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SHARP JACKSON AND CONVERSE THEOREMS OF TRIGONOMETRIC APPROXIMATION IN WEIGHTED LEBESGUE SPACES

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Abstract. In the present work we prove that improved Jackson type direct theorem of trigonometric polynomial approximation in Lebesgue spaces with Muckenhoupt weights with respect to fractional order moduli of smoothness holds. In addition, we obtain sharp converse and Marchaud inequalities of trigonometric approximation of functions and its fractional derivatives in these weighted Lebesgue spaces.

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1. Introduction

Let $L^p(\mathbb{T})$ be the Lebesgue space of 2π periodic real valued functions defined on $\mathbb{T} := [-\pi, \pi]$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{1/p} < \infty \text{ for } 1 < p < \infty.$$

It is well-known that for functions f belonging to $L^p(\mathbb{T})$, $1 < p < \infty$, the classical Jackson theorem

$$E_n(f)_p := \inf_{T \in \mathcal{T}_n} \|f - T\|_p \le c\,\omega_r\left(f, \frac{1}{n}\right)_p, \quad n \in \mathbb{N},$$

and its weak converse

$$\omega_r\left(f, \frac{1}{n}\right)_p \le \frac{C}{n^r} \sum_{\nu=1}^{n} \nu^{r-1} E_\nu(f)_p, \quad n \in \mathbb{N},$$

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improved ([32], [33]) to the inequalities

$$\frac{c}{n^{r}}\left\{\sum_{\nu=1}^{n}\nu^{\beta r-1}E_{\nu-1}^{\beta}\left(f\right)_{p}\right\}^{1/\beta}\leq c\omega_{r}\left(f,\frac{1}{n}\right)_{p} \tag{1}$$

and

$$\omega_{r}\left(f,\frac{1}{n}\right)_{p}\leq\frac{C}{n^{r}}\left\{\sum_{\nu=1}^{n}\nu^{\gamma r-1}E_{\nu-1}^{\gamma}\left(f\right)_{p}\right\}^{1/\gamma}$$

with the optimal [34] value $\gamma=\min\{2,p\}$, where[1] $r\in\mathbb{N}$, $\beta=\max\{2,p\}$, $\omega_{r}\left(f,\delta\right)_{p}:=\sup\limits_{0<h\leq\delta}\left\Vert\left(T_{h}-I\right)^{r}f\right\Vert_{p}$ is the rth *moduli of smoothness* of the function $f\in L^{p}\left(\mathbb{T}\right)$, $T_{h}f\left(\circ\right):=f\left(\circ+h\right)$ is translation operator, I is identity operator and $\mathcal{T}_{n}$ is the class of trigonometric polynomials of degree not greater than n.

These above type inequalities played an important role in investigation of properties of the conjugate functions [5], in the study of absolute convergent Fourier series [29], imbedding of function classes [26], characterizations of Lipschitz classes, and so on. As a consequence these inequalities have been investigated or generalized for many directions [30], [15], [16], [17], [8], [4] and so on. For a general treatise of approximation problems we can refer to books [31], [28], [19], [25] and [7]. It is also of interest these inequalities in weighted spaces. But moduli of smoothness $\omega_{r}\left(f,\cdot\right)_{p}$ is not suitable for approximation theorems in weighted spaces because of the translation operator T_{h} is not continuous in weighted spaces, in general. Here we will consider the weighted Lebesgue spaces. A function $\omega:\mathbb{T}\rightarrow[0,\infty]$ will be called a weight if ω is measurable and almost everywhere (a.e.) positive. For a weight ω we denote by $L^{p}\left(\mathbb{T},\omega\right)$ the weighted Lebesgue space of 2π periodic complex valued measurable functions on $\mathbb{T}$ such that $f\omega^{1/p}\in L^{p}\left(\mathbb{T}\right)$. We set $\left\Vert f\right\Vert_{p,\omega}:=\left\Vert f\omega^{1/p}\right\Vert_{p}$ for $f\in L^{p}\left(\mathbb{T},\omega\right)$. A 2π-periodic weight function ω belongs to the Muckenhoupt class A_{p}, $1<p<\infty$, if

$$\frac{1}{\left\vert J\right\vert}\int\limits_{J}\omega\left(x\right)dx\left(\frac{1}{\left\vert J\right\vert}\int\limits_{J}\omega^{-\frac{1}{p-1}}\left(x\right)dx\right)^{p-1}\leq C$$

with a finite constant C independent of J, where J is any subinterval of $\mathbb{T}$ and $\left\vert J\right\vert$ denotes the length of J.

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[1]Throughout this work by $C,c,\ldots$, we denote constants which are different in different places.

In this paper we will use the following notations: $\mathbb{R}:=(-\infty,\infty)$, $\mathbb{R}^{+}:=(0,\infty)$, $\mathbb{N}:=\{1,2,3,\ldots\}$.

For functions of class $L^p(\mathbb{T},\omega)$, $1<p<\infty$, $\omega\in A_p$, E. A. Hadjieva considered as translation the Steklov's mean operator

$$\sigma_t f(x) := \frac{1}{2t}\int_{-t}^{t} f(x+u)\,du, \quad x\in\mathbb{T}, \quad 0<t<\pi$$

and defined the Butzer-Wehrens [6] type moduli of smoothness

$$\Omega_r(f,\delta)_{p,\omega} := \sup_{0<h_i<\delta}\left\|\prod_{i=1}^{r}(I-\sigma_{h_i})f\right\|_{p,\omega}, \quad f\in L^p(\mathbb{T},\omega) \tag{2}$$

of order $r=1,2,3,\ldots$. Using (2) she was proved in [10] (see, also [13]) the Jackson type direct inequality

$$E_n(f)_{p,\omega} \le c\Omega_r\left(f,\frac{1}{n+1}\right)_{p,\omega}, \quad n+1, \quad r\in\mathbb{N} \tag{3}$$

and its weak converse

$$\Omega_r\left(f,\frac{1}{n}\right)_{p,\omega} \le \frac{C}{n^{2r}}\left(E_0(f)_{p,\omega}+\sum_{\nu=1}^{n}\nu^{2r-1}E_\nu(f)_{p,\omega}\right), \quad n\in\mathbb{N}, \tag{4}$$

where

$$E_n(f)_{p,\omega} := \inf_{T\in\mathcal{T}_n}\|f-T\|_{p,\omega}$$

is the measure of trigonometric polynomial approximation in $L^p(\mathbb{T},\omega)$.

