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# ON ONE MIXED PROBLEM OF THE PLANE THEORY OF ELASTIC MIXTURE WITH A PARTIALLY UNKNOWN BOUNDARY

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Abstract. We consider a square occupied with an elastic mixture and weakened by uniformly strong unknown holes and by uniformly strong cut out arcs, symmetric with respect to the square diagonals. The linear part of the boundary is under the action of absolutely rigid punches. An unknown part of the boundary is free from external stresses.

Using the methods of analytic functions, an exact solution of the problem and an unknown part of the boundary are found.

**რეზიუმე.** განხილულია უცნობი თანაბრადმტკიცე საზღვრიანი ხვრელით შესუსტებული ნარევის დრეკადი კვადრატი, რომლის წვეროების მიდამო ამოჭრილია უცნობი თანაბრადმტკიცე ტოლი სიდიდის რკალებით, რომლებიც სიმეტრიულია კვადრატის დიაგონალების მიმართ. საზღვრის წრფივ ნაწილებზე მოქმედებენ სწორფუძიანი გლუვი ხისტი შტამპები. საზღვრის უცნობი ნაწილი თავისუფალია გარე ძაბვებისაგან.

ანალიზურ ფუნქციათა თეორიის მეთოდების გამოყენებით მოძებნილია ამოცანის ზუსტი ამონახსნი და საზღვრის უცნობი ნაწილები.

**1**$^0$. A homogeneous equation of statics of the theory of elastic mixtures in a complex form is of the type ([4])

$$\frac{\partial^2 U}{\partial z\,\partial\overline{z}} + K\frac{\partial^2\overline{U}}{\partial\overline{z}^{\,2}} = 0, \tag{1}$$

where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, $\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\big)$, $\frac{\partial}{\partial\overline{z}} = \frac{1}{2}\big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\big)$, $U = (u_1 + iu_2, u_3 + iu_4)^\top$, $u' = (u_1, u_2)^\top$ and $u'' = (u_3, u_4)^\top$ are partial displacements,

$$K = -\frac{1}{2}\,\ell m^{-1}, \quad \ell = \begin{bmatrix} \ell_4 & \ell_5 \\ \ell_5 & \ell_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix},$$

$$\Delta_0 = m_1 m_3 - m_2^2, \quad m_k = \ell_k + \frac{1}{2}\,\ell_{3+k}, \quad k = 1,2,3, \quad \ell_1 = a_2/d_2,$$

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$$\ell_2 = -c/d_2, \quad \ell_3 = a_1/d_2, \quad d_2 = a_1a_2 - c^2, \quad a_1 = \mu_1 - \lambda_5,$$
$$a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, \quad \ell_1 + \ell_4 = b/d_1, \quad \ell_2 + \ell_5 = -c_0/d_1,$$
$$\ell_3 + \ell_6 = a/d_1, \quad a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad d_1 = ab - c_0^2,$$
$$b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho, \quad \alpha_2 = \lambda_3 - \lambda_4,$$
$$\rho = \rho_1 + \rho_2, \quad d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho.$$

Here $\mu_1, \mu_2, \mu_3, \lambda_p$, $p = \overline{1,5}$, are elastic moduli characterizing mechanical propertis of the mixture, $\rho_1$ and $\rho_2$ partial densities of the mixture. It will be assumed that the elastic constants $\mu_1, \mu_2, \mu_3, \lambda_p$, $p = \overline{1,5}$ and partial rigid densities $\rho_1$ and $\rho_2$ satisfy the conditions (inequalities) ([8]).

