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ON THE UNIFORM SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR ONE CLASS OF SINGULARLY PERTURBED REGULAR EQUATIONS

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Abstract. The boundary value problem (problem $\mathfrak{D}_\varepsilon$) for a linear differential regular equation $L_\varepsilon u = f$ with a small parameter ε ($\varepsilon > 0$) is considered. The sufficient conditions are given on the leading coefficients of the operator L_ε for uniform (with respect to ε) solvability of the problem $\mathfrak{D}_\varepsilon$, when the operator L_ε degenerates (as $\varepsilon \to 0$) into a positive definite operator L_0. In addition, the problem of calculating a minimal degree of a small parameter, as a coefficient of evaluating monomial in interpolation inequalities, which is reduced to the solution of canonical problem of minimizing linear programming, is studied.

რეზიუმე. ნაშრომში გამოკვლეულია სასაზღვრო $\mathfrak{D}_\varepsilon$ ამოცანა წრფივი რეგულარული $L_\varepsilon u = f$ სისტემისთვის მცირე ε ($\varepsilon > 0$) პარამეტრით. L_ε კოეფიციენტისთვის დადგენილია საკმარისი პირობები, რომლებიც განაპირობებენ $\mathfrak{D}_\varepsilon$ ამოცანის თანაბარ ამოხსნადობას, როცა L_ε ოპერატორი გადაგვარდება დადებითად განსაზღვრულ L_0 ოპერატორად, როცა $\varepsilon \to 0$. შესწავლილია მცირე ε პარამეტრის მინიმალური ხარისხის გამოთვლის პრობლემა, რაც დაყვანილია მამინიზირებელი წრფივი პროგრამების კანონიკური პრობლემის ამოხსნაზე.

Introduction

In the present work we prove Gárding's inequality ([1–3]) for linear differential regular operators with small parameter in higher derivatives and study the problem dealing with the uniformly positive definiteness for the operators. The latter is of great importance in evaluating remainder terms in the method of a small parameter for singularly perturbed equations (see, for e.g., [4–6]). Analogous questions for elliptic equations with small parameter in higher derivatives have been studied by Vishik and Ljusternik [4], while for pseudodifferential and difference elliptic equations with small parameter

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this problem has been considered by Frank ([7]). Note that Gárding's inequality for semielliptic and regular operators without parameter has been obtained by Mikhailov in his works [8] and[9].

Throughout the paper, the use will be made of the following notation: $\mathbb{N}$ is a set of natural numbers, $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ is a set of integers, $\mathbb{R}$ is a set of real numbers. For $n \in \mathbb{N}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $\mathscr{M} \subset \mathbb{N}_0^n$ we denote

$$|x| = \left(x_1^2 + \cdots + x_n^2\right)^{\frac{1}{2}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \dots \alpha_n!,$$

$$\beta \leq \alpha \iff \beta_j \leq \alpha_j \ (1 \leq j \leq n), \quad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \ (\beta \leq \alpha),$$

$$\alpha\beta = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n, \qquad \xi^\alpha = \xi_1^\alpha \dots \xi_n^\alpha,$$

$$\mathscr{M}^2 \equiv \mathscr{M} \times \mathscr{M} \equiv \{(\alpha, \beta) : \alpha \in \mathscr{M},\ \beta \in \mathscr{M}\}, \quad D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},$$

where $D_j = \frac{\partial}{\partial x_j}$ $(1 \leq j \leq n)$.

For a finite collection of multiindices $\mathscr{M} \subset \mathbb{N}_0^n$ and domain $G \subset \mathbb{R}^n$ we denote

$$\mathbb{W}_2^{\mathscr{M}}(G) \equiv \left\{ f \in \mathbb{L}_2(G) : \|f\|_{\mathbb{W}_2^{\mathscr{M}}(G)} \equiv \sum_{\alpha \in \langle \mathscr{M} \cup \{0\} \rangle} \|D^\alpha f\|_{\mathbb{L}_2(G)} < \infty \right\},$$

where $\langle \mathscr{M} \rangle$ is a convex hull of the collection $\mathscr{M}$, and by $\mathring{\mathbb{H}}_{\mathscr{M}}(G)$ is denoted a closure of the set $\mathbb{C}_0^\infty(G)$ with respect to the norm $\|.\|_{\mathbb{W}_2^{\mathscr{M}}(G)}$.

In this work, all functional spaces will be assumed to be real.

1. The Gárding's Inequality

Let $\Omega \subseteq \mathbb{R}^n$, $\bar{\varepsilon} \in (0,1)$, $\mathscr{N} \subset \mathbb{N}_0^n$ and $\mathscr{N}_0 \subseteq \mathscr{N}$ be finite collections of multiindices, and let

$$L_\varepsilon \equiv L_\varepsilon(x, D) \equiv \sum_{\alpha, \beta \in \mathscr{N}} D^\alpha \left(a_{\alpha,\beta}(x, \varepsilon) D^\beta\right) \ \ (a_{\alpha,\beta}(x,\varepsilon) \not\equiv 0,\ \alpha, \beta \in \mathscr{N}) \tag{1.1}$$

and

$$L_0 \equiv L_0\,(x, D) \equiv \sum_{\alpha, \beta \in \mathscr{N}_0} D^\alpha \left(a_{\alpha,\beta}\,(x, 0)\, D^\beta\right) \ \ (a_{\alpha,\beta}\,(x,0) \not\equiv 0,\ \alpha, \beta \in \mathscr{N}_0)$$

be linear differential operators with real coefficients defined on $\Omega \times [0, \bar{\varepsilon}]$.

Denote

$$\mathscr{R} \equiv \left\{(\alpha, \beta) \in \mathscr{N}^2 \backslash \mathscr{N}_0^2 : |\alpha + \beta| \equiv 0\,(mod\,2)\right\},$$

$$\mathscr{I} \equiv \left\{(\alpha, \beta) \in \mathscr{N}^2 \backslash \mathscr{N}_0^2 : |\alpha + \beta| \equiv 1\,(mod\,2)\right\},$$

$$R_\varepsilon \equiv R_\varepsilon\,(x, D) \equiv \sum_{(\alpha,\beta) \in \mathscr{R}} D^\alpha \left(a_{\alpha,\beta}\,(x, \varepsilon)\, D^\beta\right),$$

$$I_\varepsilon \equiv I_\varepsilon (x, D) \equiv \sum_{(\alpha,\beta)\in\mathscr{I}} D^\alpha \left(a_{\alpha,\beta} (x, \varepsilon) D^\beta\right),$$

$$J_\varepsilon \equiv J_\varepsilon (x, D) \equiv \sum_{\alpha,\beta\in\mathscr{N}_0} D^\alpha \left(\left(a_{\alpha,\beta} (x, \varepsilon) - a_{\alpha,\beta} (x, 0)\right) D^\beta\right).$$

On the operators L_0 and L_ε we impose the following restrictions:

I. There exists the constant $\chi_1 > 0$ such that

$$(L_0 w, w) \geq \chi_1 \sum_{\alpha\in\mathscr{N}_0\cup\{0\}} \|D^\alpha w\|^2 \quad \forall\, w \in \mathbb{C}_0^\infty (\Omega); \tag{1.2}$$

II. (a) the functions $a_{\alpha,\beta} (x, \varepsilon)$ $(\alpha, \beta \in \mathscr{N})$ are infinitely differentiable on $\overline{\Omega} \times [0, \overline{\varepsilon}]$;

(b) for every $\alpha, \beta \in \mathscr{N}_0$ function $a_{\alpha,\beta} (x, \varepsilon)$, as $\varepsilon \to 0$, uniformly with respect to x tends to $a_{\alpha,\beta} (x, 0)$;

III. for all $\alpha \in \mathscr{N}$ $\{\gamma \in \mathbb{N}_0^n : \gamma \leq \alpha\} \subseteq \gamma \in \langle\mathscr{N} \cup \{0\}\rangle$;

IV. there exist positive functions $b_{\alpha,\beta} (\varepsilon)$ $(\alpha, \beta \in \langle\mathscr{N} \cup \{0\}\rangle \setminus \langle\mathscr{N}_0 \cup \{0\}\rangle)$ (assume that $b_{\alpha,\beta} (\varepsilon) \equiv 1$ for $\alpha, \beta \in \langle\mathscr{N}_0 \cup \{0\}\rangle$), infinitely differentiable on $(0, \overline{\varepsilon}]$, such that

(a) the functions $\frac{a_{\alpha,\beta}(x,\varepsilon)}{b_{\alpha,\beta}(\varepsilon)}$ are uniformly continuous with respect to x on $\Omega \times (0, \overline{\varepsilon}]$, for $(\alpha, \beta) \in \mathscr{R}$;