And then converse inequality (4) was improved ([18], [8], [9]) to inequality

$$\Omega_r\left(f,\frac{1}{n}\right)_{p,\omega} \le \frac{c}{n^{2r}}\left(\sum_{\nu=1}^{n}\nu^{2r\gamma-1}E_{\nu-1}^{\gamma}(f)_{p,\omega}\right)^{1/\gamma}, \quad n\in\mathbb{N},$$

provided $f\in L^p(\mathbb{T},\omega)$, $1<p<\infty$, $\omega\in A_p$, $r\in\mathbb{N}$ and $\gamma:=\min\{p,2\}$.

For more general doubling weights direct and converse trigonometric and algebraic approximation problems was investigated in [23].For a general discussion of weighted polynomial approximation we can refer to the book [22]. Some direct and converse approximation by rational functions and algebraic polynomials of some weighted function spaces defined on sufficiently smooth complex domains are investigated in [1], [2], [3], [12] and [14].

But there is no results of improved direct type (1) in weighted Lebesgue space $L^p(\mathbb{T},\omega)$. In the present work we consider the improved direct and converse approximation theorems by trigonometric polynomials with respect to the fractional order weighted moduli of smoothness in the spaces $L^p(\mathbb{T},\omega)$, $1<p<\infty$, $\omega\in A_p$. For formulation of the problem we need some further notations and definitions.

If $1 \leq p < \infty$, $\omega \in A_p$, then $L^p(\mathbb{T}, \omega) \subset L^1(\mathbb{T})$. Let

$$S[f] := \sum_{k=-\infty}^{\infty} c_k(f) e^{ikx} \tag{5}$$

be the Fourier series of a function $f \in L^1(\mathbb{T})$.

For a given $f \in L^1(\mathbb{T})$, assuming

$$\int_{\mathbb{T}} f(x)\, dx = 0, \tag{6}$$

we define α*-th fractional* $(\alpha \in \mathbb{R}^+)$ *integral* of f as [35, v.2, p.134]

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k(f) (ik)^{-\alpha} e^{ikx},$$

where

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k}$$

as principal value.

Let $\alpha \in \mathbb{R}^+$ be given. We define *fractional derivative* of a function $f \in L^1(\mathbb{T})$, satisfying (6), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{\alpha-[\alpha]}(x, f)$$

provided the right hand side exists. We will say that a function $f \in L^p(\mathbb{T}, \omega)$ has fractional derivative of degree $\alpha \in \mathbb{R}^+$ if there exists a function $g \in L^p(\mathbb{T}, \omega)$ such that its Fouirer coefficients satisfy $c_k(g) = c_k(f)(ik)^\alpha$. In this case we will write $f^{(\alpha)} = g$.

It is well-known that the Steklov's mean operator is bounded [24] in $L^p(\mathbb{T}, \omega)$, $1 < p < \infty$, for $\omega \in A_p$. Using this fact and setting $x, t \in \mathbb{T}$, $r \in \mathbb{R}^+$, $\omega \in A_p$ and $f \in L^p(\mathbb{T}, \omega)$, $1 < p < \infty$, we define

$$\sigma_t^r f(x) := (I - \sigma_t)^r f(x) =$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \frac{1}{(2t)^k} \int_{-t}^{t} \cdots \int_{-t}^{t} f(x + u_1 + \ldots u_k)\, du_1 \ldots du_k,$$

where $\binom{r}{k} := \frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k > 1$, $\binom{r}{1} := r$ and $\binom{r}{0} := 1$ are Binomial coefficients.

Since [27, p.14, (1.51)]

$$\left| \binom{r}{k} \right| \leq \frac{c}{k^{\alpha+1}}, \quad k \in \mathbb{N}$$

we have

$$\sum_{k=0}^{\infty} \left| \binom{r}{k} \right| < \infty,$$

and therefore
$$\left\|\sigma_t^{\alpha} f\right\|_{p,\omega} \leq c \left\|f\right\|_{p,\omega} < \infty \tag{7}$$
provided $f \in L^p(\mathbb{T},\omega)$, $1<p<\infty$ and $\omega \in A_p$.

For $r \in \mathbb{R}^+$ we define the *fractional modulus of smoothness of index* r for $f \in L^p(\mathbb{T},\omega)$, $1<p<\infty$, $\omega \in A_p$ as
$$\Omega_r(f,\delta)_{p,\omega} := \sup_{0<h_i,t<\delta} \left\| \prod_{i=1}^{[r]} (I-\sigma_{h_i})(I-\sigma_t)^{r-[r]} f \right\|_{p,\omega}. \tag{8}$$
Since the operator σ_t is bounded in $L^p(\mathbb{T},\omega)$, $1<p<\infty$, $\omega \in A_p$ we have by (7) that
$$\Omega_r(f,\delta)_{p,\omega} \leq c\left\|f\right\|_{p,\omega}$$
where the constant $c>0$ dependent only on r and p.

Remark 1. Let $r \in \mathbb{R}^+$, $1<p<\infty$, $\omega \in A_p$ and $f \in L^p(\mathbb{T},\omega)$. The modulus of smoothness $\Omega_r(f,\delta)_{p,\omega}$, $\delta \geq 0$ has the following properties.

(i) $\Omega_r(f,\delta)_{p,\omega}$is non-negative, non-decreasing function of $\delta \geq 0$ and sub-additive in f,

(ii) $\lim\limits_{\delta\to 0}\Omega_r(f,\delta)_{p,\omega} = 0$.

Main results of this work can be formulated as following.

Proposition 1. *If $r \in \mathbb{R}^+$, $1<p<\infty$, $\omega \in A_p$ and $f \in L^p(\mathbb{T},\omega)$, then there exists a constant $c>0$ dependent only on r and p such that*
$$E_n(f)_{p,\omega} \leq c\Omega_r\Big(f,\frac{1}{n+1}\Big)_{p,\omega} \tag{9}$$
holds for $n+1 \in \mathbb{N}$.