In [3] and [4], M. O. Basheleishvili obtained the following representations:

$$U = m\varphi(z) + \frac{1}{2}\, ez\,\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2}$$

$$TU = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \\ = \frac{\partial}{\partial s(x)}\Big[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\Big], \tag{3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^\top$ and $\psi(z) = (\psi_1, \psi_2)^\top$ are arbitrary analytic vector functions;

$A = 2\mu m$, $\mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}$, $B = \mu e$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}$, $\frac{\partial}{\partial s(x)} = -n_2\frac{\partial}{\partial x_1} + n_1\frac{\partial}{\partial x_2}$, $(n_1, n_2)^\top$ are the un0it vectors of the normal, $(TU)_p$, $p = \overline{1,4}$, are the stress components ([3]);

$$(TU)_1 = r'_{11}n_1 + r'_{21}n_2, \quad r'_{11} = a\theta' + c_0\theta'' - 2\frac{\partial}{\partial x_2}(\mu_1u_2 + \mu_3u_4),$$
$$r'_{21} = -a_1\omega' - c\omega'' + 2\frac{\partial}{\partial x_1}(\mu_1u_2 + \mu_3u_4),$$
$$(TU)_2 = r'_{12}n_1 + r'_{22}n_2, \quad r'_{12} = a_1\omega' + c\omega'' + 2\frac{\partial}{\partial x_2}(\mu_1u_1 + \mu_3u_3),$$
$$r'_{22} = a\theta' + c_0\theta'' - 2\frac{\partial}{\partial x_1}(\mu_1u_1 + \mu_3u_3),$$
$$(TU)_3 = r''_{11}n_1 + r''_{21}n_2, \quad r''_{11} = c_0\theta' + b\theta'' - 2\frac{\partial}{\partial x_2}(\mu_3u_2 + \mu_2u_4),$$
$$r''_{21} = -c\omega' - a_2\omega'' + 2\frac{\partial}{\partial x_1}(\mu_3u_2 + \mu_2u_4),$$
$$(TU)_4 = r''_{12}n_1 + r''_{22}n_2, \quad r''_{12} = c\omega' + a_2\omega'' + 2\frac{\partial}{\partial x_2}(\mu_3u_1 + \mu_2u_3),$$
$$r''_{22} = c_0\theta' + b\theta'' - 2\frac{\partial}{\partial x_1}(\mu_3u_1 + \mu_2u_3),$$

$$\theta' = \operatorname{div} u' = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \theta'' = \operatorname{div} u'' = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_4}{\partial x_2},$$
$$\omega' = \operatorname{rot} u' = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \omega'' = \operatorname{rot} u'' = \frac{\partial u_4}{\partial x_1} - \frac{\partial u_3}{\partial x_2}.$$

Introduce the vectors

$$\overset{(1)}{\tau} = (r'_{11}, r''_{11})^\top, \quad \overset{(2)}{\tau} = (r'_{22}, r''_{22})^\top, \quad \tau = \overset{(1)}{\tau} + \overset{(2)}{\tau}, \tag{4}$$

$$\overset{(1)}{\eta} = (r'_{21}, r''_{21})^\top, \quad \overset{(2)}{\eta} = (r'_{12}, r''_{12})^\top, \quad \eta = \overset{(1)}{\eta} + \overset{(2)}{\eta}, \quad \varepsilon^* = \overset{(1)}{\eta} - \overset{(2)}{\eta}. \tag{5}$$

Elementary calculations result in

$$\tau = \overset{(1)}{\tau} + \overset{(2)}{\tau} = 2(2E - A - B) \operatorname{Re} \varphi'(z), \tag{6}$$

$$\varepsilon^* = \overset{(1)}{\eta} - \overset{(2)}{\eta} = 2(A - B - 2E) \operatorname{Im} \varphi'(z), \tag{7}$$

$$\overset{(1)}{\tau} - \overset{(2)}{\tau} - i\eta = 2\big[B\overline{z}\varphi''(z) + 2\mu\varphi'(z)\big], \tag{8}$$

here $\det(2E - A - B) > 0$ ([5]).

Let us consider the right rectangular system $(ns)$, where $s$ and $n$ are, respectively, the tangent and the normal to the curve $L$. Let $\alpha$ be the angle of inclination of the normal $n$ to the $ox_1$-axis, and $n = (n_1, n_2)^\top = (\cos\alpha, \sin\alpha)^\top$, $s^0 = (-n_2, n_1)^\top = (-\sin\alpha, \cos\alpha)^\top$ be the unit vector of the normal and tangent.