(b) there exists the constant $\kappa_1 > 0$ such that

$$|a_{\alpha,\beta} (x, \varepsilon)| \leq \kappa_1 b_{\alpha,\beta} (\varepsilon) \quad \forall\, x \in \Omega, \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}], (\alpha, \beta) \in \mathscr{R};$$

(c) there exists the constant $\chi_2 > 0$ such that

$$R_\varepsilon (x, \mathtt{i}\xi) \equiv \sum_{(\alpha,\beta)\in\mathscr{R}} a_{\alpha,\beta} (x, \varepsilon) (\mathtt{i}\xi)^{\alpha+\beta} \geq \chi_2 \sum_{\alpha\in\mathscr{N}\setminus\mathscr{N}_0} b_{\alpha,\alpha} (\varepsilon) \xi^{2\alpha} \tag{1.3}$$
$$\forall\, \xi \in \mathbb{R}^n, \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}];$$

(d) there exists the constant $\kappa_2 > 0$ such that

$$\frac{b_{\alpha,\beta}^2 (\varepsilon)}{b_{\alpha,\alpha} (\varepsilon) b_{\beta,\beta} (\varepsilon)} \leq \kappa_2 \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}], (\alpha, \beta) \in \mathscr{R} \cup \mathscr{I};$$

(e) for every pair $(\alpha, \beta) \in \mathscr{R} \cup \mathscr{I}$,

$$\lim_{\varepsilon\to 0} \frac{b_{\alpha,\beta}^2 (\varepsilon)}{b_{\gamma,\gamma} (\varepsilon) b_{\delta,\delta} (\varepsilon)} = 0 \quad \forall\, \gamma, \delta \in \mathbb{N}_0^n, \ \ \gamma \leq \alpha, \ \ \delta \leq \beta, \ \ \gamma + \delta \neq \alpha + \beta;$$

(f) there exists the constant $\kappa_3 > 0$ such that for all $(\alpha, \beta) \in \mathscr{I}$ and $\gamma, \delta \in \mathbb{N}_0^n$ if $\gamma \leq \alpha$, $\delta \leq \beta$ and $\gamma + \delta \neq \alpha + \beta$, then

$$\left|D^{\gamma+\delta} a_{\alpha,\beta} (x, \varepsilon)\right| \leq \kappa_3 b_{\alpha,\beta} (\varepsilon) \quad x \in \Omega, \ \ \varepsilon \in (0, \overline{\varepsilon}];$$

(g) there exists the constant $\chi_3 > 0$ such that

$$\sum_{\alpha\in\langle\mathscr{N}\cup\{0\}\rangle} b_{\alpha,\alpha} (\varepsilon) \xi^{2\alpha} \leq \chi_3 \sum_{\alpha\in\mathscr{N}\cup\{0\}} b_{\alpha,\alpha} (\varepsilon) \xi^{2\alpha} \quad \forall\, \xi \in \mathbb{R}^n, \ \ \forall\, \varepsilon \in (0, \overline{\varepsilon}].$$

For the sake of brevity of our writing, we assume that $\|.\| \equiv \|.\|_{\mathbb{L}_2(\Omega)}$ and $(.,.) \equiv (.,.)_{\mathbb{L}_2(\Omega)}$.

Lemma 1.1. *Let the operator R_ε satisfy Condition* IV(c)*. Then*

$$\left(R_\varepsilon\left(x^0, D\right)u, u\right) \geq \chi_2 \sum_{\alpha \in \mathscr{N} \setminus \mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha u\right\|^2$$
$$\forall\, u \in \mathbb{C}_0^\infty(\Omega), \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}], \quad \forall\, x^0 \in \Omega,$$

where χ_2 is the number from Condition IV(c)*.*

Proof. Follows directly from the estimate (1.3) by virtue of Parceval's equality and Fourier transformation. □

Lemma 1.2. *Let coefficients of the operator L_ε of the type* (1.1) *satisfy Conditions* IV(a) *and* IV(b) *and Condition* IV(d) *for $(\alpha, \beta) \in \mathscr{R}$. Then there exists the constant $\rho > 0$ such that for all $u \in \mathbb{C}_0^\infty(\Omega)$, whose diameter of a support is less than ρ, the estimate*

$$\left(R_\varepsilon u, u\right) \geq \chi_2 \sum_{\alpha \in \mathscr{N} \setminus \mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha u\right\|^2 - \chi_4 \sum_{\alpha \in \mathscr{N}} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha u\right\|^2 \qquad (1.4)$$
$$\forall\, \varepsilon \in (0, \overline{\varepsilon}],$$

holds; here χ_2 is the number from Condition IV(c)*, and*

$$\chi_4 \equiv \frac{\min\{\chi_1, \chi_2\}}{4\chi_3}. \qquad (1.5)$$

Proof. From Condition IV(a) it follows that for any $\tau > 0$ there exists the constant $\rho = \rho(\tau) > 0$ such that if $|x - y| < \rho$, $x, y \in \Omega, \varepsilon \in (0, \overline{\varepsilon}]$, then $|a_{\alpha,\beta}(x, \varepsilon) - a_{\alpha,\beta}(y, \varepsilon)| < \tau b_{\alpha,\beta}(\varepsilon)$.

Let $u \in \mathbb{C}_0^\infty(\Omega)$ and diameter of the support u is less than ρ, and $x^0 \in supp\,(u)$. Assume

$$R_\varepsilon^0 \equiv R_\varepsilon\left(x^0, D\right) = \sum_{(\alpha,\beta) \in \mathscr{R}} D^\alpha\left(a_{\alpha,\beta}\left(x^0, \varepsilon\right) D^\beta\right).$$

Then by the Cauchy-Bunjakovski's inequality we have

$$\left(R_\varepsilon u, u\right) = \left(R_\varepsilon^0 u, u\right) + \left(\left(R_\varepsilon - R_\varepsilon^0\right)u, u\right) =$$
$$= \left(R_\varepsilon^0 u, u\right) + \sum_{(\alpha,\beta) \in \mathscr{R}} \int_{supp(u)} [a_{\alpha,\beta}(x, \varepsilon) - a_{\alpha,\beta}\left(x^0, \varepsilon\right)] D^\alpha u D^\beta u dx \geq$$
$$\geq \left(R_\varepsilon^0 u, u\right) - \tau \sum_{(\alpha,\beta) \in \mathscr{R}} b_{\alpha,\beta}(\varepsilon)\left\|D^\alpha u\right\|\left\|D^\beta u\right\| \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}]. \qquad (1.6)$$

Using Lemma 1.1 for the operator R_ε^0, whose coefficients do not depend on x, and taking into account Condition IV(c), we obtain

$$(R_\varepsilon u, u) \geq \chi_2 \sum_{\alpha \in \mathscr{N} \setminus \mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 -$$
$$-\tau \sum_{(\alpha,\beta) \in \mathscr{R}} b_{\alpha,\beta}(\varepsilon) \|D^\alpha u\| \left\|D^\beta u\right\| \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}]. \tag{1.7}$$

Using the arithmetic inequality

$$|c_1 c_2| \leq \frac{1}{2}\left(\omega |c_1|^2 + \frac{1}{\omega}|c_2|^2\right), \quad c_1, c_2 \in \mathbb{R},$$
$$\omega \equiv \sqrt{\frac{b_{\alpha,\alpha}(\varepsilon)}{b_{\beta,\beta}(\varepsilon)}} > 0, \quad \varepsilon \in (0, \overline{\varepsilon}], \quad \alpha, \beta \in \langle \mathscr{N} \cup \{0\} \rangle, \tag{1.8}$$

by Condition IV(d), we get

$$(R_\varepsilon u, u) \geq \chi_2 \sum_{\alpha \in \mathscr{N} \setminus \mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 -$$
$$- \tau \sum_{(\alpha,\beta) \in \mathscr{R}} \frac{b_{\alpha,\beta}(\varepsilon)}{2\sqrt{b_{\alpha,\alpha}(\varepsilon)\, b_{\beta,\beta}(\varepsilon)}} \left[b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 + b_{\beta,\beta}(\varepsilon) \left\|D^\beta u\right\|^2\right] \geq$$
$$\geq \chi_2 \sum_{\alpha \in \mathscr{N} \setminus \mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 - \tau \kappa_2 K \sum_{\alpha \in \mathscr{N}} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}],$$

where $K = card\,(\mathscr{R})$. Thus choosing τ so small that $\tau\kappa_2 K \leq \chi_4$, we obtain the estimate (1.4). Thus the lemma is proved. □