Theorem 1. *Let $1<p<\infty$, $\omega \in A_p$ and $f \in L^p(\mathbb{T},\omega)$. If $n \in \mathbb{N}$, $r \in \mathbb{R}^+$ and $\beta := \max\{2,p\}$, then there is a constant $c>0$ dependent only on r and p such that*
$$\frac{c}{n^{2r}}\left\{\sum_{\nu=1}^{n} \nu^{2\beta r-1} E_\nu^{\beta}(f)_{p,\omega}\right\}^{1/\beta} \leq \Omega_r\Big(f,\frac{1}{n}\Big)_{p,\omega} \tag{10}$$
holds.

Theorem 2. *Let $1<p<\infty$, $\omega \in A_p$ and $f \in L^p(\mathbb{T},\omega)$. If $n \in \mathbb{N}$, $r \in \mathbb{R}^+$ and $\gamma := \min\{2,p\}$, then there is a constant $c>0$ dependent only on r and p such that*
$$\Omega_r\Big(f,\frac{1}{n}\Big)_{p,\omega} \leq \frac{c}{n^{2r}}\left\{\sum_{\nu=1}^{n} \nu^{2\gamma r-1} E_{\nu-1}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma} \tag{11}$$
holds.

Since $E_n(f)_{p,\omega} \downarrow 0$ we have

$$E_n(f)_{p,\omega} \leq \frac{c}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\beta r-1} E_\nu^\beta (f)_{p,\omega} \right\}^{1/\beta}$$

and therefore estimate (10) is an improvement of (3).

On the other hand since x^γ is convex for $\gamma = \min\{2,p\}$ we get

$$\left(\nu\nu^{2r-1}E_\nu(f)_{p,\omega}\right)^\gamma - \left((\nu-1)\nu^{2r-1}E_\nu(f)_{p,\omega}\right)^\gamma \leq$$

$$\leq \left(\sum_{\mu=1}^{\nu} \mu^{2r-1}E_\mu(f)_{p,\omega}\right)^\gamma - \left(\sum_{\mu=1}^{\nu-1} \mu^{2r-1}E_\mu(f)_{p,\omega}\right)^\gamma$$

and summing the last inequality with $\nu = 1, 2, 3, \ldots$

$$\sum_{\nu=1}^{n} \left\{ \left(\nu\nu^{2r-1}E_\nu(f)_{p,\omega}\right)^\gamma - \left((\nu-1)\nu^{2r-1}E_\nu(f)_{p,\omega}\right)^\gamma \right\} \leq$$

$$\leq \sum_{\nu=1}^{n} \left\{ \left(\sum_{\mu=1}^{\nu} \mu^{2r-1}E_\mu(f)_{p,\omega}\right)^\gamma - \left(\sum_{\mu=1}^{\nu-1} \mu^{2r-1}E_\mu(f)_{p,\omega}\right)^\gamma \right\},$$

whence

$$\left\{ \sum_{\nu=1}^{n} \nu^{2\gamma r-1} E_{\nu-1}^\gamma (f)_{p,\omega} \right\}^{1/\gamma} \leq 2 \sum_{\nu=1}^{n} \nu^{2r-1} E_{\nu-1}(f)_{p,\omega}.$$

The last inequality signifies that inequality (11) is better than (4).

Furthermore, in some cases, inequalities (10) and (11) give more precise results:

If

$$E_n(f)_{p,\omega} \asymp \frac{1}{n^{2r}}, \quad n \in \mathbb{N}$$

then from (3) and (4) we have

$$\Omega_r\left(f, \frac{1}{n}\right)_{p,\omega} \asymp \frac{1}{n^{2r}} \left|\log \frac{1}{n}\right|$$

and from (10) and (11)

$$c\frac{1}{n^{2r}} \left|\log \frac{1}{n}\right|^{1/\beta} \leq \Omega_r\left(f, \frac{1}{n}\right)_{p,\omega} \leq C\frac{1}{n^{2r}} \left|\log \frac{1}{n}\right|^{1/\gamma}.$$

If $p > 2$ and $n \in \mathbb{N}$, then there is [8, Theorem 4] a function $f_0 \in L^p(\mathbb{T}, \omega)$ such that

$$\Omega_1\left(f_0, \frac{1}{n}\right)_{p,\omega} \geq \frac{c}{n^2} \left\{ \sum_{\nu=1}^{n} \nu^3 E_{\nu-1}^2 (f)_{p,\omega} \right\}^{1/\gamma}$$

and hence inequality (11) is sharp in the sense that it can't improved in their natural terms.

As a corollary of Proposition 1, Theorems 1 and 2 we have the following sharp Marchaud and its converse inequalities

Corollary 2. *Let* $1 < p < \infty$, $\omega \in A_p$ *and* $f \in L^p(\mathbb{T}, \omega)$. *If* $r, l \in \mathbb{R}^+$, $r < l$, $\gamma = \min\{2, p\}$, $\beta = \max\{2, p\}$ *and* $0 < t \leq 1/2$, *then there are constants* $c, C > 0$ *depending only on* r *and* p *such that*

$$ct^{2r}\left\{ \int_t^1 \left[\frac{\Omega_l(f,u)_{p,\omega}}{u^{2r}} \right]^{\beta} \frac{du}{u} \right\}^{1/\beta} \leq$$

$$\leq \Omega_r(f,t)_{p,\omega} \leq Ct^{2r}\left\{ \int_t^1 \left[\frac{\Omega_l(f,u)_{p,\omega}}{u^{2r}} \right]^{\gamma} \frac{du}{u} \right\}^{1/\gamma}$$

hold.

We denote by $W_p^{\alpha}(\mathbb{T}, \omega)$, $\alpha > 0$, $1 < p < \infty$, the linear space of functions $f \in L^p(\mathbb{T}, \omega)$ such that $f^{(\alpha)} \in L^p(\mathbb{T}, \omega)$ a.e.

Theorem 3. *Let* $1 < p < \infty$, $\omega \in A_p$ *and* $f \in L^p(\mathbb{T}, \omega)$. *If*

$$\sum_{k=1}^{\infty} k^{\gamma\alpha - 1} E_k^{\gamma}(f)_{p,\omega} < \infty \tag{12}$$

for some $\alpha \in \mathbb{R}^+$ *and* $\gamma = \min\{2, p\}$, *then* $f \in W_p^{\alpha}(\mathbb{T}, \omega)$. *Furthermore, for* $n \in \mathbb{N}$ *we have*

$$E_n\left(f^{(\alpha)}\right)_{p,\omega} \leq c\left(n^{\alpha} E_n(f)_{p,\omega} + \left\{ \sum_{\nu=n+1}^{\infty} \nu^{\alpha\gamma-1} E_{\nu}^{\gamma}(f)_{p,\omega} \right\}^{1/\gamma}\right),$$

where the constant $c > 0$ *dependent only on* r *and* p.