Introduce the vectors

$$\sigma_n = \begin{pmatrix} (TU)_1 n_1 + (TU)_2 n_2 \\ (TU)_3 n_1 + (TU)_4 n_2 \end{pmatrix} = \overset{(1)}{\tau} \cos^2\alpha + \overset{(2)}{\tau} \sin^2\alpha + \eta \sin\alpha\cos\alpha, \tag{9}$$

$$\sigma_s = \begin{pmatrix} (TU)_2 n_1 - (TU)_1 n_2 \\ (TU)_4 n_1 - (TU)_3 n_2 \end{pmatrix} = \frac{1}{2}\big(\overset{(2)}{\tau} - \overset{(1)}{\tau}\big) \sin 2\alpha + \frac{1}{2}\,\eta \cos 2\alpha - \frac{1}{2}\,\varepsilon^*, \tag{10}$$

$$\sigma_s^* = \sigma_s + \frac{1}{2}\,\varepsilon^*, \tag{11}$$

$$\sigma_t = \begin{pmatrix} \big[r'_{21} n_1 - r'_{11} n_2, r'_{22} n_1 - r'_{12} n_2\big]^\top s \\ \big[r''_{21} n_1 - r''_{11} n_2, r''_{22} n_1 - r''_{12} n_2\big]^\top s \end{pmatrix} =$$
$$= \overset{(1)}{\tau} \sin^2\alpha + \overset{(2)}{\tau} \cos^2\alpha - \eta \sin\alpha\cos\alpha. \tag{12}$$

From (9)–(12) and (6)–(8) on $L$ we obtain

$$\sigma_n + \sigma_t = \overset{(1)}{\tau} + \overset{(2)}{\tau} = 2(2E - A - B) \operatorname{Re} \varphi'(t), \tag{13}$$

$$\sigma_n - i\sigma_s = (2E - A)\overline{\varphi'(t)} - B\varphi'(t) + \big[B\overline{t}\varphi''(t) + 2\mu\psi'(t)\big] e^{2i\alpha}, \tag{14}$$

After elementary calculation we obtain

$$\sigma_n + 2\mu\Big(\frac{\partial U_s}{\partial s} + \frac{U_n}{\rho_0}\Big) + i\Big[\sigma_s - 2\mu\Big(\frac{\partial U_n}{\partial s} - \frac{U_s}{\rho_0}\Big)\Big] = 2\varphi'(t), \tag{15}$$

where $\frac{1}{\rho_0}$ is the curvature of the curve $L$ at the point $t$;

$$U_n = \begin{pmatrix} u_1 n_1 + u_2 n_2 \\ u_3 n_1 + u_4 n_2 \end{pmatrix}, \quad U_s = \begin{pmatrix} u_2 n_1 - u_1 n_2 \\ u_4 n_1 - u_3 n_2 \end{pmatrix}. \tag{16}$$

Direct calculations allow us to verify that on $L$

$$(\sigma_n + i\sigma_s)e^{i\alpha} = i\frac{\partial}{\partial s}\Big[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\Big], \tag{17}$$

whence it follows that

$$\Big[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\Big]_L = -i\int_L e^{i\alpha}(\sigma_n + i\sigma_s)ds. \tag{18}$$

Formulas (2), (3), (6), (8), (13), (14) and (15) are analogous to the Kolosov-Muskhelishvili's formulas for the linear theory of elastic mixtures ([6]).

**2**$^0$. In the present work we study an analogous problem which in the case of the plane theory of elasticity has been studied by R. Bantsuri ([1]). To solve the problem we use the formulas due to Kolosov-Muskhelishvili and the method described in [1].