Lemma 1.3 (see [10], pp. 83–84)**.** *Let* $d > 0$ *and*

$$Q_\sigma \equiv \{x : d(\sigma_j - 1) < x_j < d(\sigma_j + 1),\ j = 1, \ldots, n\} \quad (\sigma \in \mathbb{Z}^n).$$

Then there exists the function $\zeta \in \mathbb{C}_0^\infty(Q_{\mathbf{0}})$ *such that* $0 \leq \zeta \leq 1$ *and*

$$\sum_{\sigma \in \mathbb{Z}^n} (\zeta(x - \sigma))^2 = 1,$$

or, what is the same, if we put $\zeta_\sigma(x) \equiv \zeta(x - \sigma)$*, then* $\zeta_\sigma \in \mathbb{C}_0^\infty(Q_\sigma)$ *and*

$$\sum_{\sigma \in \mathbb{Z}^n} (\zeta_\sigma(x))^2 = 1.$$

Theorem 1.1. *Let coefficients of the operator* L_ε *of the type* (1.1) *satisfy Conditions* III, IV(a), IV(b), IV(c) *and Conditions* IV(d) *and* IV(e) *for* $(\alpha, \beta) \in \mathscr{R}$. *Then for every* $\chi_5 > 0$ *there exists the constant* $\varepsilon_1 \in (0, \overline{\varepsilon}]$ *such that*

$$(R_\varepsilon u, u) \geq \chi_2 \sum_{\alpha \in \mathscr{N} \setminus \mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 - \chi_4 \sum_{\alpha \in \mathscr{N}} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 -$$

$$-\chi_5 \sum_{\alpha\in\langle\mathscr{N}\cup\{0\}\rangle} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha u\right\|^2 \quad \forall\, u\in\mathbb{C}_0^\infty(\Omega)\,,\ \ \forall\,\varepsilon\in(0,\varepsilon_1]. \tag{1.9}$$

where χ_2 *is the number from Condition* IV(c)*, and* χ_4 *is introduced by the relation* (1.5)*.*

Proof. Let $d>0$ and $2\sqrt{n}d<\rho$, where ρ is the number from Lemma 1.2, and Q_σ are the cubes from Lemma 1.3 with sides $2d$ (*diam* $Q_\sigma<\rho$) and $\zeta_\sigma(x)\equiv\zeta(x-\sigma)\in\mathbb{C}_0^\infty(Q_\sigma)$. Then

$$\left|D^\alpha\zeta_\sigma(x)\right|=\left|D^\alpha\zeta(x-\sigma)\right|\le K_0\quad\forall\,\alpha\in\mathscr{N},\ \ \forall\,x\in\mathbb{R}^n \tag{1.10}$$

with some constant $K_0>0$. By the Leibniz formula, for all $u\in\mathbb{C}_0^\infty(\Omega)$

$$\begin{aligned}(R_\varepsilon u,u)&=\sum_{(\alpha,\beta)\in\mathscr{R}}\sum_{\sigma\in\mathbb{Z}^n}\int_\Omega a_{\alpha,\beta}(x,\varepsilon)\left(\zeta_\sigma(x)\right)^2D^\alpha uD^\beta udx=\\&=\sum_{(\alpha,\beta)\in\mathscr{R}}\sum_{\sigma\in\mathbb{Z}^n}\int_\Omega a_{\alpha,\beta}(x,\varepsilon)\,D^\alpha(\zeta_\sigma u)\,D^\beta(\zeta_\sigma u)\,dx+B_\varepsilon(u)=\\&=\sum_{\sigma\in\mathbb{Z}^n}(R_\varepsilon\zeta_\sigma u,\zeta_\sigma u)+B_\varepsilon(u)\,,\end{aligned} \tag{1.11}$$

where

$$B_\varepsilon(u)=$$
$$=-\sum_{(\alpha,\beta)\in\mathscr{R}}\sum_{\sigma\in\mathbb{Z}^n}\sum_{\substack{\gamma\le\alpha,\delta\le\beta\\\gamma+\delta\ne\alpha+\beta}}\int_\Omega a_{\alpha,\beta}(x,\varepsilon)\binom{\alpha}{\gamma}\binom{\beta}{\delta}D^{\alpha-\gamma}\zeta_\sigma D^\gamma uD^{\beta-\delta}\zeta_\sigma D^\delta udx. \tag{1.12}$$

For $B_\varepsilon(u)$, by Condition IV(b) and the estimate (1.10), we have

$$\left|B_\varepsilon(u)\right|\le K_1\sum_{(\alpha,\beta)\in\mathscr{R}}\sum_{\substack{\gamma\le\alpha,\delta\le\beta\\\gamma+\delta\ne\alpha+\beta}}\sum_{\sigma\in\mathbb{Z}^n}b_{\alpha,\beta}(\varepsilon)\int_{Q_\sigma}\left|D^\gamma u\right|\left|D^\delta u\right|dx \tag{1.13}$$
$$\forall\,\varepsilon\in(0,\overline{\varepsilon}]$$

with some constant $K_1>0$. Since

$$\sum_{\sigma\in\mathbb{Z}^n}\int_{Q_\sigma}\left|D^\gamma u\right|\left|D^\delta u\right|dx=2^n\int_\Omega\left|D^\gamma u\right|\left|D^\delta u\right|dx$$

(see [10], p. 85), by the estimate (1.8) we obtain

$$\left|B_\varepsilon(u)\right|\le$$
$$\le2^nK_1\sum_{(\alpha,\beta)\in\mathscr{R}}\sum_{\substack{\gamma\le\alpha,\delta\le\beta\\\gamma+\delta\ne\alpha+\beta}}b_{\alpha,\beta}(\varepsilon)\int_\Omega\left|D^\gamma u\right|\left|D^\delta u\right|dx\le$$

$$\leq 2^n K_1 \sum_{(\alpha,\beta)\in\mathscr{R}} \sum_{\substack{\gamma\leq\alpha,\delta\leq\beta \\ \gamma+\delta\neq\alpha+\beta}} \frac{b_{\alpha,\beta}(\varepsilon)}{2\sqrt{b_{\gamma,\gamma}(\varepsilon)\, b_{\delta,\delta}(\varepsilon)}} \left[b_{\gamma,\gamma}(\varepsilon)\left\|D^\gamma u\right\|^2 + b_{\delta,\delta}(\varepsilon)\left\|D^\delta u\right\|^2\right]$$

whence by Conditions III and IV(e), for any χ_5, there exists the constant $\widetilde{\varepsilon} \in (0, \overline{\varepsilon}]$ such that

$$|B_\varepsilon(u)| \leq \frac{\chi_5}{2} \sum_{\alpha\in\langle\mathscr{N}\cup\{0\}\rangle} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha u\right\|^2 \quad \forall\, \varepsilon \in (0, \widehat{\varepsilon}]. \tag{1.14}$$

From equality (1.11), by means of (1.14), we have

$$(R_\varepsilon u, u) \geq \sum_{\sigma\in\mathbb{Z}^n} (R_\varepsilon \zeta_\sigma u, \zeta_\sigma u) - \\ - \frac{\chi_5}{2} \sum_{\alpha\in\langle\mathscr{N}\cup\{0\}\rangle} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha u\right\|^2 \quad \forall\, \varepsilon \in (0, \widehat{\varepsilon}]. \tag{1.15}$$

Since *supp* $\zeta_\sigma u \subset Q_\sigma$ and *diam* $Q_\sigma < \rho$, by Lemma 1.2 we find that

$$(R_\varepsilon \zeta_\sigma u, \zeta_\sigma u) \geq \chi_2 \sum_{\alpha\in\mathscr{N}\setminus\mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha(\zeta_\sigma u)\right\|^2 - \\ - \chi_4 \sum_{\alpha\in\mathscr{N}} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha(\zeta_\sigma u)\right\|^2. \tag{1.16}$$

Since

$$\sum_{\sigma\in\mathbb{Z}^n} \left\|D^\alpha(\zeta_\sigma u)\right\|^2 = \sum_{\sigma\in\mathbb{Z}^n} \int_\Omega \left|D^\alpha(\zeta_\sigma u)\right|^2 dx = \sum_{\sigma\in\mathbb{Z}^n} \int_\Omega \zeta_\sigma^2 \left|D^\alpha u\right|^2 dx + M_\varepsilon(u) = \\ = \left\|D^\alpha u\right\|^2 + M_\varepsilon(u) \quad \forall\, \alpha \in \mathscr{N},$$

where $M_\varepsilon(u)$ is the quadratic form of the type $B_\varepsilon(u)$. Estimating $M_\varepsilon(u)$ analogously to $B_\varepsilon(u)$, we find that there exists the constant $\varepsilon_1 \in (0, \widehat{\varepsilon}]$ such that

$$|M_\varepsilon(u)| \leq \frac{\chi_5}{4K_2} \sum_{\alpha\in\langle\mathscr{N}\cup\{0\}\rangle} b_{\alpha,\alpha}(\varepsilon)\left\|D^\alpha u\right\|^2 \\ \forall\, \varepsilon \in (0, \varepsilon_1] \quad (K_2 \equiv card\ \mathscr{N}). \tag{1.17}$$