Corollary 3. *Under the conditions of Theorem* 3 *we have for* $n \in \mathbb{N}$ *and* $r \in \mathbb{R}^+$

$$\Omega_r\left(f^{(\alpha)}, \frac{1}{n}\right)_{p,\omega} \leq$$

$$\leq c\left(\left(\sum_{\nu=n+1}^{\infty} \nu^{\alpha\gamma-1} E_{\nu}^{\gamma}(f)_{p,\omega}\right)^{\frac{1}{\gamma}} + \frac{1}{n^{2r}}\left(\sum_{\nu=1}^{n} \nu^{\gamma(2r+\alpha)-1} E_{\nu}^{\gamma}(f)_{p,\omega}\right)^{\frac{1}{\gamma}}\right)$$

with a constant $c > 0$ *dependent only on* r *and* p.

2. Proof of Theorem 1

We need the following weighted version of Marcinkiewicz multiplier and Littlewood-Paley theorems [20, Theorems 1 and 2]:

Theorem A. *Let a sequence $\{\lambda_\mu\}$ of real numbers be satisfy*

$$|\lambda_\mu| \leq A, \qquad \sum_{\mu=2^{m-1}}^{2^m-1} |\lambda_\mu - \lambda_{\mu+1}| \leq A \tag{13}$$

for all $\mu, m \in \mathbb{N}$. If $1 < p < \infty$, $\omega \in A_p$ and $f \in L^p(\mathbb{T}, \omega)$ with the Fourier series (5), then there is a function $F \in L^p(\mathbb{T}, \omega)$ such that the series $\sum_{k=-\infty}^{\infty} \lambda_k c_k e^{ikx}$ is Fourier series for F and

$$\|F\|_{p,\omega} \leq cA \|f\|_{p,\omega} \tag{14}$$

where c does not depend on f.

Theorem B. *Let $\nu \in \mathbb{N}$, $1 < p < \infty$, $\omega \in A_p$ and $f \in L^p(\mathbb{T}, \omega)$ with the Fourier series (5) satisfying (6), then there is constants c, C such that*

$$c\left\| \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{1/2} \right\|_{p,\omega} \leq$$

$$\leq \left\| \sum_{|\mu|=2^{\nu-1}}^{\infty} c_\nu e^{i\nu x} \right\|_{p,\omega} \leq C \left\| \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{1/2} \right\|_{p,\omega}, \tag{15}$$

where

$$\Delta_\mu := \Delta_\mu(x, f) := \sum_{|\nu|=2^{\mu-1}}^{2^\mu - 1} c_\nu e^{i\nu x} \text{ with } c_\nu := c_\nu(f).$$

Lemma 1. *If $0 < \alpha \leq \beta$, $\omega \in A_p$, $1 < p < \infty$ and $f \in L^p(\mathbb{T}, \omega)$, then*

$$\Omega_\beta(f, \cdot)_{p,\omega} \leq c\Omega_\alpha(f, \cdot)_{p,\omega}. \tag{16}$$

Proof. If $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}$, then it is easy to see from (8) that

$$\Omega_\beta(f, \cdot)_{p,\omega} \leq c\Omega_\alpha(f, \cdot)_{p,\omega}. \tag{17}$$

Now, we assume $0 < \alpha \leq \beta < 1$. In this case, putting $\Phi(\cdot) := \sigma_t^\alpha f(\cdot)$ we have

$$\sigma_t^{\beta-\alpha}\Phi(\cdot) = \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{(2t)^j} \int_{-t}^{t} \cdots \int_{-t}^{t} \Phi(\cdot + u_1 + \dots u_j)\, du_1 \dots du_j =$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{(2t)^j} \int_{-t}^{t} \cdots \int_{-t}^{t} \left[\sum_{k=0}^{\infty} (-1)^{\alpha-k} \binom{\alpha}{k} \frac{1}{(2t)^k} \times \right.$$

$$\left. \times \int_{-t}^{t} \cdots \int_{-t}^{t} f(\cdot + u_1 + \dots u_j + u_{j+1} + \dots u_{j+k})\, du_1 \dots du_j du_{j+1} \dots du_{j+k} \right] =$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \beta-\alpha \\ j \end{pmatrix} \begin{pmatrix} \alpha \\ k \end{pmatrix} \times$$

$$\times \left[\frac{1}{(2t)^{j+k}} \int_{-t}^{t} \cdots \int_{-t}^{t} f\left(\cdot + u_1 + \ldots u_{j+k}\right) du_1 \ldots du_{j+k} \right] =$$

$$= \sum_{\upsilon=0}^{\infty} (-1)^{\upsilon} \begin{pmatrix} \beta \\ \upsilon \end{pmatrix} \frac{1}{(2t)^{\upsilon}} \int_{-t}^{t} \cdots \int_{-t}^{t} f\left(\cdot + u_1 + \ldots u_{\upsilon}\right) du_1 \ldots du_{\upsilon} = \sigma_t^{\beta} f(\cdot) \text{ a.e.}$$

Then

$$\left\| \sigma_t^{\beta} f(\cdot) \right\|_{p,\omega} = \left\| \sigma_t^{\beta-\alpha} \Phi(\cdot) \right\|_{p,\omega} \leq c \left\| \sigma_t^{\alpha} f(\cdot) \right\|_{p,\omega}$$

and

$$\Omega_{\beta}(f, \cdot)_{p,\omega} \leq c \Omega_{\alpha}(f, \cdot)_{p,\omega}. \tag{18}$$

Remaining cases will follow from (17) and (18). □

Proof of Proposition 1. From (3) and (16) we have

$$E_n(f)_{p,\omega} \leq c\Omega_{[r]+1}\Big(f, \frac{1}{n+1}\Big)_{p,\omega} \leq C\Omega_r\Big(f, \frac{1}{n+1}\Big)_{p,\omega}, \quad n+1 \in \mathbb{N}$$

and the assertion (9) follows. □

Proof of Theorem 1. Let $r \in \mathbb{R}^+$, $1 < \beta < \infty$, $c_\mu := c_\mu(f)$ and $n \in \mathbb{N}$. We suppose that the number $m \in \mathbb{N}$ satisfies $2^m \leq n \leq 2^{m+1}$. We put

$$\delta_{n,r,\beta} := \left\{ \sum_{\nu=1}^{n} \frac{\nu^{2\beta r-1}}{n^{2\beta r}} E_{\nu}^{\beta}(f)_{p,\omega} \right\}^{1/\beta}.$$