On the plane $z = x_1 + ix_2$, let an elastic quadratic plate of mixtue with vertices lying on the coordinate axes be weakened by an unknown hole. Moreover, neighborhoods of the vertices are cut out by uniformly strong, equal unknown arcs which are symmetric with respect to the coordinate axes. The boundaries of a doubly connected domain under consideration are assumed to be symmetric with respect to the coordinate axes.

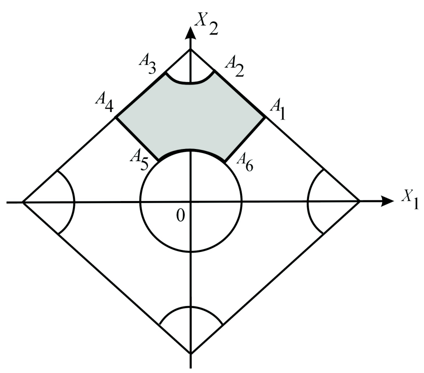


Fig. 1.

The length of the square side we denote by $2a$. Let every segment of the boundary be under the action of absolutely smooth rigid punches with a linear base. Suppose that the normal force concentrated at the midpoint is equal to $p = (p_1, p_2)^\top$.

An unknown part of the boundary is free from external stresses. Assume that the vector $\sigma_s$ is equal to zero on the entire boundary, the vector $U_n$ takes on the segments a constant value, and on the unknown part of the boundary $\sigma_n$ is equal to zero.

We consider the following problem: Find a stressed state of the body and of an unknown part of the boundary under the condition that the vector (12), i.e., $\sigma_t$, is constant, $\sigma_t = K^0 = \text{const}$.

To investigate the above-posed problem, we consider a curvilinear hexagon $A_1A_2A_3A_4A_5A_6$ which is denoted by $D$, $D \subset (x_2 > 0)$. Assume that $A_1A_2 \| A_4A_5$ and $A_3A_4 \| A_6A_1$. $A_6A_1$ and $A_4A_5$ are, respectively, the parts of the segments $x_2 = \pm x_1$ $(x_2 > 0)$. $A_2A_3$ is an unknown arc, and $A_5A_6$ ia a part of the hole boundary. $A_1A_2$ and $A_3A_4$ are, respectively, the parts of the square sides. Denote $\Gamma_1 = A_1A_2$, $\Gamma_2 = A_3A_4$, $\Gamma_3 = A_4A_5$, $\Gamma_4 = A_6A_1$, $\Gamma = \bigcup\limits_{j=1}^{4} \Gamma_j$.

Since $A_1A_2 \| A_4A_5$ and $A_3A_4 \| A_6A_1$, the value of the principal vector on the segments $A_4A_5$ and $A_6A_1$ is equal to $-\frac{1}{2}p$. Note also that the above-posed problem is of cyclic symmetry, and on the segments $A_4A_5$ and $A_6A_1$, $U_n = \sigma_s = 0$.

On the basis of analogous Kolosov-Muskhelishvili's formulas (3), (6), (13), (15) and (18) our problem is reduced to finding two analytic vector functions $\varphi(z)$ and $\psi(z)$ in $D$ by the boundary conditions

$$\operatorname{Im} \varphi'(t) = 0, \quad t \in \Gamma, \tag{19}$$

$$\operatorname{Re} \left[ e^{-i\alpha(t)} (A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} \right] = c(t), \tag{20}$$

$$(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} = B^0(t), \;\; t \in A_2A_3 \cup A_5A_6, \tag{21}$$

$$\operatorname{Re} \varphi'(t) = H, \quad t \in A_2A_3 \cup A_5A_6, \tag{22}$$

where $\alpha(t)$ is the angle made by the outer normal $n$ and the $ox_1$-axis,

$$C(t) = -\operatorname{Re} i \int_{A_1}^{t} \sigma_n(t_0) \exp i(\alpha(t_0) - \alpha(t)) ds_0, \;\; t \in \Gamma, \tag{23}$$

$$\begin{aligned} B^0(t) &= -i \int_{A_1}^{t} \sigma_n(t_0) e^{\frac{\pi}{4} i} ds_0, \;\; t \in A_2A_3, \\ B^0(t) &= -\sum_{j=1}^{3} i \int_{A_1}^{t} \sigma_n(t_0) e^{i\alpha_j(t)} ds_0, \;\; t \in A_5A_6, \end{aligned} \tag{24}$$

$$H = \frac{1}{2}(2E - A - B)^{-1} K^0. \tag{25}$$

$K^0 = (K_1^0, K_2^0)^\top$ is to be defined when solving the problem.

