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SOME EMBEDDINGS INTO THE MORREY SPACES ON THE LAGUERRE HYPERGROUP

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Abstract. Let $\mathbb{K} = [0, \infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this paper we obtain some embeddings into the Morrey space $L_{p,\lambda}(\mathbb{K})$. As applications we prove that the fractional maximal operator M_β is bounded from the Morrey space $L_{p,\lambda}(\mathbb{K})$ to $L_\infty(\mathbb{K})$ for $0 \le \lambda < 2\alpha + 4$, $0 \le \beta < 2\alpha + 4 - \lambda$ and $p = \frac{2\alpha+4-\lambda}{\beta}$.

რეზიუმე. ვთქვათ, $\mathbb{K} = [0, \infty) \times \mathbb{R}$ არის ჰეიზენბერგის ჯგუფებზე რადიალური ფუნქციების სივრცის ფუნდამენტური მრავალნაირობა-ლაგერის ჰიპერჯგუფი. სტატიაში დადგენილია მორის $L_{p,\lambda}(\mathbb{K})$ სივრცეში ჩართვები. ამ შედეგზე დაყრდნობით დამტკიცებულია წილადური მაქსიმალური M_β ფუნქციის შემოსაზღვრულობა ზემოთ აღნიშნული სივრციდან L_∞ სივრცეში, როცა $0 \le \lambda < 2\alpha + 4$, $0 \le \beta < 2\alpha + 4 - \lambda$ და $p = \frac{2\alpha+4-\lambda}{\beta}$.

Introduction

In this paper, we define the Morrey space and fractional maximal function using harmonic analysis on Laguerre hypergroups which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [2], [7], [8], [13]–[16]). We study some embeddings into the Morrey space on the Laguerre hypergroup. As applications we obtain the boundedness of the fractional maximal operator in the Morrey space on the Laguerre hypergroup.

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces introduced by C. Morrey [12] in 1938 play an important role, see [4], [11].

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In section 3, we give some embeddings into the Morrey space on the Laguerre hypergroup. In section 4, we prove the

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boundedness of the fractional maximal operator M_β from the spaces $L_{p,\lambda}(\mathbb{K})$ to $L_\infty(\mathbb{K})$ for $p = \frac{2\alpha+4-\lambda}{\beta}$.

1. Preliminaries

Consider the following partial differential operators system:

$$\begin{cases} D_1 = \frac{\partial}{\partial t}, \\ D_2 = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}, \\ (x,t) \in]0,\infty[\times\mathbb{R} \text{ and } \alpha \in [0,\infty[. \end{cases}$$

For $\alpha = n-1$, $n \in \mathbb{N}\setminus\{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$, the initial problem

$$\begin{cases} D_1 u = i\lambda u, \\ D_2 u = -4|\lambda|\left(m + \frac{\alpha+1}{2}\right)u; \\ u(0,0) = 1, \ \frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all} \quad t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^{(\alpha)}\left(|\lambda|x^2\right), \quad (x,t)\in\mathbb{K},$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre functions defined on $\mathbb{R}_+$ by

$$\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2}L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α (see [2]).

Let $\alpha \ge 0$ be a fixed number, $\mathbb{K} = [0,\infty)\times\mathbb{R}$ and m_α be the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}, \quad \alpha \ge 0.$$

For every $1 \le p \le \infty$, we denote by $L_p(\mathbb{K})$ the spaces of complex-valued functions f, measurable on $\mathbb{K}$ such that

$$\|f\|_{L_p} = \left(\int_{\mathbb{K}} |f(x,t)|^p \, dm_\alpha(x,t)\right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{L_\infty} = ess\sup_{(x,t)\in\mathbb{K}}|f(x,t)| \quad \text{if} \quad p=\infty.$$

For $1 \le p < \infty$ we denote by $WL_p(\mathbb{K})$, the weak $L_p(\mathbb{K})$ spaces defined as the set of locally integrable functions $f(x,t)$, $(x,t)\in\mathbb{K}$ with the finite norm

$$\|f\|_{WL_p} = \sup_{r>0} r\,\left(m_\alpha\left\{(x,t)\in\mathbb{K} \,:\, |f(x,t)|>r\right\}\right)^{1/p}.$$