Then by (15)

$$\delta_{n,r,\beta} \leq \left\{ \sum_{\nu=1}^{m+1} \sum_{|\mu|=2^{\nu-1}}^{2^{\nu}-1} \frac{\mu^{2\beta r-1}}{n^{2\beta r}} E_{\mu}^{\beta}(f)_{p,\omega} \right\}^{1/\beta} \leq$$

$$\leq \left\{ \sum_{\nu=1}^{m+1} \frac{2^{2\nu\beta r}}{n^{2\beta r}} E_{2^{\nu-1}-1}^{\beta}(f)_{p,\omega} \right\}^{1/\beta} \leq \left\{ \sum_{\nu=1}^{m+1} \frac{2^{2\nu\beta r}}{n^{2\beta r}} \left\| \sum_{|\mu|=2^{\nu-1}}^{\infty} c_\mu e^{i\mu x} \right\|_{p,\omega}^{\beta} \right\}^{1/\beta} \leq$$

$$\leq C \left\{ \sum_{\nu=1}^{m+1} \frac{2^{2\nu\beta r}}{n^{2\beta r}} \left\| \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{1/2} \right\|_{p,\omega}^{\beta} \right\}^{1/\beta}.$$

Setting $1 < p \leq 2$, $\beta = 2$, using generalized Minkowski's inequality and Abel's transformation we find

$$\delta_{n,r,2} \leq C \left\{ \sum_{\nu=1}^{m+1} \frac{2^{4\nu r}}{n^{4r}} \left(\int_{\mathbb{T}} \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{p/2} \omega(x)\, dx \right)^{2/p} \right\}^{1/2} \leq$$

$$\leq C\Bigg(\int\limits_{\mathbb{T}}\left(\sum_{\nu=1}^{m+1}\frac{2^{4\nu r}}{n^{4r}}\sum_{\mu=\nu}^{\infty}\left|\Delta_{\mu}\right|^{2}\right)^{p/2}\omega\left(x\right)dx\Bigg)^{1/p}\leq$$

$$\leq C\Bigg(\int\limits_{\mathbb{T}}\left(\sum_{\nu=1}^{m}\frac{2^{4\nu r}}{n^{4r}}\left|\Delta_{\nu}\right|^{2}+\frac{2^{4r(m+1)}}{n^{4r}}\sum_{\mu=m+1}^{\infty}\left|\Delta_{\mu}\right|^{2}\right)^{p/2}\omega\left(x\right)dx\Bigg)^{1/p}\leq$$

$$\leq C\Bigg(\int\limits_{\mathbb{T}}\left(\sum_{\nu=1}^{m}\frac{2^{4\nu r}}{n^{4r}}\left|\Delta_{\nu}\right|^{2}\right)^{p/2}\omega\left(x\right)dx\Bigg)^{1/p}+$$

$$+c\Bigg(\int\limits_{\mathbb{T}}\left(\sum_{\mu=m+1}^{\infty}\left|\Delta_{\mu}\right|^{2}\right)^{p/2}\omega\left(x\right)dx\Bigg)^{1/p}=:CI_{1}+cI_{2}.$$

Using (15) and (9) we can estimate I_2 as follows

$$I_{2}=\left\|\left(\sum_{\mu=m+1}^{\infty}\left\|\Delta_{\mu}\right\|^{2}\right)^{1/2}\right\|_{p,\omega}\leq c\left\|\sum_{\|\mu\|=2^{m}}^{\infty}c_{\mu}e^{i\mu x}\right\|_{p,\omega}\leq$$

$$\leq cE_{2^{m}-1}\left(f\right)_{p,\omega}\leq c\Omega_{r}\left(f,\frac{1}{n}\right)_{p,\omega}.$$

On the other hand

$$I_{1}=\Bigg(\int\limits_{\mathbb{T}}\left(\sum_{\nu=1}^{m}\frac{2^{4\nu r}}{n^{4r}}\left\|\Delta_{\nu}\right\|^{2}\right)^{p/2}\omega\left(x\right)dx\Bigg)^{1/p}\leq\left\|\sum_{\nu=1}^{m}\frac{2^{2\nu r}}{n^{2r}}\left|\Delta_{\nu}\right|\right\|_{p,\omega}\leq$$

$$\leq\left\|\sum_{\nu=1}^{m}\sum_{|\mu|=2^{\nu-1}}^{2^{\nu}-1}\frac{2^{2\nu r}}{n^{2r}}\left|c_{\mu}e^{i\mu x}\right|\right\|_{p,\omega}=$$

$$=\left\|\sum_{\nu=1}^{m}\sum_{|\mu|=2^{\nu-1}}^{2^{\nu}-1}\frac{2^{2\nu r}}{|\mu|^{2r}}\frac{\left(\frac{|\mu|}{n}\right)^{2r}}{\left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^{r}}\left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^{r}\left|c_{\mu}e^{i\mu x}\right|\right\|_{p,\omega}. \tag{19}$$

We define

$$h_{\mu}:=\begin{cases}\frac{2^{2\nu r}}{|\mu|^{2r}}, & \text{for } 1\leq|\mu|\leq 2^{m}-1,\ \nu=1,\ldots,m,\\ \frac{2^{2mr}}{|\mu|^{2r}}, & \text{for } 2^{m}\leq|\mu|\leq n,\\ 0, & \text{for } |\mu|>n,\end{cases}$$

and

$$\lambda_{\mu}:=\begin{cases}\frac{\left(\frac{|\mu|}{n}\right)^{2r}}{\left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^{r}}, & \text{for } 1\leq|\mu|\leq n,\\ 0, & \text{for } |\mu|>n.\end{cases}$$