Thus from the estimate (1.15), by virtue of (1,16) and (1.17), we immediately obtain (1.9). □

Remark 1.1. Note that if the coefficients $a_{\alpha,\beta}$ $((\alpha,\beta)\in\mathscr{I})$ do not depend on x or $a_{\alpha,\beta}(x,\varepsilon) = a_{\beta,\alpha}(x,\varepsilon)$ for all $(\alpha,\beta)\in\mathscr{I}$, then

$$(I_\varepsilon u, u) = 0 \quad \forall\, u \in \mathbb{C}_0^\infty(\Omega), \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}].$$

Theorem 1.2. *Let Conditions* IV(e) *and* IV(f) *be fulfilled. Then for every* $\chi_6 > 0$ *there exists the constant* $\varepsilon_2 \in (0, \overline{\varepsilon}]$ *such that*

$$(I_\varepsilon u, u) \geq -\chi_6 \sum_{\alpha \in \langle \mathscr{N} \cup \{0\} \rangle} b_{\alpha,\alpha}(\varepsilon) \left\| D^\alpha u \right\|^2 \tag{1.18}$$
$$\forall\, u \in \mathbb{C}_0^\infty(\Omega), \quad \forall\, \varepsilon \in (0, \varepsilon_2].$$

Proof. Let $(\alpha, \beta) \in \mathscr{I}$, consequently $|\alpha| + |\beta| \equiv 1\,(mod\, 2)$, as well. Since $a_{\alpha,\beta} \equiv a_{\alpha,\beta}(x, \varepsilon)$ $((\alpha, \beta) \in \mathscr{I})$ are the real functions, therefore

$$2\left(D^\alpha\left(a_{\alpha,\beta} D^\beta u\right), u\right) = \left(D^\alpha\left(a_{\alpha,\beta} D^\beta u\right), u\right) - \left(D^\beta\left(a_{\alpha,\beta} D^\alpha u\right), u\right).$$

Consequently, by the Leibniz formula,

$$\begin{aligned}
2\left(D^\alpha a_{\alpha,\beta} D^\beta u, u\right) &= \left(a_{\alpha,\beta} D^{\alpha+\beta} u, u\right) + \sum_{\substack{\gamma \leq \alpha \\ \gamma \neq \alpha}} \binom{\alpha}{\gamma} \left(D^{\alpha-\gamma} a_{\alpha,\beta} D^{\beta+\gamma} u, u\right) - \\
&- \left(a_{\alpha,\beta} D^{\alpha+\beta} u, u\right) - \sum_{\substack{\delta \leq \beta \\ \delta \neq \beta}} \binom{\beta}{\delta} \left(D^{\beta-\delta} a_{\alpha,\beta} D^{\alpha+\delta} u, u\right) = \\
&= (-1)^{|\beta|} \sum_{\substack{\gamma \leq \alpha \\ \gamma \neq \alpha}} \binom{\alpha}{\gamma} \left(D^\gamma u, D^\beta\left(D^{\alpha-\gamma} a_{\alpha,\beta} u\right)\right) - \\
&- (-1)^{|\alpha|} \sum_{\substack{\delta \leq \beta \\ \delta \neq \beta}} \binom{\beta}{\delta} \left(D^\delta u, D^\alpha\left(D^{\beta-\delta} a_{\alpha,\beta} u\right)\right) = \\
&= (-1)^{|\beta|} \sum_{\substack{\gamma \leq \alpha \\ \gamma \neq \alpha}} \binom{\alpha}{\gamma} \sum_{\delta \leq \beta} \binom{\beta}{\delta} \left(D^\gamma u, D^{\alpha-\gamma+\beta-\delta} a_{\alpha,\beta} D^\delta u\right) - \\
&- (-1)^{|\alpha|} \sum_{\substack{\delta \leq \beta \\ \delta \neq \beta}} \binom{\beta}{\delta} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \left(D^\delta u, D^{\alpha-\gamma+\beta-\delta} a_{\alpha,\beta} D^\gamma u\right).
\end{aligned}$$

Let us estimate the summands appearing in the above sums. Since by the estimate (1.8) and Condition IV(f), for every $\gamma, \delta \in \mathbb{N}_0^n, \gamma \leq \alpha, \delta \leq \beta, \gamma + \delta \neq \alpha + \beta$

$$\begin{aligned}
&\left|\left(D^\gamma u, D^{\alpha-\gamma+\beta-\delta} a_{\alpha,\beta} D^\delta u\right)\right| \leq \kappa_3 b_{\alpha,\beta}(\varepsilon) \left\| D^\gamma u \right\| \left\| D^\delta u \right\| \leq \\
&\leq \frac{\kappa_3 b_{\alpha,\beta}(\varepsilon)}{2\sqrt{b_{\gamma,\gamma}(\varepsilon)\, b_{\delta,\delta}(\varepsilon)}} \left[b_{\gamma,\gamma}(\varepsilon) \left\| D^\gamma u \right\|^2 + b_{\delta,\delta}(\varepsilon) \left\| D^\delta u \right\|^2\right],
\end{aligned}$$

therefore with a constant $K' > 0$,

$$\big(D^\alpha a_{\alpha,\beta} D^\beta u, u\big) \geq$$
$$\geq -\kappa_3 K' \sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\ \gamma+\delta\neq\alpha+\beta}} \frac{b_{\alpha,\beta}(\varepsilon)}{2\sqrt{b_{\gamma,\gamma}(\varepsilon)\, b_{\delta,\delta}(\varepsilon)}} \Big[b_{\gamma,\gamma}(\varepsilon)\left\|D^\gamma u\right\|^2 + b_{\delta,\delta}(\varepsilon)\left\|D^\delta u\right\|^2\Big]$$

whence we immediately find that

$$(I_\varepsilon u, u) \geq -\kappa_3 K' \sum_{(\alpha,\beta)\in\mathscr{I}} \sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\ \gamma+\delta\neq\alpha+\beta}} \frac{b_{\alpha,\beta}(\varepsilon)}{2\sqrt{b_{\gamma,\gamma}(\varepsilon)\, b_{\delta,\delta}(\varepsilon)}} \times$$
$$\times \Big[b_{\gamma,\gamma}(\varepsilon)\left\|D^\gamma u\right\|^2 + b_{\delta,\delta}(\varepsilon)\left\|D^\delta u\right\|^2\Big]. \tag{1.19}$$

By virtue of Condition IV(e), there exists the constant $\varepsilon_2 \in (0, \overline{\varepsilon}]$ such that

$$\frac{\kappa_3 K' b_{\alpha,\beta}(\varepsilon)}{2K^2\sqrt{b_{\gamma,\gamma}(\varepsilon)\, b_{\delta,\delta}(\varepsilon)}} \leq \chi_6$$
$$\forall\, \varepsilon \in (0, \varepsilon_2],\ \ \gamma, \delta \in \mathbb{N}_0^n,\ \ (\alpha,\beta) \in \mathscr{I},\ \ \gamma \leq \alpha,\ \ \delta \leq \beta,\ \ \gamma+\delta \neq \alpha+\beta,$$

where $K = card\ \langle \mathscr{N} \cup \{0\}\rangle$. Thus from the estimate (1.19) follows (1.18). □

Theorem 1.3. *Let Condition* II(b) *be fulfilled. Then for every* $\chi_7 > 0$ *there exists the constant* $\varepsilon_3 \in (0, \overline{\varepsilon}]$ *such that*

$$(J_\varepsilon u, u) \geq -\chi_7 \sum_{\alpha\in\mathscr{N}_0} \left\|D^\alpha u\right\|^2 \quad \forall\, u \in \mathbb{C}_0^\infty(\Omega), \quad \forall\, \varepsilon \in (0, \varepsilon_3]. \tag{1.20}$$

Proof. It follows from Condition II(b) that for every $\tau > 0$ there exists the number $\varepsilon_3 \in (0, \overline{\varepsilon}]$ such that

$$\left|a_{\alpha,\beta}(x,\varepsilon) - a_{\alpha,\beta}(x,0)\right| < \tau \quad \forall\, x \in \Omega,\ \ \forall\, \alpha, \beta \in \mathscr{N}_0,\ \ \forall\, \varepsilon \in (0, \varepsilon_3].$$