Note that

$$L_p(\mathbb{K}) \subset WL_p(\mathbb{K}) \quad \text{and} \quad \|f\|_{WL_p} \le \|f\|_{L_p} \text{ for all } f \in L_p(\mathbb{K}).$$

Let $|(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{1/4}$ be the homogeneous norm of $(x,t) \in \mathbb{K}$. For $r > 0$ we will denote by $\delta_r(x,t) = (rx, r^2 t)$ the dilation of $(x,t) \in \mathbb{K}$, and by $B_r(x,t)$ the ball centered at (x,t) with radius r, i.e., the set of $B_r(x,t) = \{(y,s) \in \mathbb{K} : |(x-y, t-s)|_{\mathbb{K}} < r\}$, and by B_r the ball $B_r(0,0)$.

We denote by

$$f_r(x,t) = r^{-(2\alpha+4)} f\left(\delta_{\frac{1}{r}}(x,t)\right)$$

the dilated of the function f defined on $\mathbb{K}$ preserving the mean of f with respect to the measure dm_α, in the sense that

$$\int\limits_{\mathbb{K}} f_r(x,t) dm_\alpha(x,t) = \int\limits_{\mathbb{K}} f(x,t) dm_\alpha(x,t), \quad \forall r > 0 \text{ and } \quad f \in L_1(\mathbb{K}).$$

For $(x,t), (y,s) \in \mathbb{K}$ and $\theta \in [0, 2\pi[$, $r \in [0,1]$ let

$$((x,t),(y,s))_{\theta,r} = \left(\left(x^2 + y^2 + 2xyr\cos\theta\right)^{1/2}, t + s + xyr\sin\theta \right).$$

Let $\Sigma = \Sigma_2$ be the unit sphere in $\mathbb{K}$. We denote by ω_2 the surface area of Σ and by Ω_2 the volume of the unit ball in $\mathbb{K}$.

Lemma 1 ([6, 7]). *The following equalities are valid*

$$\omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)} \quad \text{and} \quad \Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

Note that for any $x \in \mathbb{K}$ and $r > 0$, the area of the sphere $S_r(x,t)$ is $r^{2\alpha+3}\omega_2$ and its volume is $r^{2\alpha+4}\Omega_2 = r^{2\alpha+4}\frac{\omega_2}{2\alpha+4}$.

For $f \in L_1(\mathbb{K})$ the Fourier-Laguerre transform $\mathcal{F}$ is defined by

$$\mathcal{F}(f)(\lambda, m) = \int\limits_{\mathbb{K}} \varphi_{-\lambda,m}(x,t) f(x,t) dm_\alpha(x,t)$$

such that

$$\|\mathcal{F}(f)\|_{L_\infty} \le \|f\|_{L_1}$$

(see [2, 14]).

The generalized translation operators $T^{(\alpha)}_{(x,t)}$ on the Laguerre hypergroup are given for a suitable function f by

$$T^{(\alpha)}_{(x,t)} f(y,s) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f\left(((x,t),(y,s))_{\theta,1}\right) d\theta, & \text{if } \alpha = 0, \\ \frac{\alpha}{\pi}\int_0^1 \left(\int_0^{2\pi} f\left(((x,t),(y,s))_{\theta,r}\right) d\theta\right) r(1-r^2)^{\alpha-1} dr, & \text{if } \alpha > 0. \end{cases}$$

The generalized translation operators $T^{(\alpha)}_{(x,t)}$ on the Laguerre hypergroup satisfies the following properties (see [2, 14])

$$T^{(\alpha)}_{(x,t)} f(y,s) = T^{(\alpha)}_{(y,s)} f(x,t), \quad T^{(\alpha)}_{(0,0)} f(y,s) = f(y,s),$$

$$\|T^{(\alpha)}_{(x,t)} f\|_{L_p(\mathbb{K})} \le \|f\|_{L_p(\mathbb{K})} \quad \text{for all} \quad f \in L_p(\mathbb{K}),\ 1 \le p \le \infty, \tag{1}$$

$$\mathcal{F}\big(T^{(\alpha)}_{(x,t)}f\big)(\lambda,m)=\mathcal{F}(f)(\lambda,m)\ \varphi_{\lambda,m}(x,t).$$

The translation operator $T^{(\alpha)}_{(x,t)}$ is defined by

$$T^{(\alpha)}_{(x,t)}f(y,s)=\int\limits_{\mathbb{K}} f(z,v)W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv,$$

where $dzdv$ is the Lebesgue measure on $\mathbb{K}$, and W_α is an appropriate kernel satisfying

$$\int\limits_{\mathbb{K}} W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv=1$$

(see [13]).