In this case, for $|\mu| = 1, 2, 3, \ldots$, $\{h_\mu\}$ satisfy (13) with $A = 2^{2r}$ and also $\{\lambda_\mu\}$ satisfy (13) with $A = (1 - \sin 1)^{-r}$. By (19) we get

$$I_1 \leq \left\| \sum_{|\mu|=1}^{2^m-1} \frac{2^{2\nu r}}{|\mu|^{2r}} \frac{\left(\frac{|\mu|}{n}\right)^{2r}}{\left(1 - \frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r} \left(1 - \frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r \left|c_\mu e^{i\mu x}\right| \right\|_{p,\omega} =$$

$$= \left\| \sum_{|\mu|=1}^{\infty} h_\mu \lambda_\mu \left(1 - \frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r \left|c_\mu e^{i\mu x}\right| \right\|_{p,\omega}.$$

Now, using Theorem A twice in the last norm we obtain

$$I_1 \leq \frac{c2^{2r}}{(1-\sin 1)^r} \left\| \sum_{|\mu|=1}^{\infty} \left(1 - \frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r \left|c_\mu e^{i\mu x}\right| \right\|_{p,\omega} \leq$$

$$\leq \frac{c2^{2r}}{(1-\sin 1)^r} \left\| \left(I - \sigma_{1/n}\right)^r f \right\|_{p,\omega} =$$

$$= \frac{c2^{2r}}{(1-\sin 1)^r} \left\| \left(I - \sigma_{1/n}\right)^{[r]} \left(I - \sigma_{1/n}\right)^{r-[r]} f \right\|_{p,\omega} \leq$$

$$\leq \frac{c2^{2r}}{(1-\sin 1)^r} \sup_{0<h_i,t<\frac{1}{n}} \left\| \prod_{i=1}^{[r]} \left(I - \sigma_{h_i}\right) \left(I - \sigma_t\right)^{r-[r]} f \right\|_{p,\omega} \leq C\Omega_r\left(f, \frac{1}{n}\right)_{p,\omega}.$$

Therefore

$$\delta_{n,r,2} \leq C\Omega_r\left(f, \frac{1}{n}\right)_{p,\omega}.$$

If $p > 2$, $\beta = p$, then

$$\delta_{n,r,p} \leq C\left\{ \sum_{\nu=1}^{m+1} \frac{2^{2\nu pr}}{n^{2pr}} \left\| \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{1/2} \right\|_{p,\omega}^p \right\}^{1/p} =$$

$$= C\left\{ \sum_{\nu=1}^{m+1} \frac{2^{2\nu pr}}{n^{2pr}} \left[\int_{\mathbb{T}} \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{p/2} \omega(x)\, dx \right] \right\}^{1/p} \leq$$

$$\leq C\left\{ \left[\int_{\mathbb{T}} \sum_{\nu=1}^{m+1} \frac{2^{2\nu pr}}{n^{2pr}} \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{p/2} \omega(x)\, dx \right] \right\}^{1/p} \leq$$

$$\leq C\left\{ \left[\int_{\mathbb{T}} \left(\sum_{\nu=1}^{m+1} \frac{2^{4\nu r}}{n^{4r}} \sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{p/2} \omega(x)\, dx \right] \right\}^{1/p}$$

and hence

$$\delta_{n,r,p} \leq C\Omega_r\left(f, \frac{1}{n}\right)_{p,\omega}.$$

Proof of Theorem 1 is completed. $\square$

3. Proof of Theorem 2

Let $f \in L^p(\mathbb{T}, \omega)$, $1 < p < \infty$, $\omega \in A_p$ and $\int_0^{2\pi} f(x)\,dx = 0$. We assume that f has Fourier series (5) with $c_\mu := c_\mu(f)$. We choose a $m \in \mathbb{N}$ so that $2^m \leq n < 2^{m+1}$ hold. Let us denote $S_n(x) := S_n(x, f) := \sum_{k=-n}^{n} c_k e^{ikx}$ and $S_*(x, f) := \sup_{n \geq 1} S_n(x, f)$ for a given $x \in \mathbb{T}$. Since S_* is bounded operator [11] in $L^p(\mathbb{T}, \omega)$, $1 < p < \infty$, we have

$$\|f - S_n\|_{p,\omega} \leq c E_n(f)_{p,\omega}. \tag{20}$$

As is well-known $\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} f := \prod_{i=1}^{[r]} (I - \sigma_{h_i})(I - \sigma_t)^{r-[r]} f$ has Fourier series

$$\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} f(\cdot) \sim$$
$$\sim \sum_{\nu=-\infty}^{\infty} \left(1 - \frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right) \ldots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) c_\nu e^{i\nu\cdot}$$

and

$$\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} f(\cdot) =$$
$$= \sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} (f(\cdot) - S_{2^{m-1}}(\cdot, f)) + \sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} S_{2^{m-1}}(\cdot, f).$$

From (20) and $E_n(f)_{p,\omega} \downarrow 0$ we have

$$\left\|\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} (f(\cdot) - S_{2^{m-1}}(\cdot, f))\right\|_{p,\omega} \leq$$
$$\leq c \|f(\cdot) - S_{2^{m-1}}(\cdot, f)\|_{p,\omega} \leq c E_{2^{m-1}}(f)_{p,\omega} \leq$$
$$\leq \frac{c}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\gamma r - 1} E^\gamma_{\nu-1}(f)_{p,\omega} \right\}^{1/\gamma}.$$

On the other hand from (15) we get

$$\left\|\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} S_{2^{m-1}}(\cdot, f)\right\|_{p,\omega} \leq c \left\| \left\{ \sum_{\mu=1}^{m} |\delta_\mu|^2 \right\}^{1/2} \right\|_{p,\omega},$$

where

$$\delta_\mu := \sum_{|\nu|=2^{\mu-1}}^{2^\mu - 1} \left(1 - \frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right) \ldots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) c_\nu e^{i\nu x}.$$

If $p > 2$

$$\left\| \left\{ \sum_{\mu=1}^{m} |\delta_\mu|^2 \right\}^{1/2} \right\|_{p,\omega} \leq \left\{ \sum_{\mu=1}^{m} \|\delta_\mu\|^2_{p,\omega} \right\}^{1/2}$$

and if $1 < p \le 2$ using generalized Minkowski's inequality we obtain

$$\left\| \left\{ \sum_{\mu=1}^{m} |\delta_\mu|^2 \right\}^{1/2} \right\|_{p,\omega} \le \left\{ \sum_{\mu=1}^{m} \|\delta_\mu\|_{p,\omega}^{p} \right\}^{1/p}$$

and therefore

$$\left\| \left\{ \sum_{\mu=1}^{m} |\delta_\mu|^2 \right\}^{1/2} \right\|_{p,\omega} \le \left\{ \sum_{\mu=1}^{m} \|\delta_\mu\|_{p,\omega}^{\gamma} \right\}^{1/\gamma}.$$