Therefore

$$\left|\left(\left(a_{\alpha,\beta}(x,\varepsilon) - a_{\alpha,\beta}(x,0)\right) D^\beta u, D^\alpha u\right)\right| \leq \frac{\tau}{2}\Big[\left\|D^\alpha u\right\|^2 + \left\|D^\beta u\right\|^2\Big]$$
$$\forall\, \alpha, \beta \in \mathscr{N}_0,\ \ \forall\, \varepsilon \in (0, \varepsilon_3],$$

and hence

$$(J_\varepsilon u, u) \geq -\tau K^2 \sum_{\alpha\in\mathscr{N}_0} \left\|D^\alpha u\right\|^2 \quad \forall\, u \in \mathbb{C}_0^\infty(\Omega),\ \ \forall\, \varepsilon \in (0, \varepsilon_3],$$

where K is the power of the set $\mathscr{N}_0$. Thus for $\tau = \chi_7/K^2$ we obtain the estimate (1.20). □

The main result of this section is the following

Theorem 1.4. *Let the coefficients of the operator L_ε of the type* (1.1) *satisfy Conditions* I, II, III *and* IV. *Then there exist the constants $\overline{\overline{\varepsilon}} \in (0, \overline{\varepsilon}]$, $C_1 > 0$ and $C_2 > 0$ such that for all $u \in \mathbb{C}_0^\infty(\Omega)$ the estimate*

$$\sum_{\alpha \in \langle \mathscr{N} \rangle \setminus \langle \mathscr{N}_0 \rangle} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 + \\ + \sum_{\alpha \in \langle \mathscr{N}_0 \cup \{0\} \rangle} \|D^\alpha u\|^2 \le C_1 (L_\varepsilon u, u) \le C_2 \|L_\varepsilon u\|^2 \quad \forall\, \varepsilon \in (0, \overline{\overline{\varepsilon}}]. \tag{1.21}$$

is valid.

Proof. Let $\chi_5 > 0, \chi_6 > 0, \chi_7 > 0$ and $\chi_5 + \chi_6 + \chi_7 \le \frac{\min\{\chi_1, \chi_2\}}{4\chi_3}$. Since by Theorems 1.1, 1.2 and 1.3 there exist the constants $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0, \overline{\varepsilon}]$ such that the estimates (1.9), (1.18) and (1.20) are fulfilled, by virtue of Condition I we have

$$(L_\varepsilon u, u) = (L_0 u, u) + (R_\varepsilon u, u) + (I_\varepsilon u, u) + (J_\varepsilon u, u) \ge \\ \ge \chi_1 \sum_{\alpha \in \mathscr{N}_0 \cup \{0\}} \|D^\alpha u\|^2 + \chi_2 \sum_{\alpha \in \mathscr{N} \setminus \mathscr{N}_0} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 - \\ -\chi_4 \sum_{\alpha \in \mathscr{N}} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 - \chi_5 \sum_{\alpha \in \langle \mathscr{N} \cup \{0\} \rangle} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 - \\ -\chi_6 \sum_{\alpha \in \langle \mathscr{N} \cup \{0\} \rangle} b_{\alpha,\alpha}(\varepsilon) \|D^\alpha u\|^2 - \chi_7 \sum_{\alpha \in \mathscr{N}_0} \|D^\alpha u\|^2 \\ \forall\, u \in \mathbb{C}_0^\infty(\Omega), \quad \forall\, \varepsilon \in (0, \overline{\overline{\varepsilon}}], \overline{\overline{\varepsilon}} \equiv \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

whence by Condition IV(g), with regard for (1.5), we obtain the first part of the estimate (1.21) with the constant $C_1 \equiv \frac{2\chi_3}{\min\{\chi_1, \chi_2\}}$. Since for every $\omega > 0$,

$$(L_\varepsilon u, u) \le \frac{1}{2}\left(\omega \|u\|^2 + \frac{1}{\omega}\|L_\varepsilon u\|^2\right) \quad \forall\, u \in \mathbb{C}_0^\infty(\Omega), \tag{1.22}$$

using the already proven part of the estimate (1.21), we obtain the second part of (1.21). □

2. Interpolation Inequalities

2.1. Let $p, q \in \mathbb{N}$, $A \in \mathbb{R}^{p \times q}$, $\lambda, c \in \mathbb{R}^p$ and $\mu, b \in \mathbb{R}^q$. Following the standard terminology of the theory of linear programming, we introduce the following

Definition 2.1 (see [11], p. 110)**.** The canonic problem of minimizing the linear programming consists in finding a nonnegative vector λ which

$$\text{minimizes} \quad \lambda c \tag{2.1}$$

under the condition

$$\lambda A = b. \tag{2.2}$$

Definition 2.2 (see [11], p. 113)**.** The double problem to the problem (2.1), (2.4) consists in the finding of the vector μ (without restrictions with respect to the sign), which

$$\text{maximizes} \quad \mu b \tag{2.3}$$

under the condition

$$A\mu \leq c. \tag{2.4}$$

Definition 2.3 (see [11], pp. 113–114)**.** The problem (2.1), (2.2) ((2.3), (2.4)) is said to be admissible if there exists the vector λ (μ) satisfying the condition (2.2) ((2.4)). Such vector is called admissible. The admissible vector λ (μ) is said to be optimal if it minimizes a linear form of λc (maximizes μb), and the value of that maximum (minimum) is called the value of the problem of linear programming.

Theorem 2.1 (the basic duality theorem, see [11], p. 114)**.** *If the problems* (2.1), (2.2) *and* (2.3), (2.4) *are admissible, then they have optimal vectors, and the values of the problems coincide.*

2.2. By $\mathfrak{F}(\mathbb{N}_0^n)$ we denote a set of of finite subsets of the set $\mathbb{N}_0^n$.

Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$ and φ be the negative function defined on $\mathscr{M}$.

For the collection $\mathscr{K} \subseteq \mathscr{M}$ and the vector $\beta \in \langle \mathscr{K} \rangle$ we consider the following canonic minimization problem: find a collection of negative numbers λ_α ($\alpha \in \mathscr{K}$) which

$$\text{minimizes} \quad \sum_{\alpha \in \mathscr{K}} \lambda_\alpha \varphi(\alpha) \tag{2.5}$$

under the condition

$$\begin{cases} \sum\limits_{\alpha \in \mathscr{K}} \lambda_\alpha \alpha = \beta, \\ \sum\limits_{\alpha \in \mathscr{K}} \lambda_\alpha = 1, \end{cases} \tag{2.6}$$

and its dual problem: find an $n+1$-dimensional vector $(\overline{\mu}, \mu_{n+1}) \equiv (\mu_1, \dots, \mu_n, \mu_{n+1})$ which

$$\text{maximizes} \quad \overline{\mu}\beta + \mu_{n+1} \tag{2.7}$$

under the condition

$$\overline{\mu}\alpha + \mu_{n+1} \leq \varphi(\alpha) \quad \forall\, \alpha \in \mathscr{K}. \tag{2.8}$$

Obviously, both problems are admissible, hence by virtue of Theorem 2.1, they have optimal solutions. The value of the above-formulated problems we denote by $\varphi_{\mathscr{K}}^{opt}(\beta)$.

Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$, $\varphi : \mathscr{M} \mapsto \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$, $\overline{\varepsilon} \in (0,1)$ and $\gamma_0 \in [1, +\infty)$. For the vector $\alpha^0 \in \langle \mathscr{M} \rangle$ we put

$$\widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right) \equiv \min \left\{ q \in \mathbb{R} : \forall \varepsilon \in (0, \overline{\varepsilon}],\ \forall \xi \in \mathbb{R}^n,\ \xi \geq \mathbf{0},\ \varepsilon^q \xi^{\alpha^0} \leq \gamma_0 \sum_{\alpha \in \mathscr{M}} \varepsilon^{\varphi(\alpha)} \xi^{\alpha} \right\}.$$

Theorem 2.2. *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$ and $\varphi : \mathscr{M} \mapsto \mathbb{R}_+$. Then*

$$\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) = \widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right) \quad \forall\, \alpha^0 \in \langle \mathscr{M} \rangle .$$

To prove Theorem 2.2, we cite the following auxiliary statements.

