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NECESSARY AND SUFFICIENT CONDITIONS FOR THE SCHUR HARMONIC CONVEXITY OF THE GENERALIZED MUIRHEAD MEAN

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Abstract. For $x, y > 0$, $a, b \in R$ and $a+b \neq 0$, the generalized Muirhead mean $M(a, b; x, y)$ is defined by $M(a, b; x, y) = \left(\frac{x^a y^b + x^b y^a}{2}\right)^{\frac{1}{a+b}}$. In this paper, we prove that $M(a, b; x, y)$ is Schur harmonic convex with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ if and only if $(a, b) \in \{(a, b) : a + b > 0\} \cup \{(a, b) : a \leq 0, b \leq 0, (a - b)^2 + (a + b) \leq 0, a^2 + b^2 \neq 0\}$ and Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ if and only if $(a, b) \in \{(a, b) : a \geq 0, a + b < 0, (a - b)^2 + (a + b) \geq 0\} \cup \{(a, b) : b \geq 0, a + b < 0, (a - b)^2 + (a + b) \geq 0\}$.

რეზიუმე. ნაშრომში გამოკვლეულია შურის ჰარმონიული ამოზნექილობა მუირჰედის განზოგადებული საშუალოსათვის $M(a, b; x, y) = \left(\frac{x^a y^b + x^b y^a}{2}\right)^{\frac{1}{a+b}}$. $(x, y) \in (0, \infty) \times (0, \infty)$-ის მიმართ ფიქსირებული ნამდვილი a და b-სათვის პირობით $a + b \neq 0$.

1. Introduction

For $x, y > 0$, $a, b \in R$ and $a + b \neq 0$, the generalized Muirhead mean $M(a, b; x, y)$ was introduced by T. Trif [1] as follows.

$$M(a, b; x, y) = \left(\frac{x^a y^b + x^b y^a}{2}\right)^{\frac{1}{a+b}}. \tag{1.1}$$

It is easy to see that the generalized Muirhead mean $M(a, b; x, y)$ is continuous on the domain $\{(a, b; x, y) : a + b \neq 0; x, y > 0\}$ and differentiable with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for fixed $a, b \in R$ with $a + b \neq 0$. It is of symmetry between a and b and between x and y. Many mean values

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are special cases of the generalized Muirhead mean, for example,

$$\begin{aligned} M_a(x,y) &= M(a,0;x,y) \quad \text{is the power or Hölder mean,}\\ A(x,y) &= M(0,1;x,y) \quad \text{is the arithmetic mean,}\\ G(x,y) &= M(a,a;x,y) \quad \text{is the geometric mean} \end{aligned}$$

and

$$H(x,y) = M(0,-1;x,y) \quad \text{is the harmonic mean.}$$

In paper [1], T. Trif investigated the monotonicity of $M(a,b;x,y)$ with respect to a or b, and established a comparison theorem and a Minkowski-type inequality involving the generalized Muirhead mean $M(a,b;x,y)$. The aim of this paper is to investigate the Schur harmonic convexity and concavity of $M(a,b;x,y)$ with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for fixed $a,b\in R$ with $a+b\neq 0$.

For convenience of readers, we recall the notations and definitions as follows.

For $x=(x_1,x_2), y=(y_1,y_2)\in(0,\infty)\times(0,\infty)$ and $\alpha\in R$, we denote by

$$x+y=(x_1+y_1,x_2+y_2),$$
$$xy=(x_1y_1,x_2y_2),$$
$$\alpha x=(\alpha x_1,\alpha x_2)$$

and

$$\frac{1}{x}=\Big(\frac{1}{x_1},\frac{1}{x_2}\Big).$$

Definition 1.1. A set $E_1\subseteq R^2$ is called a convex set if $\frac{x+y}{2}\in E_1$ whenever $x,y\in E_1$. A set $E_2\subseteq(0,\infty)\times(0,\infty)$ is called a harmonic convex set if $\frac{2xy}{x+y}\in E_2$ whenever $x,y\in E_2$.

It is easy to see that $E\subseteq(0,\infty)\times(0,\infty)$ is a harmonic convex set if and only if $\frac{1}{E}=\{\frac{1}{x}:x\in E\}$ is a convex set.