By Abel's transformation we get

$$\|\delta_\mu\|_{p,\omega} \le \sum_{|\nu|=2^{\mu-1}}^{2^\mu-2} \left| \left(1-\frac{\sin \upsilon t}{\upsilon t}\right)^{r-[r]} \left(1-\frac{\sin \upsilon h_1}{\upsilon h_1}\right) \dots \left(1-\frac{\sin \upsilon h_{[r]}}{\upsilon h_{[r]}}\right) - \right.$$
$$\left. - \left(1-\frac{\sin(\nu+1)t}{(\nu+1)t}\right)^{r-[r]} \left(1-\frac{\sin(\nu+1)h_1}{(\nu+1)h_1}\right) \dots \left(1-\frac{\sin(\nu+1)h_{[r]}}{(\nu+1)h_{[r]}}\right) \right|$$
$$\times \left\| \sum_{|l|=2^{\mu-1}}^{\nu} \left|c_l e^{ilx}\right| \right\|_{p,\omega} +$$
$$+ \left| \left(1-\frac{\sin(2^\mu-1)t}{(2^\mu-1)t}\right)^{r-[r]} \left(1-\frac{\sin(2^\mu-1)h_1}{(2^\mu-1)h_1}\right) \dots \left(1-\frac{\sin(2^\mu-1)h_{[r]}}{(2^\mu-1)h_{[r]}}\right) \right|$$
$$\times \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} \left|c_l e^{ilx}\right| \right\|_{p,\omega}$$

and

$$\left\| \sum_{|l|=2^{\mu-1}}^{\nu} c_l e^{ilx} \right\|_{p,\omega} \le c E_{2^{\mu-1}-1}(f)_{p,\omega},$$

$$\left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} \left|c_l e^{ilx}\right| \right\|_{p,\omega} \le C E_{2^{\mu-1}-1}(f)_{p,\omega}.$$

Since $x^r\left(1-\frac{\sin x}{x}\right)^r$ is non decreasing for positive x we have

$$\|\delta_\mu\|_{p,\omega} \le c 2^{2\mu r} t^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 E_{2^{\mu-1}-1}(f)_{p,\omega}$$

and hence

$$\left\| \sigma^r_{t,h_1,h_2,\dots,h_{[r]}} S_{2^{m-1}}(\cdot,f) \right\|_{p,\omega} \le$$
$$\le c t^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 \left\{ \sum_{\mu=1}^{m} 2^{2\mu r\gamma} E^{\gamma}_{2^{\mu-1}-1}(f)_{p,\omega} \right\}^{1/\gamma} \le$$
$$\le c t^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 \left\{ 2^{2\gamma r} E_0^{\gamma}(f)_{p,\omega} \right\}^{1/\alpha} +$$

$$+ct^{2(r-[r])}h_1^2h_2^2\dots h_{[r]}^2\left\{\sum_{\mu=2}^{m}\sum_{\nu=2^{\mu-2}}^{2^{\mu-1}-1}\nu^{2\gamma r-1}E_{\nu-1}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma}\leq$$

$$\leq ct^{2(r-[r])}h_1^2h_2^2\dots h_{[r]}^2\left\{\sum_{\nu=1}^{2^{m-1}-1}\nu^{2\gamma r-1}E_{\nu-1}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma}.$$

Therefore we find

$$\Omega_r\left(f,\frac{1}{n}\right)_{p,\omega}\leq\frac{c}{n^{2r}}\left\{\sum_{\nu=1}^{n}\nu^{2\gamma r-1}E_{\nu-1}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma}$$

and Theorem 2 is proved.

4. Proof of Theorem 3

We will use the following lemmas

Lemma A [18]. *Let $\{f_n\}$ be a sequence such that every f_n is absolutely continuous, and $\omega\in A_p$, $1<p<\infty$. If the sequence $\{f_n\}$ converges to the function f in $L^p(\mathbb{T},\omega)$, $1<p<\infty$, norm and the sequence of first derivatives $\{f_n'\}$ converges to some function g* in *$L^p(\mathbb{T},\omega)$, $1<p<\infty$, norm, then f is absolutely continuous and $f'(x)=g(x)$* a.e.

Lemma B [21, Theorem 1]. *Let $T_n\in\mathcal{T}_n$, $1<p<\infty$, $\omega\in A_p$ and $\alpha\in\mathbb{R}^+$. Then there exists a constant $c>0$ independent of n such that*

$$\left\|T_n^{(\alpha)}\right\|_{p,\omega}\leq cn^{\alpha}\left\|T_n\right\|_{p,\omega}$$

holds.

Proof of Theorem 3. Let T_n be a polynomial of class $\mathcal{T}_n$ such that $E_n(f)_{p,\omega}=\|f-T_n\|_{p,\omega}$ and we set

$$\mathcal{U}_0(x):=T_1(x)-T_0(x);\,\mathcal{U}_\nu(x):=T_{2^\nu}(x)-T_{2^{\nu-1}}(x),\quad\nu=1,2,3,\dots.$$

Hence

$$T_{2^N}(x)=T_0(x)+\sum_{\nu=0}^{N}\mathcal{U}_\nu(x),\qquad N=0,1,2,\dots.$$

For a given $\varepsilon>0$, by (12) there exists $\eta\in\mathbb{N}$ such that

$$\sum_{\nu=2^\eta}^{\infty}\nu^{\gamma\alpha-1}E_\nu^{\gamma}(f)_{p,\omega}<\varepsilon. \tag{21}$$

From Lemma B we have

$$\left\|\mathcal{U}_\nu^{(\alpha)}\right\|_{p,\omega}\leq c2^{\nu\alpha}\left\|\mathcal{U}_\nu\right\|_{p,\omega}\leq C2^{\nu\alpha}E_{2^{\nu-1}}(f)_{p,\omega},\quad\nu\in\mathbb{N}.$$