Lemma 2.1 (see [12], p. 29)**.** *Let $c_\alpha \geq 0$ $(\alpha \in \mathscr{M})$, $\lambda_\alpha \geq 0$ $(\alpha \in \mathscr{M})$ and*

$$\sum_{\alpha \in \mathscr{M}} \lambda_\alpha = 1.$$

Then

$$\prod_{\alpha \in \mathscr{M}} c_\alpha^{\lambda_\alpha} \leq \sum_{\alpha \in \mathscr{M}} \lambda_\alpha c_\alpha.$$

Proposition 2.1. *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$, $\varphi : \mathscr{M} \mapsto \mathbb{R}_+$, $\lambda_\alpha \geq 0$ $(\alpha \in \mathscr{M})$,*

$$\sum_{\alpha \in \mathscr{M}} \lambda_\alpha = 1, \quad \alpha^0 \equiv \sum_{\alpha \in \mathscr{M}} \lambda_\alpha \alpha, \quad q \equiv \sum_{\alpha \in \mathscr{M}} \lambda_\alpha \varphi(\alpha).$$

Then

$$\varepsilon^q \xi^{\alpha^0} \leq \sum_{\alpha \in \mathscr{M}} \varepsilon^{\varphi(\alpha)} \xi^{\alpha} \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}],\ \ \forall\, \xi \in \mathbb{R}^n,\ \ \xi \geq \mathbf{0}.$$

Proof. By Lemma 2.1, we have

$$\varepsilon^q \xi^{\alpha^0} = \varepsilon^{\sum\limits_{\alpha \in \mathscr{M}} \lambda_\alpha \varphi(\alpha)} \xi^{\sum\limits_{\alpha \in \mathscr{M}} \lambda_\alpha \alpha} = \left(\prod_{\alpha \in \mathscr{M}} \varepsilon^{\lambda_\alpha \varphi(\alpha)} \right) \left(\prod_{\alpha \in \mathscr{M}} \xi^{\lambda_\alpha \alpha} \right) =$$
$$= \prod_{\alpha \in \mathscr{M}} \left(\varepsilon^{\varphi(\alpha)} \xi^{\alpha} \right)^{\lambda_\alpha} \leq \sum_{\alpha \in \mathscr{M}} \lambda_\alpha \varepsilon^{\varphi(\alpha)} \xi^{\alpha} \leq \sum_{\alpha \in \mathscr{M}} \varepsilon^{\varphi(\alpha)} \xi^{\alpha}$$
$$\forall\, \varepsilon \in (0, \overline{\varepsilon}],\ \ \forall\, \xi \in \mathbb{R}^n,\ \ \xi \geq \mathbf{0}.$$
□

Proof of Theorem 2.2. Let $\alpha^0 \in \langle \mathscr{M} \rangle$. Since it follows from Proposition 2.1 that $\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) \geq \widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right)$, it suffices to show that $\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) \leq \widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right)$. Assume to the contrary that $\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) > \widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right)$. From the definition of the function $\widetilde{\varphi}_{\mathscr{M}}^{opt}$ we have

$$\varepsilon^{\widetilde{\varphi}_{\mathscr{M}}^{opt}(\alpha^0)} \xi^{\alpha^0} \leq \gamma_0 \sum_{\alpha \in \mathscr{M}} \varepsilon^{\varphi(\alpha)} \xi^{\alpha} \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}],\ \overline{\varepsilon} < 1,\ \forall\, \xi \in \mathbb{R}^n,\ \xi \geq \mathbf{0}. \tag{2.9}$$

Let $\lambda_\alpha^0 \geq 0$ $(\alpha \in \mathscr{M})$ and $\left(\overline{\mu^0}, \mu_{n+1}^0\right)$ be optimal vectors of the problems (2.5), (2.6) (for $\mathscr{K} = \mathscr{M}$ and $\beta = \alpha^0$) and (2.7), (2.8), respectively. Then by Theorem 2.1,

$$\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) = \sum_{\alpha \in \mathscr{M}} \lambda_\alpha^0 \varphi\left(\alpha\right) = \overline{\mu^0}\alpha^0 + \mu_{n+1}^0.$$

Thus substituting $\xi_j = \varepsilon^{-\mu_j^0}$ in inequality (2.9) and multiplying both parts by $\varepsilon^{-\mu_{n+1}^0}$, we obtain

$$\varepsilon^{\widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right) - \varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right)} \leq \gamma_0 \sum_{\alpha \in \mathscr{M}} \varepsilon^{\varphi(\alpha) - \overline{\mu^0}\alpha + \mu_{n+1}^0} \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}],\ \overline{\varepsilon} < 1. \tag{2.10}$$

Since by our assumption $\widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right) - \varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) < 0$ and by the condition (2.8) for all $\alpha \in \mathscr{M}$ $\varphi\left(\alpha\right) - \overline{\mu^0}\alpha + \mu_{n+1}^0 \geq 0$, therefore as $\varepsilon \to 0$, the right-hand side of (2.10) is bounded, whereas the left-hand side tends to infinity. The obtained contradiction proves that $\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) \leq \widetilde{\varphi}_{\mathscr{M}}^{opt}\left(\alpha^0\right)$. $\square$

Remark 2.1. The problem (2.5), (2.6) can be solved by the well-known simplex-method (for the variety of the method, see [13]; note that there exists a polynomial algorithm for solving optimization problems of linear programming [14]).

Denote

$$\mathscr{E}_\varphi\left(\mathscr{M}\right) \equiv \left\{\alpha \in \mathscr{M} : \varphi\left(\alpha\right) = \varphi_{\mathscr{M}}^{opt}\left(\alpha\right)\right\}, \tag{2.11}$$
$$\mathscr{V}\left(\mathscr{M}\right) \equiv \left\{\alpha \in \mathscr{M} : \alpha \notin \left\langle \mathscr{M} \setminus \{\alpha\}\right\rangle\right\}.$$

We call $\mathscr{E}_\varphi\left(\mathscr{M}\right)$ an essential part of the collection $\mathscr{M}$. From the theorem below it immediately follows that $\mathscr{V}\left(\mathscr{M}\right) \subseteq \mathscr{E}_\varphi\left(\mathscr{M}\right)$.

Theorem 2.3 (see [15] and [16])**.** *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$ and $\alpha^0 \in \mathbb{N}_0^n$. Then for the existence of a number $\gamma > 0$ such that*

$$\xi^{\alpha^0} \leq \gamma \sum_{\alpha \in \mathscr{M}} \xi^\alpha \quad \forall\, \xi \in \mathbb{R}^n,\ \xi \geq \mathbf{0},$$

it is necessary and sufficient that $\alpha^0 \in \langle \mathscr{M} \rangle$.

Denote

$$\mathscr{B}_\varphi\left(\mathscr{M}\right) \equiv \mathscr{V}\left(\mathscr{M}\right) \cup \left\{\alpha \in \mathscr{M} \setminus \mathscr{V}\left(\mathscr{M}\right) : \varphi_{\mathscr{M}\setminus\{\alpha\}}^{opt}\left(\alpha\right) > \varphi_{\mathscr{M}}^{opt}\left(\alpha\right)\right\} \tag{2.12}$$

We call $\mathscr{B}_\varphi\left(\mathscr{M}\right)$ a base part of the collection $\mathscr{M}$.

Proposition 2.2. *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$, $\varphi : \mathscr{M} \mapsto \mathbb{R}_+$ and $\alpha^0 \in \mathscr{M} \setminus \mathscr{B}_\varphi\left(\mathscr{M}\right)$. Then*

$$\varphi_{\mathscr{M}}^{opt}\left(\alpha^1\right) = \varphi_{\mathscr{M}\setminus\{\alpha^0\}}^{opt}\left(\alpha^1\right) \quad \forall\, \alpha^1 \in \mathscr{M} \setminus \left\{\alpha^0\right\}.$$

Proof. Since $\alpha^0 \in \mathscr{M} \backslash \mathscr{B}_\varphi(\mathscr{M}) \subseteq \mathscr{M} \backslash \mathscr{V}(\mathscr{M})$, therefore $\alpha^0 \in \langle \mathscr{M} \backslash \{\alpha^0\} \rangle$. Let λ_α^0 $\left(\alpha \in \mathscr{M} \backslash \{\alpha^0\}\right)$ be an optimal collection of the problem (2.5), (2.6) for $\mathscr{K} = \mathscr{M} \backslash \{\alpha^0\}$ and $\beta = \alpha^0$, and λ_α^1 $(\alpha \in \mathscr{M})$ be an optimal collection of the problem (2.5), (2.6) for $\mathscr{K} = \mathscr{M}$ and $\beta = \alpha^1$. Then

$$\sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^0 = 1, \qquad \sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^0 \alpha = \alpha^0,$$

$$\varepsilon^{\varphi^{opt}_{\mathscr{M} \backslash \{\alpha^0\}}(\alpha^0)} \xi^{\alpha^0} \leq \sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^0 \varepsilon^{\varphi(\alpha)} \xi^\alpha \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}], \ \forall\, \xi \in \mathbb{R}^n, \ \xi \geq \mathbf{0}.$$

and

$$\sum_{\alpha \in \mathscr{M}} \lambda_\alpha^1 = 1, \qquad \sum_{\alpha \in \mathscr{M}} \lambda_\alpha^1 \alpha = \alpha^1,$$

$$\varepsilon^{\varphi^{opt}_{\mathscr{M}}(\alpha^1)} \xi^{\alpha^1} \leq \sum_{\alpha \in \mathscr{M}} \lambda_\alpha^1 \varepsilon^{\varphi(\alpha)} \xi^\alpha \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}], \ \forall\, \xi \in \mathbb{R}^n, \ \xi \geq \mathbf{0}.$$