Definition 1.2. Let $E\subseteq R^2$ be a convex set, a real-valued function $f:E\to R$ is said to be convex on E if $f(\frac{x+y}{2})\leq\frac{f(x)+f(y)}{2}$ for all $x,y\in E$. Moreover, f is said to be concave if $-f$ is convex.

Definition 1.3. Let $E\subseteq(0,\infty)\times(0,\infty)$ be a harmonic convex set, a real-valued function $f:E\to(0,\infty)$ is said to be harmonic convex (or harmonic concave, respectively) on E if

$$f\left(\frac{2xy}{x+y}\right)\leq(\text{or }\geq,\text{respectively})\frac{2f(x)f(y)}{f(x)+f(y)}$$

for all $x,y\in E$.

Definitions 1.2 and 1.3 have the following consequences.

Remark 1.1. If $E_1 \subseteq (0,\infty)\times(0,\infty)$ is a harmonic convex set and $f : E_1 \to (0,\infty)$ is a harmonic convex function, then

$$F(x) = \frac{1}{f(\frac{1}{x})} : \frac{1}{E_1} \to (0,\infty)$$

is a concave function. Conversely, if $E_2 \subseteq (0,\infty)\times(0,\infty)$ is a convex set and $F : E_2 \to (0,\infty)$ is a convex function, then

$$f(x) = \frac{1}{F(\frac{1}{x})} : \frac{1}{E_2} \to (0,\infty)$$

is a harmonic concave function.

Definition 1.4. Let $E \subseteq R^2$ be a set, a real-valued function $F : E \to R$ is said to be Schur convex on E if

$$F(x_1, x_2) \leq F(y_1, y_2)$$

for each pair of two-tuples $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $x \prec y$, i.e.

$$x_{[1]} \leq y_{[1]}$$

and

$$x_{[1]} + x_{[2]} = y_{[1]} + y_{[2]},$$

where $x_{[i]}$ denotes the ith largest component in x. A function F is said to be Schur concave if $-F$ is Schur convex.

Definition 1.5. Let $E \subseteq (0,\infty)\times(0,\infty)$ be a set, a real-valued function $F : E \to R$ is said to be Schur harmonic convex (or Schur harmonic concave, respectively) on E if

$$F(x_1, x_2) \leq (\text{or} \geq, \text{respectively}) F(y_1, y_2)$$

for each pair of $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$.

Definitions 1.4 and 1.5 have the following consequences.

Remark 1.2. Let $E \subseteq (0,\infty)\times(0,\infty)$ be a set, and $H = \frac{1}{E} = \{\frac{1}{x} : x \in E\}$, then $f : E \to (0,\infty)$ is Schur harmonic convex (or concave, respectively) on E if and only if $\frac{1}{f(\frac{1}{x})}$ is a Schur concave (or Schur convex, respectively) on H.

Schur convexity was introduced by I. Schur in 1923 [2] and it has many important applications in analytic inequalities [3-7], theory of statistical experiments [8], graphs and matrices [9], combinatorial optimization [10], reliability [11], gamma functions [12], information-theoretic topics [13], stochastic orderings [14] and other related fields. Recently, the Schur multiplicative convexity was investigated in [15-18], but no one has ever researched the Schur harmonic convexity.

Our aim in what follows is to discuss the Schur harmonic convexity and concavity of the generalized Muirhead mean $M(a,b;x,y)$ with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for fixed $a,b\in R$ with $a+b\neq 0$, our main result is the following Theorem 1.1.

Theorem 1.1. *The generalized Muirhead mean* $M(a,b;x,y)$ *is Schur harmonic convex with respect to* $(x,y)\in(0,\infty)\times(0,\infty)$ *if and only if* $(a,b)\in\{(a,b):a+b>0\}\cup\{(a,b):a\leq 0,b\leq 0,(a-b)^2+(a+b)\leq 0,a^2+b^2\neq 0\}$ *and Schur harmonic concave with respect to* $(x,y)\in(0,\infty)\times(0,\infty)$ *if and only if* $(a,b)\in\{(a,b):a\geq 0,a+b<0,(a-b)^2+(a+b)\geq 0\}\cup\{(a,b):b\geq 0,a+b<0,(a-b)^2+(a+b)\geq 0\}$.