Taking into account that $\sigma_n(t) = 0$ for $t \in A_2A_3 \cup A_5A_6$ and

$$\alpha(t) = \alpha_j = \frac{\pi}{4} + \frac{\pi(j-1)}{2}, \quad t \in \Gamma_j, \quad j = \overline{1,4}, \tag{26}$$

we obtain

$$C(t)=\begin{cases} \frac{1}{2}\,p, & t\in\Gamma_2\cup\Gamma_3,\\ 0, & t\in\Gamma_1\cup\Gamma_4,\end{cases} \tag{27}$$

$$B^0=\begin{cases} \frac{1}{2}\,ipe^{\frac{\pi}{4}i}, & t\in A_2A_3,\\ -\frac{1}{2}\,pe^{\frac{\pi}{4}i}, & t\in A_5A_6.\end{cases} \tag{28}$$

Assume that $U_k(z)$ is continuous in the closed domain $D$, and $\varphi_k(z)$, $\psi_k(z)$, $k=1,2$, are continuously extendable on the boundary of $D$, except possibly the points $A_2A_3$, $A_5,A_6$ in the neighborhood of which they admit the estimate of the type

$$|\varphi_k'(z)|,|\psi_k(z)|<N|z-A_j|^{-\beta},\quad \beta<\frac{1}{2}, \tag{$*$}$$
$$j=2,3,5,6,\quad N=\text{const},\quad k=1,2.$$

Equalities (19) and (22) is the Keldysh-Sedov problem for the domain $D$ ([7]). It is proved that the problem (19) and (22) has a unique solution ([7])

$$\varphi(z)=Hz-\frac{1-i}{4}\,e^{\frac{\pi}{4}i}(2E-A)^{-1}p. \tag{29}$$

Substituting the values $\varphi(t)$, $C(t)$ and $B^0(t)$ into the boundary conditions (20) and (21), we obtain

$$\operatorname{Re}e^{-i\alpha_j}\Big(\frac{1}{2}\,K^0t+2\mu\overline{\psi(t)}\Big)=\frac{1}{4}\,p,\quad t\in\Gamma_j,\quad j=\overline{1,4}, \tag{30}$$

$$e^{-\frac{\pi}{4}i}\Big(\frac{1}{2}\,K^0t+2\mu\overline{\psi(t)}\Big)=\begin{cases}\frac{(1+i)p}{4}, & t\in A_2A_3,\\ -\frac{(1+i)p}{4}, & t\in A_5A_6.\end{cases} \tag{31}$$

If $t\in\Gamma_j$, then $(t-A_i)=i|t-A_j|e^{i\alpha_j}$, whence

$$\operatorname{Re}e^{-i\alpha_j}t=\operatorname{Re}e^{-i\alpha_j}A_j,$$

that is,

$$\operatorname{Re}e^{i\alpha_j}=\begin{cases}a, & t\in\Gamma_1\cup\Gamma_2,\\ 0, & t\in\Gamma_3\cup\Gamma_4.\end{cases} \tag{32}$$