Since $\varphi^{opt}_{\mathscr{M} \backslash \{\alpha^0\}}\left(\alpha^0\right) = \varphi^{opt}_{\mathscr{M}}\left(\alpha^0\right) \leq \varphi\left(\alpha^0\right)$ $\left(\alpha^0 \notin \mathscr{B}_\varphi(\mathscr{M})\right)$, therefore

$$\varepsilon^{\varphi^{opt}_{\mathscr{M}}(\alpha^1)} \xi^{\alpha^1} \leq \sum_{\alpha \in \mathscr{M}} \lambda_\alpha^1 \varepsilon^{\varphi(\alpha)} \xi^\alpha \leq \sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^1 \varepsilon^{\varphi(\alpha)} \xi^\alpha +$$

$$+ \lambda_{\alpha^0}^1 \sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^0 \varepsilon^{\varphi(\alpha)} \xi^\alpha \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}], \ \forall\, \xi \in \mathbb{R}^n, \ \xi \geq \mathbf{0}.$$

This implies that $\varphi^{opt}_{\mathscr{M}}\left(\alpha^1\right) \geq \varphi^{opt}_{\mathscr{M} \backslash \{\alpha^0\}}\left(\alpha^1\right)$, since

$$\sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^1 + \lambda_{\alpha^0}^1 \sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^0 = 1,$$

$$\sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^1 \alpha + \lambda_{\alpha^0}^1 \sum_{\alpha \in \mathscr{M} \backslash \{\alpha^0\}} \lambda_\alpha^0 \alpha = \alpha^1.$$

The converse inequality, i.e., $\varphi^{opt}_{\mathscr{M}}\left(\alpha^1\right) \leq \varphi^{opt}_{\mathscr{M} \backslash \{\alpha^0\}}\left(\alpha^1\right)$, is trivial. □

Proposition 2.3. *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$ and $\varphi : \mathscr{M} \mapsto \mathbb{R}_+$. Then*

$$\varphi^{opt}_{\mathscr{M}}(\alpha) = \varphi^{opt}_{\mathscr{B}_\varphi(\mathscr{M})}(\alpha) \quad \forall\, \alpha \in \langle \mathscr{M} \rangle.$$

Proof follows directly from Proposition 2.2.

Proposition 2.4. *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$ and $\varphi : \mathscr{M} \mapsto \mathbb{R}_+$. Then*

$$\varepsilon^{\varphi(\alpha^0)} \xi^{\alpha^0} \leq \sum_{\alpha \in \mathscr{B}_\varphi(\mathscr{M})} \varepsilon^{\varphi(\alpha)} \xi^\alpha$$

$$\forall\, \varepsilon \in (0, \overline{\varepsilon}], \ \overline{\varepsilon} < 1, \ \forall\, \xi \in \mathbb{R}^n, \ \xi \geq \mathbf{0}, \ \forall\, \alpha^0 \in \langle \mathscr{M} \rangle.$$

Proof follows from Proposition 2.3.

For the collection $\mathscr{K} \subseteq \mathscr{M}$ and for the nonnegative function ϕ defined on $\mathscr{K}$ we put

$$\mathscr{P}_{\mathscr{K},\phi}(\varepsilon,\xi) \equiv \sum_{\alpha\in\mathscr{K}} \varepsilon^{\phi(\alpha)}\xi^{\alpha} \qquad \varepsilon \in (0,\overline{\varepsilon}],\ \xi \in \mathbb{R}^n.$$

Theorem 2.4. *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$ and $\varphi : \mathscr{M} \mapsto \mathbb{R}_+$. Then there exist the numbers $\gamma_1 > 0$ and $\gamma_2 > 0$ such that for every $\varepsilon \in (0,\overline{\varepsilon}]$ and $\xi \in \mathbb{R}^n, \xi \geq \mathbf{0}$, the following conditions are valid:*

(a) $\mathscr{P}_{\mathscr{B}_\varphi(\mathscr{M}),\varphi}(\varepsilon,\xi) = \mathscr{P}_{\mathscr{B}_\varphi(\mathscr{M}),\varphi^{opt}_{\mathscr{M}}}(\varepsilon,\xi) \leq \mathscr{P}_{\mathscr{E}_\varphi(\mathscr{M}),\varphi^{opt}_{\mathscr{M}}}(\varepsilon,\xi) =$
$= \mathscr{P}_{\mathscr{E}_\varphi(\mathscr{M}),\varphi}(\varepsilon,\xi) \leq \mathscr{P}_{\mathscr{M},\varphi}(\varepsilon,\xi) \leq \mathscr{P}_{\mathscr{M},\varphi^{opt}_{\mathscr{M}}}(\varepsilon,\xi)$;

(b) $\mathscr{P}_{\mathscr{M},\varphi^{opt}_{\mathscr{M}}}(\varepsilon,\xi) \leq \gamma_1 \mathscr{P}_{\mathscr{B}_\varphi(\mathscr{M}),\varphi^{opt}_{\mathscr{M}}}(\varepsilon,\xi)$;

(c) $\mathscr{P}_{\langle\mathscr{M}\rangle,\varphi^{opt}_{\mathscr{M}}}(\varepsilon,\xi) \leq \gamma_2 \mathscr{P}_{\mathscr{B}_\varphi(\mathscr{M}),\varphi^{opt}_{\mathscr{M}}}(\varepsilon,\xi)$.

Proof of the statement of item (a) is trivial (follows directly from definitions of $\mathscr{B}_\varphi(\mathscr{M})$, $\mathscr{E}_\varphi(\mathscr{M})$ and $\varphi^{opt}_{\mathscr{M}}$), items (b) and (c) follow directly from Proposition 2.4.

Proposition 2.5. *Let $\mathscr{M} \in \mathfrak{F}(\mathbb{N}_0^n)$ and $\varphi : \mathscr{M} \mapsto \mathbb{R}_+$. Then*

$$\varphi^{opt}_{\mathscr{M}}\left(\alpha^1+\alpha^2\right) \leq \frac{1}{2}\left(\varphi^{opt}_{\mathscr{M}}\left(2\alpha^1\right)+\varphi^{opt}_{\mathscr{M}}\left(2\alpha^2\right)\right) \quad \forall\, \alpha^1,\ \alpha^2 \in \langle\mathscr{M}\rangle. \tag{2.13}$$

Proof. Let $\alpha^1, \alpha^2 \in \langle\mathscr{M}\rangle$. Since $\alpha^1+\alpha^2 = \frac{2\alpha^1+2\alpha^2}{2}$, by Proposition 2.1 we have

$$\varepsilon^{\frac{1}{2}\left(\varphi^{opt}_{\mathscr{M}}(2\alpha^1)+\varphi^{opt}_{\mathscr{M}}(2\alpha^2)\right)}\xi^{\alpha^1+\alpha^2} \leq \frac{1}{2}\,\varepsilon^{\varphi^{opt}_{\mathscr{M}}(2\alpha^1)}\xi^{2\alpha^1} + \frac{1}{2}\,\varepsilon^{\varphi^{opt}_{\mathscr{M}}(2\alpha^2)}\xi^{2\alpha^2},$$

whence, using Theorem 2.2, by the definition of $\widetilde{\varphi}^{opt}_{\mathscr{M}}$, we obtain $\widetilde{\varphi}^{opt}_{\mathscr{M}}\left(\alpha^1+\alpha^2\right) \leq \frac{1}{2}\left(\widetilde{\varphi}^{opt}_{\mathscr{M}}\left(2\alpha^1\right)+\widetilde{\varphi}^{opt}_{\mathscr{M}}\left(2\alpha^2\right)\right)$. Consequently, by Theorem 2.2 we obtain inequality (2.13). □

3. The Main Result

Let coefficients of the operator L_ε (see formula 1.1) be of the form

$$a_{\alpha,\beta}(x,\varepsilon) \equiv \varepsilon^{\psi(\alpha,\beta)}\eta_{\alpha,\beta}(x,\varepsilon) \quad \left(\eta_{\alpha,\beta}(x,0) \not\equiv 0,\ \alpha,\beta \in \mathscr{N}\right), \tag{3.1}$$

where ψ is the nonnegative function defined on $\mathscr{N} \times \mathscr{N}$, and $\eta_{\alpha,\beta}$ is the function defined on $\Omega \times (0,\overline{\varepsilon}]$.