2. Lemmas

In this section we introduce and establish several Lemmas, which are used in the proof of Theorem 1.1.

Lemma 2.1 [19]. *Let* $E\subseteq R^2$ *be a symmetric convex set with nonempty interior intE and* $\varphi:E\to R$ *be a continuous symmetric function on* E. *If* φ *is differentiable on intE, then* φ *is Schur convex* (*or Schur concave, respectively*) *on* E *if and only if*

$$(y-x)\left(\frac{\partial\varphi}{\partial y}-\frac{\partial\varphi}{\partial x}\right)\geq 0\ (or\leq 0, respectively)$$

for all $(x,y)\in\operatorname{int}E$.

Lemma 2.2. *Let* $E\subseteq(0,\infty)\times(0,\infty)$ *be a symmetric harmonic convex set with nonempty interior intE and* $\varphi:E\to(0,\infty)$ *be a continuous symmetric function on* E. *If* φ *is differentiable on intE, then* φ *is Schur harmonic convex* (*or Schur harmonic concave, respectively*) *on* E *if and only if*

$$(y-x)\left(y^2\frac{\partial\varphi}{\partial y}-x^2\frac{\partial\varphi}{\partial x}\right)\geq 0\ (or\leq 0, respectively)$$

for all $(x,y)\in intE$.

Proof. Lemma 2.2 follows from Lemma 2.1 and Remark 1.2 together with the elementary computation. □

Lemma 2.3. *Let* $a,b\in R$, $a+b\neq 0$ *and* $f(t)=\frac{1}{a+b}(bt^{b+1}+at^{a+1}-at^b-bt^a)$. *Then the following statements hold:*

(1) *If* $a>b$ *and* $a+b>0$, *then* $f(t)\geq 0$ *for* $t\in[1,\infty)$;

(2) *If* $a>0,a+b<0$ *and* $(a-b)^2+(a+b)>0$, *then* $f(t)\leq 0$ *for* $t\in[1,\infty)$;

(3) *If* $a>0$ *and* $(a-b)^2+(a+b)<0$, *then there exist* $t_1,t_2\in(1,\infty)$ *such that* $f(t_1)<0$ *and* $f(t_2)>0$;

(4) *If* $a > b, a < 0$ *and* $(a-b)^2+(a+b) > 0$*, then there exist* $t_3, t_4 \in (1, \infty)$ *such that* $f(t_3) < 0$ *and* $f(t_4) > 0$*;*

(5) *If* $a > b, a < 0$ *and* $(a-b)^2+(a+b) < 0$*, then* $f(t) \geq 0$ *for* $t \in [1, \infty)$*.*

Proof. Let $f_1(t) = t^{-b} f(t)$ and $f_2(t) = t^{2-a+b} f_1''(t)$, then simple computation yields

$$f_1(1) = f(1) = 0, \tag{2.1}$$

$$f_1'(t) = \frac{1}{a+b}[a(a-b+1)t^{a-b} - b(a-b)t^{a-b-1} + b],$$

$$f_1'(1) = \frac{(a-b)^2 + (a+b)}{a+b}, \tag{2.2}$$

$$f_1''(t) = \frac{1}{a+b}[a(a-b)(a-b+1)t^{a-b-1} - b(a-b)(a-b-1)t^{a-b-2}],$$

$$f_2(1) = f_1''(1) = \frac{a-b}{a+b}[(a-b)^2 + (a+b)] \tag{2.3}$$

and

$$f_2'(t) = \frac{a(a-b)(a-b+1)}{a+b}. \tag{2.4}$$

(1) If $a > b$ and $a + b > 0$, then from (2.4), (2.3) and (2.2) we see that

$$f_2'(t) > 0, \tag{2.5}$$

$$f_2(1) > 0 \tag{2.6}$$

and

$$f_1'(1) > 0. \tag{2.7}$$

Now, (2.5)-(2.7) together with (2.1) imply that $f(t) \geq 0$ for $t \in [1, \infty)$.