For all $(\lambda,m)\in\mathbb{R}\times\mathbb{N}$, the function $\varphi_{\lambda,m}(x,t)$ satisfies the following product formula

$$\varphi_{\lambda,m}(x,t)\,\varphi_{\lambda,m}(y,s)=T^{(\alpha)}_{(x,t)}\varphi_{\lambda,m}(y,s).$$

By using the generalized translation operators $T^{(\alpha)}_{(x,t)}$, $(x,t)\in\mathbb{K}$, we define a generalized convolution product $*$ on $\mathbb{K}$ by

$$\big(\delta_{(x,t)}*\delta_{(y,s)}\big)(f)=T^{(\alpha)}_{(x,t)}f(y,s),$$

where $\delta_{(x,t)}$ is the Dirac measure at (x,t).

We define the convolution product on the space $M_b(\mathbb{K})$ of bounded Radon measures on $\mathbb{K}$ by

$$(\mu*\nu)(f)=\int\limits_{\mathbb{K}\times\mathbb{K}} T^{(\alpha)}_{(x,t)}f(y,s)\,d\mu(x,t)\,d\nu(y,s).$$

If $\mu=h\cdot m_\alpha$ and $\nu=g\cdot m_\alpha$, then we have

$$\mu*\nu=(h*\check{g})\cdot m_\alpha,\quad\text{with}\quad \check{g}(y,s)=g(y,-s),$$

where h and g belong to the space $L_1(\mathbb{K})$ of the integrable functions on $\mathbb{K}$ with respect to the measure $dm_\alpha(x,t)$, and $h*g$ is the convolution product defined by

$$(h*g)(x,t)=\int\limits_{\mathbb{K}} T^{(\alpha)}_{(x,t)}h(y,s)\,g(y,-s)\,dm_\alpha(y,s),\quad\text{for all}\quad (x,t)\in\mathbb{K}.$$

Note that, for the convolution operators the Young inequality is valid: If $1\le p,\,r\le q\le\infty$, $1/p'+1/q=1/r$, $f\in L_p(\mathbb{K})$, and $g\in L_r(\mathbb{K})$, then $f*g\in L_q(\mathbb{K})$ and

$$\|f*g\|_{L_q}\le\|f\|_{L_p}\|g\|_{L_r}\,, \tag{2}$$

where $p'=p/(p-1)$.

$(M_b(\mathbb{K}),*,i)$ is an involutive Banach algebra, where i is the involution on $\mathbb{K}$ given by $i(x,t)=(x,-t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [3], [10]). Hence $(\mathbb{K},*,i)$ is a hypergroup in the sense of Jewett and the functions $\varphi_{\lambda,m}$ are characters of $\mathbb{K}$. If $\alpha = n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathcal{H}_n$.

2. Some Embeddings into the Morrey Spaces

In this section we study some embeddings into the Morrey space $L_{p,\lambda}(\mathbb{K})$. Note that the Hardy-Littlewood maximal function on the Laguerre hypergroup

$$Mf(x,t) = \sup_{r>0} \frac{1}{m_\alpha B_r} \int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)| \, dm_\alpha(y,s)$$

was introduced and investigated by Vagif Guliyev and Miloud Assal in [7].
In [7] the following theorems was proved.

Theorem 1 ([7])**.** 1. *If $f \in L_1(\mathbb{K})$, then $Mf \in WL_1(\mathbb{K})$ and*

$$\|Mf\|_{WL_1} \le C_1 \|f\|_{L_1},$$

where $C_1 > 0$ is independent of f.
2. *If $f \in L_p(\mathbb{K})$, $1 < p \le \infty$, then $Mf \in L_p(\mathbb{K})$ and*

$$\|Mf\|_{L_p} \le C_p \|f\|_{L_p},$$

where $C_p > 0$ is independent of f.

Corollary 1. *If $f \in L_{loc}(\mathbb{K})$, then*

$$\lim_{r\to 0} \frac{1}{m_\alpha B_r} \int\limits_{B_r} \left| T^{(\alpha)}_{(x,t)} f(y,s) - f(x,t) \right| dm_\alpha(y,s) = 0$$

for a. e. $(x,t) \in \mathbb{K}$.