Taking into account equality (26), we can rewrite the boundary conditions (30) and (32) as follows:

$$\operatorname{Re}\Big[\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(t)\Big]=\begin{cases}\frac{p}{4}, & t\in\Gamma_1,\\ -\frac{p}{4}, & t\in\Gamma_3,\end{cases} \tag{33}$$

$$\operatorname{Im}\left[\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}-2\mu e^{\frac{\pi}{4}i}\psi(t)\right]=\begin{cases}\dfrac{p}{4}\,, & t\in\Gamma_2,\\ -\dfrac{p}{4}\,, & t\in\Gamma_4,\end{cases}\tag{34}$$

$$\operatorname{Re}K^0te^{-\frac{\pi}{4}i}=\begin{cases}aK^0\,, & t\in\Gamma_1,\\ 0, & t\in\Gamma_3,\end{cases}\tag{35}$$

$$\operatorname{Im}K^0te^{-\frac{\pi}{4}i}=\begin{cases}aK^0\,, & t\in\Gamma_2,\\ 0, & t\in\Gamma_4.\end{cases}\tag{36}$$

The boundary conditions (31) can be represented in the form

$$\operatorname{Re}\Big(\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}\dfrac{1}{4}\,p, & t\in A_2A_3,\\ -\dfrac{1}{4}\,p, & t\in A_5A_6,\end{cases}\tag{37}$$

$$\operatorname{Im}\Big(\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}-2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}\dfrac{1}{4}\,p, & t\in A_2A_3,\\ -\dfrac{1}{4}\,p, & t\in A_5A_6.\end{cases}\tag{38}$$

Combining equalities (36) and (34), we find that

$$\operatorname{Im}\Big(\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}-\dfrac{1}{4}\,(p-4aK^0), & t\in\Gamma_2,\\ \dfrac{1}{4}\,p, & t\in\Gamma_4.\end{cases}\tag{39}$$

Analogously, from (33) and (35) we obtain

$$\operatorname{Re}\Big(\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}-2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}-\dfrac{1}{4}\,(p-4aK^0), & t\in\Gamma_1,\\ \dfrac{1}{4}\,p, & t\in\Gamma_3.\end{cases}\tag{40}$$

Thus on the basis of (33), (34) and (37)-(40), we can write

$$\begin{aligned}&\operatorname{Re}\Big(\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}\dfrac{1}{4}\,p, & t\in\Gamma_1\cup A_2A_3,\\ -\dfrac{1}{4}\,p, & t\in\Gamma_3\cup A_5A_6,\end{cases}\\ &\operatorname{Im}\Big(\frac{1}{2}\,K^0te^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}-\dfrac{1}{4}\,(p-4aK^0), & t\in\Gamma_2,\\ \dfrac{1}{4}\,p, & t\in\Gamma_4,\end{cases}\end{aligned}\tag{41}$$

$$
\begin{aligned}
&\operatorname{Re}\Big(\frac{1}{2}\,K^0t-2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}-\dfrac{1}{4}\,(p-4aK^0) & t\in\Gamma_1,\\ \dfrac{1}{4}\,p, & t\in\Gamma_3,\end{cases}\\
&\operatorname{Im}\Big(\frac{1}{2}\,K^0t-2\mu e^{\frac{\pi}{4}i}\psi(t)\Big)=\begin{cases}\dfrac{1}{4}\,p, & t\in\Gamma_2\cup A_2A_3,\\ -\dfrac{1}{4}\,p, & t\in\Gamma_4\cup A_5A_6.\end{cases}
\end{aligned}
\tag{42}
$$

Let the function $z=\omega(\zeta)$, $\zeta=\xi_1+i\xi_2$ map conformally the domain $D$ onto the upper plane $\operatorname{Im}\zeta>0$. By $a_j$ we denote the image of the point $A_j$, i.e., $a_j=(\omega(A_j))^{-1}$, $j=\overline{1,6}$, and assume that $a_6=1$, $a_5=-1$. Moreover, we assume that the midpoint of the arc $A_2A_3$ transfers into $\zeta=\infty$.

Since the domain $D$ is symmetric with respect to the $ox_2$-axis, we can assume that $-a_2=a_3$, $-a_1=a_4$.