(2) If $a > 0, a + b < 0$ and $(a-b)^2 + (a+b) > 0$, then from (2.4), (2.3) and (2.2) we see that

$$f_2'(t) < 0, \tag{2.8}$$

$$f_2(1) < 0 \tag{2.9}$$

and

$$f_1'(1) < 0. \tag{2.10}$$

Now, (2.8)-(2.10) together with (2.1) imply that $f(t) \leq 0$ for $t \in [1, \infty)$.

(3) If $a > 0$ and $(a-b)^2 + (a+b) < 0$, then (2.2) leads to $f_1'(1) > 0$, this result and the continuity of $f_1'(t)$ imply that there exists $\delta_1 > 0$ such that

$$f_1'(t) > 0 \tag{2.11}$$

for $t \in [1, 1 + \delta_1)$. From (2.11) and (2.1) we know that $f(t) > 0$ for $t \in (1, 1 + \delta_1)$.

On the other hand, it is easy to see that $\lim_{t \to +\infty} f(t) = -\infty$. Hence Lemma 2.3(3) is true.

(4) If $a > b, a < 0$ and $(a-b)^2+(a+b) > 0$, then (2.2) leads to $f_1'(1) < 0$, this result and the continuity of $f_1'(t)$ imply that there exists $\delta_2 > 0$ such that

$$f_1'(t) < 0 \tag{2.12}$$

for $t \in [1, 1+\delta_2)$. From (2.12) and (2.1) we know that $f(t) < 0$ for $t \in (1, 1+\delta_2)$.

On the other hand, if let $h(t) = a + bt^{b-a} - bt^{-1} - at^{b-a-1}$, then $f(t) = \frac{t^{a+1}}{a+b}h(t)$ and $\lim_{t\to+\infty} h(t) = a < 0$, this result and $a+b < 0$ imply that there exists $M \geq 1$ such that $f(t) > 0$ for $t > M$. Hence Lemma 2.3(4) is true.

(5) If $a > b, a < 0$ and $(a-b)^2 + (a+b) < 0$, then from (2.4), (2.3) and (2.2) we know that (2.5), (2.6) and (2.7) hold. Then (2.5)-(2.7) together with (2.1) lead to $f(t) \geq 0$ for $t \in [1, \infty)$. □

3. Proof of Theorem 1.1

We use Lemma 2.2 to discuss the nonnegativity and nonpositivity of $(y-x)(y^2\frac{\partial M(a,b;x,y)}{\partial y} - x^2\frac{\partial M(a,b;x,y)}{\partial x})$ for all $(x,y) \in (0,\infty)\times(0,\infty)$ and for fixed $(a,b) \in R^2$ with $a+b \neq 0$. Since $(y-x)(y^2\frac{\partial M(a,b;x,y)}{\partial y} - x^2\frac{\partial M(a,b;x,y)}{\partial x}) = 0$ for $x = y$ and it is symmetric with respect to x and y, without loss of generality we assume $y > x$ in the following discussion.

Let

$$E_1 = \{(a,b) : a+b > 0\} \cup \{(a,b) : a \leq 0, b \leq 0,$$
$$(a-b)^2 + (a+b) \leq 0, a^2+b^2 \neq 0\},$$

$$\begin{aligned} E_2 &= \{(a,b) : a \geq 0, a+b < 0, (a-b)^2 + (a+b) \geq 0\} \cup \\ &\cup \{(a,b) : b \geq 0, a+b < 0, (a-b)^2 + (a+b) \geq 0\} \end{aligned}$$

and

$$\begin{aligned} E_3 &= \{(a,b) : a > 0, (a-b)^2 + (a+b) < 0\} \cup \\ &\cup \{(a,b) : b > 0, (a-b)^2 + (a+b) < 0\} \cup \\ &\cup \{(a,b) : a > b, a < 0, (a-b)^2 + (a+b) > 0\} \cup \\ &\cup \{(a,b) : b > a, b < 0, (a-b)^2 + (a+b) > 0\}. \end{aligned}$$

Then $E_1 \cup E_2 \cup E_3 = \{(a,b) : a \in R, b \in R, a+b \neq 0\}$, and it is obvious that Theorem 1.1 is true if once we prove that $M(a,b;x,y)$ is Schur harmonic convex, Schur harmonic concave, and neither Schur harmonic convex nor Schur harmonic concave with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for $(a,b) \in E_1, E_2$ and E_3, respectively. We divide our proof into three cases.