Denote

$$\mathscr{M}(\mathscr{N}) \equiv \{\alpha+\beta : \alpha,\beta \in \mathscr{N}\}, \quad \mathscr{M}(\mathscr{R}) \equiv \{\alpha+\beta : (\alpha,\beta) \in \mathscr{R}\},$$
$$\varphi(\nu) \equiv \min_{\substack{\alpha,\beta\in\mathscr{N}\\ \alpha+\beta=\nu}} \psi(\alpha,\beta) \quad \nu \in \mathscr{M}(\mathscr{N}).$$
$$\overline{\mathscr{R}} \equiv \{(\alpha,\beta) \in \mathscr{R} : \alpha+\beta \in \mathscr{E}_\varphi(\mathscr{M}(\mathscr{R}))\}, \quad \mathscr{B} \equiv \mathscr{B}_\varphi(\mathscr{N}\backslash\mathscr{N}_0),$$

where $\mathscr{E}_{\varphi}(\mathscr{M}(\mathscr{R}))$ is the essential part of the collection $\mathscr{M}(\mathscr{R})$ (see 2.11) and $\mathscr{B}_{\varphi}(\mathscr{N}\backslash\mathscr{N}_0)$ is the base part of the collection $\mathscr{N}\backslash\mathscr{N}_0$ (see 2.12).

On the coefficients of the operator L_ε we impose the following restrictions:

II$'$. (a) The functions $\eta_{\alpha,\beta}(x,\varepsilon)$ $(\alpha,\beta\in\mathscr{N})$ are infinitely differentiable on $\overline{\Omega}\times[0,\overline{\varepsilon}]$;

(b) for every $\alpha,\beta\in\mathscr{N}_0$, the function $\eta_{\alpha,\beta}(x,\varepsilon)$, as $\varepsilon\to0$, tends uniformly with respect to x to $\eta_{\alpha,\beta}(x,0)$;

IV$'$. (a) the functions $\eta_{\alpha,\beta}(x,\varepsilon)$ are uniformly continuous with respect to x on $\Omega\times(0,\overline{\varepsilon}]$ for $(\alpha,\beta)\in\mathscr{R}$;

(b) there exists the constant $\kappa_1>0$ such that

$$|\eta_{\alpha,\beta}(x,\varepsilon)|\le\kappa_1\quad\forall\,x\in\Omega,\ \ \forall\,\varepsilon\in(0,\overline{\varepsilon}],\ (\alpha,\beta)\in\mathscr{R};$$

(c) there exists the constant $\chi_2>0$ such that

$$\sum_{(\alpha,\beta)\in\overline{\mathscr{R}}}\eta_{\alpha,\beta}(x,0)\,(\mathtt{i}\xi)^{\alpha+\beta}\ge\chi_2\sum_{\alpha\in\mathscr{B}}b_{\alpha,\alpha}(\varepsilon)\,\xi^{2\alpha}\quad\forall\,\xi\in\mathbb{R}^n,\ \ \forall\,\varepsilon\in(0,\overline{\varepsilon}];$$

(f) there exists the constant $\kappa_3>0$ such that for all $(\alpha,\beta)\in\mathscr{I}$ and $\gamma,\delta\in\mathbb{N}_0^n$, if $\gamma\le\alpha$, $\delta\le\beta$ and $\gamma+\delta\ne\alpha+\beta$, then

$$\left|D^{\gamma+\delta}\eta_{\alpha,\beta}(x,\varepsilon)\right|\le\kappa_3,\quad x\in\Omega,\ \ \varepsilon\in(0,\overline{\varepsilon}];$$

V. For all $\alpha,\beta\in\mathscr{M}(\mathscr{N})$, $\alpha\le\beta$, $\alpha\ne\beta$

$$\varphi^{opt}_{\mathscr{M}(\mathscr{N})}(\alpha)<\varphi^{opt}_{\mathscr{M}(\mathscr{N})}(\beta).$$

Remark 3.1. Let the coefficients of the operator L_ε be of the type (3.1), and let Conditions II$'$, III, IV$'$(a), IV$'$(b), IV$'$(f) and V be fulfilled. It is not difficult to prove that if in Condition IV we assume $b_{\alpha,\beta}(\varepsilon)\equiv\varepsilon^{\varphi^{opt}_{\mathscr{M}(\mathscr{N})}(\alpha+\beta)}$, then Conditions IV(c) and IV$'$(c) are equivalent, while Conditions IV(d), IV(e) and IV(g) by virtue of Condition V (see also Theorem 2.4 and Proposition 2.5) are fulfilled automatically.

Taking into account Remark 3.1 and Theorem 1.4, we can prove the following

Theorem 3.1. *Let the coefficients of the type* (3.1) *of the operator* L_ε *satisfy Conditions* I, II, III, IV' *and* V. *Then there exist the constants* $\overline{\overline{\varepsilon}}\in(0,\overline{\varepsilon}]$, $C_1>0$ *and* $C_2>0$ *such that for all* $u\in\mathring{\mathbb{H}}_{\mathscr{N}}(\Omega)$ *the estimate*

$$\|u\|^2_\varepsilon\equiv\sum_{\alpha\in\langle\mathscr{N}\rangle\backslash\langle\mathscr{N}_0\rangle}\varepsilon^{\varphi^{opt}_{\mathscr{M}(\mathscr{N})}(2\alpha)}\|D^\alpha u\|^2+\sum_{\alpha\in\langle\mathscr{N}_0\cup\{0\}\rangle}\|D^\alpha u\|^2\le$$
$$\le C_1\,(L_\varepsilon u,u)\quad\forall\,\varepsilon\in(0,\overline{\overline{\varepsilon}}]$$

is valid.

Consider the following boundary value problem.

Problem $\mathfrak{D}_\varepsilon$. Find a solution $u \in \overset{\circ}{\mathbb{H}}_{\mathscr{N}}(\Omega)$ of the equation

$$L_\varepsilon u = f, \quad f \in \mathbb{L}_2(\Omega). \tag{3.2}$$

Definition 3.1. see [4] and [5] The problem $\mathfrak{D}_\varepsilon$ is said to be uniformly solvable if there exists a number $\varepsilon_0 > 0$ for which

(a) the problem $\mathfrak{D}_\varepsilon$ is solvable for $\varepsilon \in (0, \varepsilon_0]$, i.e., for every $f \in \mathbb{L}_2(\Omega)$, equation (3.2) has a solution $u_\varepsilon \in \overset{\circ}{\mathbb{H}}_{\mathscr{N}}(\Omega)$;

(b) there exist the number $C_0 > 0$ and the functional space B_ε $\left(\overset{\circ}{\mathbb{H}}_{\mathscr{N}}(\Omega) \subset B_\varepsilon\right)$ with the norm $\|.\|_{B_\varepsilon}$ such that for all $u \in \overset{\circ}{\mathbb{H}}_{\mathscr{N}}(\Omega)$

$$\|u\|_{B_\varepsilon} \leq C_0 \|f\|_{\mathbb{L}_2(\Omega)}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Definition 3.2 (see, for e.g., (17))**.** Let $\mathscr{M} \subset \mathbb{N}_0^n$ be a finite collection of multiindices. Then the polyhedron $\langle \mathscr{M} \rangle$ is said to be complete, if it has vertex both at the origin and on every coordinate axis. A complete polyhedron $\langle \mathscr{M} \rangle$ is said to be completely rectilinear, if outer normals of the $n-1$-dimensional sides have only positive coordinates.

Theorem 3.2. *Let $\mathscr{N} \subset \mathbb{N}_0^n$, $\langle \mathscr{N} \rangle$ be a completely rectilinear polyhedron, $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the displacement conditions* (*see* [17]) *and the operator L_ε satisfy the conditions of Theorem* 3.1*. Then the problem $\mathfrak{D}_\varepsilon$ is uniformly solvable.*

Proof. Item (a) of Definition 3.1 (i.e., solvability of the problem $\mathfrak{D}_\varepsilon$) under the conditions I and IV$'$ has been proved in [18] (see also [9] and [19]), while item (b) of Definition 3.1 follows from Theorem 3.1 by virtue of inequality (1.22). □

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