Introducing the vectors

$$
\begin{aligned}
\Phi_0(\zeta)&=2\Big[\frac{1}{2}\,K^0\omega(\zeta)e^{-\frac{\pi}{4}i}+2\mu e^{\frac{\pi}{4}i}\psi(\omega(\zeta))\Big],\quad \operatorname{Im}\zeta>0,\\
F(\zeta)&=2i\Big[\frac{1}{2}\,K^0\omega(\zeta)e^{-\frac{\pi}{4}i}-2\mu e^{\frac{\pi}{4}i}\psi(\omega(\zeta))\Big],\quad \operatorname{Im}\zeta>0,
\end{aligned}
\tag{43}
$$

the boundary conditions of the problems (41) and (42) are reduced to the Keldysh-Sedov problem

$$
\begin{aligned}
&\operatorname{Re}\Phi_0(\xi_1)=\begin{cases}\dfrac{1}{2}\,p, & \xi_1\in(-\infty,-a_2)\cup(a_1,\infty),\\ -\dfrac{1}{2}\,p, & \xi_1\in(-a_1,1),\end{cases}\\
&\operatorname{Im}\Phi_0(\xi_1)=\begin{cases}-\dfrac{1}{2}\,(p-4aK^0), & \xi_1\in(-a_2,-a_1),\\ \dfrac{1}{2}\,p, & \xi_1\in(1,a_1),\end{cases}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&\operatorname{Re}F(\xi_1)=\begin{cases}-\dfrac{1}{2}\,p, & \xi_1\in(-\infty,-a_1)\cup(a_2,\infty),\\ \dfrac{1}{2}\,p, & \xi_1\in(-1,a_1),\end{cases}\\
&\operatorname{Im}F(\xi_1)=\begin{cases}\dfrac{1}{2}\,p, & \xi_1\in(-a_1,-1),\\ -\dfrac{1}{2}\,(p-4aK^0), & \xi_1\in(a_1,a_2).\end{cases}
\end{aligned}
\tag{45}
$$

The solution of the problems (44) and (45) is given in terms of the following formulas ([7])

$$
\Phi_0(\zeta)=\frac{\chi_1(\zeta)}{2\pi i}\,p\Big(\int\limits_{-\infty}^{-a_1}\frac{d\xi}{|\chi_1(\xi)|(\xi-\zeta)}-\int\limits_{-a_1}^{a_1}\frac{d\xi}{|\chi_1(\xi)|(\xi-\zeta)}+
$$

$$+\int_{a_1}^{\infty}\frac{d\xi}{|\chi_1(\xi)|(\xi-\zeta)}-\frac{4aK^0}{p}\int_{-a_2}^{-a_1}\frac{d\xi}{|\chi_1(\xi)|(\xi-\zeta)}+C\bigg), \tag{46}$$

$$F(\zeta)=\frac{\chi_2(\zeta)}{2\pi i}\,p\bigg(-\int_{-\infty}^{-a_1}\frac{d\xi}{|\chi_2(\xi)|(\xi-\zeta)}+\int_{-a_1}^{a_1}\frac{d\xi}{|\chi_2(\xi)|(\xi-\zeta)}-$$

$$-\int_{a_1}^{\infty}\frac{d\xi}{|\chi_2(\xi)|(\xi-\zeta)}-\frac{4aK^0}{p}\int_{-a_2}^{-a_1}\frac{d\xi}{|\chi_2(\xi)|(\xi-\zeta)}+C\bigg), \tag{47}$$

where

$$\chi_1(\zeta)=\sqrt{\frac{\zeta+a_2)(\zeta-1)}{(\zeta+a_1)(\zeta-a_1)}},\quad \chi_2(\zeta)=\sqrt{\frac{\zeta-a_2)(\zeta+1)}{(\zeta+a_1)(\zeta-a_1)}},\ \ \mathrm{Im}\,\zeta>0,$$
$$\chi_1(\infty)=\chi_2(\infty)=1.$$