Definition 1 ([5])**.** Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha + 4$. We denote by $L_{p,\lambda}(\mathbb{K})$ the Morrey space on the Laguerre hypergroup as the set of locally integrable functions $f(x,t)$, $(x,t) \in \mathbb{K}$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \sup_{r>0,\,(x,t)\in\mathbb{K}} \left(r^{-\lambda} \int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|^p dm_\alpha(y,s) \right)^{1/p}.$$

Note that

$$L_{p,0}(\mathbb{K}) = L_p(\mathbb{K}),$$

and if $\lambda < 0$ or $\lambda > 2\alpha + 4$, then $L_{p,\lambda}(\mathbb{K}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{K}$.

Definition 2 ([5])**.** Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha + 4$. We denote by $WL_{p,\lambda}(\mathbb{K})$ the weak Morrey space on the Laguerre hypergroup as the set of locally integrable functions $f(x,t)$, $(x,t) \in \mathbb{K}$ with finite norm

$$\|f\|_{WL_{p,\lambda}} = \sup_{\tau>0} \tau \sup_{r>0,\,(x,t)\in\mathbb{K}} \left(r^{-\lambda} \int\limits_{\{(y,s)\in B_r:\, T^{(\alpha)}_{(x,t)}|f(y,s)|>\tau\}} dm_\alpha(y,s) \right)^{1/p}.$$

We note that

$$L_{p,\lambda}(\mathbb{K}) \subset WL_{p,\lambda}(\mathbb{K}) \;\text{ and }\; \|f\|_{WL_{p,\lambda}} \le \|f\|_{L_{p,\lambda}}.$$

Lemma 2. *Let $1 \le p < \infty$. Then*

$$L_{p,2\alpha+4}(\mathbb{K}) = L_\infty(\mathbb{K})$$

and

$$\|f\|_{L_{p,2\alpha+4}} = \Omega_2^{1/p} \, \|f\|_{L_\infty}.$$

Proof. Let $f \in L_\infty(\mathbb{K})$. Then

$$\left(r^{-2\alpha-4} \int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|^p dm_\alpha(y,s) \right)^{1/p} \le \Omega_2^{1/p} \, \|f\|_{L_\infty}.$$

Therefore $f \in L_{p,2\alpha+4}(\mathbb{K})$ and

$$\|f\|_{L_{p,2\alpha+4}} \le \Omega_2^{1/p} \|f\|_{L_\infty}.$$

Let $f \in L_{p,2\alpha+4}(\mathbb{K})$. By the Lebesgue's Theorem we have (see Corollary 1)

$$\lim_{t\to 0} (m_\alpha(B_r))^{-1} \int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|^p dm_\alpha(y,s) = |f(x,t)|^p.$$

Then

$$|f(x,t)| = \left(\lim_{t\to 0} (m_\alpha(B_r))^{-1} \int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|^p dm_\alpha(y,s) \right)^{1/p} \le$$

$$\le \Omega_2^{-1/p} \, \|f\|_{L_{p,2\alpha+4}}.$$

Therefore $f \in L_\infty(\mathbb{K})$ and

$$\|f\|_{L_\infty} \le \Omega_2^{-1/p} \, \|f\|_{L_{p,2\alpha+4}}.$$

□

Lemma 3. *Let $1 \le p < \infty$, $0 \le \lambda < 2\alpha + 4$. Then for $\beta = \frac{2\alpha+4-\lambda}{p}$*

$$L_{p,\lambda}(\mathbb{K}) \subset L_{1,2\alpha+4-\beta}(\mathbb{K}) \quad \textit{and} \quad \|f\|_{L_{1,2\alpha+4-\beta}} \le \Omega_2^{1/p'} \|f\|_{L_{p,\lambda}},$$

where $1/p + 1/p' = 1$.

