. (Russian) *Nauka, Moscow*, 1970.
9. V. D. Kupradze and M. A. Aleksidze, The method of functional equations for the approximate solution of certain boundary-value problems. (Russian) *Z. Vycisl. Mat. i Mat. Fiz.* **4** (1964), 683–715.
10. M. A. Aleksidze, Fundamental functions in approximate solutions of the boundary value problems, part I. (Russian) *Nauka, Moscow*, 1991.
11. G. Fairweather and A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems, Numerical treatment of boundary integral equations. *Adv. Comput. Math.* **9** (1998), No. 1–2, 69–95.
12. L. Lazarashvili and M. Zakradze, On the method of fundamental solutions and some aspects of its application. *Proc. A. Razmadze Math. Inst.* **123** (2000), 41–51.
13. M. V. Zakradze, On a matrix of fundamental solutions of the equation of the dynamic of elasticity theory. (Russian) *Trudy Inst. Vychisl. Mat. Akad. Nauk Gruzin. SSR*, **26** (1986), No. 1, 55–64.

(Received 30.06.2009)

Authors' address:

N. Muskhelishvili Institute of Computational Mathematics
8, Akuri Str., Tbilisi 0171
Georgia