Case 1. $(a,b) \in E_1$. Let $E_{11} = \{(a,b) : a > b, a+b > 0\}$, $E_{12} = \{(a,b) : b > a, a+b > 0\}$, $E_{13} = \{(a,b) : a > b, a < 0, (a-b)^2 + (a+b) < 0\}$,

$E_{14} = \{(a,b) : b > a, b < 0, (a-b)^2 + (a+b) < 0\}$. Then (1.1) leads to the following identity

$$(y-x)\left(y^2\frac{\partial M(a,b;x,y)}{\partial y} - x^2\frac{\partial M(a,b;x,y)}{\partial x}\right) = \\ = \frac{(y-x)M(a,b;x,y)x^{a+b+1}}{(a+b)(x^ay^b+x^by^a)}\left[a(\frac{y}{x})^{a+1}+b(\frac{y}{x})^{b+1}-a(\frac{y}{x})^{b}-b(\frac{y}{x})^{a}\right] \quad (3.1)$$

for $(a,b) \in E_{11}$.

From (3.1), Lemma 2.2, Lemma 2.3(1), Lemma 2.3(5) and the assumption $y > x$ we know that $M(a,b;x,y)$ is Schur harmonic convex with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_{11} \cup E_{13}$. Then the symmetry and continuity of $M(a,b;x,y)$ with respect to (a,b) show that $M(a,b;x,y)$ is Schur harmonic convex with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_1$.

Case 2. $(a,b) \in E_2$. Let $E_{21} = \{(a,b) : a > 0, a+b < 0, (a-b)^2 + (a+b) > 0\}$ and $E_{22} = \{(a,b) : b > 0, a+b < 0, (a-b)^2 + (a+b) > 0\}$.

From (3.1), Lemma 2.2, Lemma 2.3(2) and the assumption $y > x$ we know that $M(a,b;x,y)$ is Schur harmonic concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_{21}$. Then the continuity and symmetry of $M(a,b;x,y)$ with respect to (a,b) show that $M(a,b;x,y)$ is Schur harmonic concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_2$.

Case 3. $(a,b) \in E_3$. Let $E_{31} = \{(a,b) : a > 0, (a-b)^2 + (a+b) < 0\}$, $E_{32} = \{(a,b) : b > 0, (a-b)^2 + (a+b) < 0\}$, $E_{33} = \{(a,b) : a > b, a < 0, (a-b)^2+(a+b) > 0\}$, $E_{34} = \{(a,b) : b > a, b < 0, (a-b)^2+(a+b) > 0\}$.

From (3.1), Lemma 2.2, Lemma 2.3(3), Lemma 2.3(4) and the assumption $y > x$ we clearly see that $M(a,b;x,y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_{31} \cup E_{33}$. Then the symmetry of $M(a,b;x,y)$ with respect to (a,b) imply that $M(a,b;x,y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_3$.

Corollary 3.1. *Let* $I = \{(a,b) : a+b > 0\} \cup \{(a,b) : a \le 0, b \le 0, (a-b)^2 + (a+b) \le 0, a^2+b^2 \ne 0\}$, $J = \{(a,b) : a \ge 0, a+b < 0, (a-b)^2 + (a+b) \ge 0\} \cup \{(a,b) : b \ge 0, a+b < 0, (a-b)^2 + (a+b) \ge 0\}$ *and* $H(x,y) = \frac{2xy}{x+y}$, *then*

(1) $M(a,b:x,y) \ge H(x,y)$ *for all* $(x,y) \in (0,\infty) \times (0,\infty)$ *if and only if* $(a,b) \in I$;

(2) $M(a,b:x,y) \le H(x,y)$ *for all* $(x,y) \in (0,\infty) \times (0,\infty)$ *if and only if* $(a,b) \in J$.

Proof. We clearly see that

$$\left(\frac{1}{H(x,y)}, \frac{1}{H(x,y)}\right) \prec \left(\frac{1}{x}, \frac{1}{y}\right) \tag{3.2}$$

for all $(x, y) \in (0, \infty) \times (0, \infty)$.

Therefore, Corollary 3.1 follows from Theorem 1.1 and (3.2) together with (1.1). □

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