Proof. Let $f \in L_{p,\lambda}(\mathbb{K})$, $1 \le p < \infty$, $0 \le \lambda < 2\alpha + 4$ and $\beta p = 2\alpha + 4 - \lambda$. By the Hölder's inequality we have

$$\int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)| dm_\alpha(y,s) \le$$

$$\le \left(\int\limits_{B_r} \left(T^{(\alpha)}_{(x,t)} |f(y,s)| \right)^p dm_\alpha(x,t) \right)^{1/p} (m_\alpha(B_r))^{1/p'} \le$$

$$\le \Omega_2^{1/p'} r^{(2\alpha+4)/p'} \left(\int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|^p dm_\alpha(y,s) \right)^{1/p}.$$

Moreover

$$r^{\beta-2\alpha-4} \int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)| dm_\alpha(y,s) \le$$

$$\le \Omega_2^{1/p'} r^{\beta-(2\alpha+4)/p} \left(\int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|^p dm_\alpha(y,s) \right)^{1/p} \le$$

$$\le \Omega_2^{1/p'} \left(r^{-\lambda} \int\limits_{B_r} T^{(\alpha)}_{(y,s)} |f(x,t)|^p dm_\alpha(y,s) \right)^{1/p} \le$$

$$\le \Omega_2^{1/p'} \|f\|_{L_{p,\lambda}}.$$

Therefore $f \in L_{1,2\alpha+4-\beta}(\mathbb{K})$ and

$$\|f\|_{L_{1,2\alpha+4-\beta}} \le \Omega_2^{1/p'} \|f\|_{L_{p,\lambda}}. \qquad \square$$

3. Some Applications

In this section by the results of section 3 we get boundedness of the fractional maximal operator in the Morrey space on the Laguerre hypergroup.

For the $0 \le \beta < 2\alpha+4$ we define the following fractional maximal functions

$$M_{p,\beta} f(x) \equiv \left(M_\beta |f|^p \right)^{1/p} (x)$$

$$= \sup_{r>0} \left((m_\alpha B_r)^{-1+\beta/(2\alpha+4)} \int\limits_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|^p \, dm_\alpha(y,s) \right)^{1/p}.$$

In the case $\beta = 0$ we denote $M_{p,0} f$ by $M_p f$. Note that $M_1 f = Mf$.

Lemma 4. *Let $1 \le p < \infty$, $0 \le \beta < 2\alpha + 4$ and $f \in L_{p,2\alpha+4-\beta}(\mathbb{K})$. Then $M_{p,\beta} f \in L_\infty(\mathbb{K})$ and the following equality*

$$\|M_{p,\beta} f\|_{L_\infty} = \Omega_2^{(\frac{\beta}{2\alpha+4}-1)\frac{1}{p}} \|f\|_{L_{p,2\alpha+4-\beta}}$$

is valid.

Proof.

$$\|M_{p,\beta}f\|_{L_\infty} =$$

$$= \Omega_2^{(\frac{\beta}{2\alpha+4}-1)\frac{1}{p}} \sup_{(x,t)\in\mathbb{K}, r>0} \left(r^{\beta-2\alpha-2} \int_{B_r} T_{(x,t)}^{(\alpha)} |f(y,s)|^p \, dm_\alpha(y,s) \right)^{1/p} =$$

$$= \Omega_2^{(\frac{\beta}{2\alpha+4}-1)\frac{1}{p}} \|f\|_{L_{p,2\alpha+4-\beta}} . \qquad \square$$

In the case $\beta = 0$ from Lemma 4 we get for $M_p f$ the following property is valid.

Corollary 2. *Let* $1 \le p < \infty$. *Then*

$$\|M_p f\|_{L_\infty} = \Omega_2^{-\frac{1}{p}} \|f\|_{L_\infty} .$$

In the case $p = 1$ from Lemmas 3 and 4 we get for $M_\beta f$ the following property is valid.

Corollary 3. *Let* $0 \le \lambda < 2\alpha + 4$ *and* $0 \le \beta < 2\alpha + 4 - \lambda$. *Then the operator* M_β *is bounded from* $L_{p,\lambda}(\mathbb{K})$ *to* $L_\infty(\mathbb{K})$ *for* $p = \frac{2\alpha+4-\lambda}{\beta}$. *Moreover*

$$\|M_\beta f\|_{L_\infty} = \Omega_2^{\frac{\beta}{2\alpha+4}-1} \|f\|_{L_{1,2\alpha+4-\beta}} \le \Omega_2^{\frac{\beta}{2\alpha+4}-\frac{1}{p}} \|f\|_{L_{p,\lambda}}$$

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