On the other hand it is easily seen that

$$2^{\nu\alpha}E_{2^{\nu-1}}(f)_{p,\omega} \leq c2^{2\alpha}\left\{\sum_{\mu=2^{\nu-2}+1}^{2^{\nu-1}} \mu^{\gamma\alpha-1}E_{\mu}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma}, \quad \nu = 2,3,4,\ldots.$$

For the positive integers satisfying $K < N$

$$T_{2^N}^{(\alpha)}(x) - T_{2^K}^{(\alpha)}(x) = \sum_{\nu=K+1}^{N} U_{\nu}^{(\alpha)}(x), \quad x \in \mathbb{T}$$

and hence if K, N are large enough we obtain from (21)

$$\left\|T_{2^N}^{(\alpha)}(x) - T_{2^K}^{(\alpha)}(x)\right\|_{p,\omega} \leq \sum_{\nu=K+1}^{N} \left\|\mathcal{U}_{\nu}^{(\alpha)}(x)\right\|_{p,\omega} \leq c\sum_{\nu=K+1}^{N} 2^{\nu\alpha}E_{2^{\nu-1}}(f)_{p,\omega} \leq$$

$$\leq C4^{\alpha}\sum_{\nu=K+1}^{N}\left\{\sum_{\mu=2^{\nu-2}}^{2^{\nu-1}} \mu^{\gamma\alpha-1}E_{\mu}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma} \leq$$

$$\leq c\left\{\sum_{\mu=2^{K-1}+1}^{2^{N-1}} \mu^{\gamma\alpha-1}E_{\mu}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma} < c4^{\alpha}\varepsilon^{1/\gamma}.$$

Therefore $\left\{T_{2^N}^{(\alpha)}\right\}$ is a Cauchy sequence in $L^p(\mathbb{T},\omega)$. Then there exists a $\varphi \in L^p(\mathbb{T},\omega)$ satisfying

$$\left\|T_{2^N}^{(\alpha)} - \varphi\right\|_{p,\omega} \to 0, \quad \text{as } N \to \infty.$$

On the other hand we have

$$\|T_{2^N} - f\|_{p,\omega} = E_{2^N}(f)_{p,\omega} \to 0, \quad \text{as } N \to \infty.$$

Then from Lemma A we obtain that $I_{\alpha-[\alpha]}(\cdot, f)$ is absolutely continuous on $\mathbb{T}$ and $\left(I_{\alpha-[\alpha]}(\cdot, f)\right)' = f^{(\alpha)} \in L^p(\mathbb{T},\omega)$. Therefore $f \in W_p^{\alpha}(\mathbb{T},\omega)$.

We note that

$$E_n\left(f^{(\alpha)}\right)_{p,\omega} \leq \left\|f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{p,\omega} \leq$$

$$\leq \left\|S_{2^{m+2}}f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{p,\omega} + \left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)} - S_{2^k}f^{(\alpha)}\right]\right\|_{p,\omega}. \tag{22}$$

By Lemma B we get for $2^m < n < 2^{m+1}$

$$\left\|S_{2^{m+2}}f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{p,\omega} \leq c2^{(m+2)\alpha}E_n(f)_{p,\omega} \leq cn^{\alpha}E_n(f)_{p,\omega}. \tag{23}$$

By (15) we find

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)} - S_{2^k}f^{(\alpha)}\right]\right\|_{p,\omega} \leq$$

$$\leq c\left\|\left\{\sum_{k=m+2}^{\infty}\left|\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\alpha}c_{\nu}e^{i\nu x}\right|^2\right\}^{1/2}\right\|_{p,\omega},\ c_{\nu}:=c_{\nu}(f),$$

and therefore standard computations imply

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)}-S_{2^k}f^{(\alpha)}\right]\right\|_{p,\omega}\leq c\left(\sum_{k=m+2}^{\infty}\left\|\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\alpha}c_{\nu}e^{i\nu x}\right\|_{p,\omega}^{\gamma}\right)^{1/\gamma}.$$

Putting

$$|\delta_{\nu}^{*}|:=\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\alpha}c_{\nu}e^{i\nu x}=\sum_{\nu=2^k+1}^{2^{k+1}}\nu^{\alpha}2\operatorname{Re}\left(c_{\nu}e^{i(\nu x+\alpha\pi/2)}\right)$$

we have

$$\|\delta_{\nu}^{*}\|_{p,\omega}=\left\|\sum_{\nu=2^k+1}^{2^{k+1}}\nu^{\alpha}U_{\nu}(x)\right\|_{p,\omega},$$

where $U_{\nu}(x)=2\operatorname{Re}\left(c_{\nu}e^{i(\nu x+\alpha\pi/2)}\right)$. Using Abel's transformation we get

$$\|\delta_{\nu}^{*}\|_{p,\omega}\leq\sum_{\nu=2^k+1}^{2^{k+1}-1}|\nu^{\alpha}-(\nu+1)^{\alpha}|\left\|\sum_{l=2^k+1}^{\nu}U_l(x)\right\|_{p,\omega}+$$

$$+\left|\left(2^{k+1}\right)^{\alpha}\right|\left\|\sum_{l=2^k+1}^{2^{k+1}-1}U_l(x)\right\|_{p,\omega}.$$

For $2^k+1\leq\nu\leq 2^{k+1}$, $(k\in\mathbb{N})$ we have

$$\left\|\sum_{l=2^k+1}^{\nu}U_l(x)\right\|_{p,\omega}\leq cE_{2^k}(f)_{p,\omega}$$

and since

$$(\nu+1)^{\alpha}-\nu^{\alpha}\leq\begin{cases}\alpha(\nu+1)^{\alpha-1}, & \alpha\geq 1,\\ \alpha\nu^{\alpha-1}, & 0\leq\alpha<1,\end{cases}$$

we obtain

$$\|\delta_{\nu}^{*}\|_{p,\omega}\leq C2^{k\alpha}E_{2^k-1}(f)_{p,\omega}.$$

Therefore

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)}-S_{2^k}f^{(\alpha)}\right]\right\|_{p,\omega}\leq c\left\{\sum_{k=m+2}^{\infty}2^{k\alpha\gamma}E_{2^k-1}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma}\leq$$

$$\leq c\left\{\sum_{\nu=n+1}^{\infty}\nu^{\gamma\alpha-1}E_{\nu}^{\gamma}(f)_{p,\omega}\right\}^{1/\gamma}\tag{24}$$

and using (22), (23) and (24) Theorem 3 is proved. □

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