It is not difficult to state that

$$\chi_1(\xi_1)=\begin{cases}|\chi_1(\xi_1)|, & \xi_1\in(-\infty,-a_2)\cup(-a_1,1)\cup(a_1,\infty),\\ -|\chi_1(\xi_1)|i, & \xi_1\in(-a_2,-a_1)\cup(1,a_1),\end{cases}$$

$$\chi_2(\xi_1)=\begin{cases}|\chi_2(\xi_1)|, & \xi_1\in(-\infty,-a_1)\cup(-1,a_1)\cup(a_2,\infty),\\ |\chi_2(\xi_1)|i, & \xi_1\in(-a_1,-1)\cup(a_1,a_2),\end{cases},$$

$$|\chi_1(\xi_1)|=|\chi_2(-\xi_1)|. \tag{48}$$

Since the functions $\chi_1(\xi_1)$ and $\chi_2(\xi_1)$ at the points $\xi_1=\pm a_1$ have singularity of order 1/2, in the neighborhood of the points $\xi_1=\pm a_1$, for $\Phi_0(\zeta)$ and $F(\zeta)$ to be bounded it is necessary and sufficient that the following conditions be fulfilled:

$$p\bigg(\int_{-\infty}^{-a_1}\frac{d\xi}{|\chi_1(\xi)|(\xi\pm a_1)}-\int_{-a_1}^{a_1}\frac{d\xi}{|\chi_1(\xi)|(\xi\pm a_1)}+\int_{a_1}^{\infty}\frac{d\xi}{|\chi_1(\xi)|(\xi\pm a_1)}\bigg)=$$

$$=4aK^0\int_{-a_2}^{-a_1}\frac{d\xi}{|\chi_1(\xi)|(\xi\pm a_1)}-C, \tag{49}$$

$$p\bigg(-\int_{-\infty}^{-a_1}\frac{d\xi}{|\chi_2(\xi)|(\xi\pm a_1)}+\int_{-a_1}^{a_1}\frac{d\xi}{|\chi_2(\xi)|(\xi\pm a_1)}-\int_{a_1}^{\infty}\frac{d\xi}{|\chi_2(\xi)|(\xi\pm a_1)}\bigg)=$$

$$=-4aK^0\int_{a_1}^{a_2}\frac{d\xi}{|\chi_2(\xi)|(\xi\pm a_1)}-C. \tag{50}$$

If in equation (50) we replace $\xi_1$ by $-\xi_1$ and take into account the relation (48), we obtain the condition which coincides with (49).

Note that the system of these two equations contains unknown parameters $a_1$, $a_2$, $c_1$, $c_2$, $K_1^0$ and $K_2^0$ ($c = (c_1, c_2)^\top$, $L^0 = (K_1^0, K_2^0)^\top$). Finally, excluding from the above system the vector $c = (c_1, c_2)^\top$, we obtain

$$4aK^0 \int\limits_{-a_2}^{-a_1} \frac{d\xi}{|\chi_1(\xi)|(\xi^2 - a_1^2)} = p\bigg( - \int\limits_{-a_1}^{a_1} \frac{d\xi}{|\chi_1(\xi)|(\xi^2 - a_1^2)} + \\ + \int\limits_{-\infty}^{-a_1} \frac{d\xi}{|\chi_1(\xi)|(\xi^2 - a_1^2)} + \int\limits_{a_1}^{\infty} \frac{d\xi}{|\chi_1(\xi)|(\xi^2 - a_1^2)}\bigg). \tag{51}$$

(51) is the equation containing unknown parameters $a_1$, $a_2$ and $K^0 = (K_1^0, K_2^0)$. Fixing $K^0 = (K_1^0, K_2^0)$, we obtain the sysetm of two nonlinear equations with respect to $a_1$ and $a_2$. But its solution is connected with great difficulties, but we have managed to find how to get the value of an unknown parameter in the interval $1 < a_1 < a_2 < +\infty$ and to find an unknown vector $K^0$.

Introduce the notation

$$L = \int\limits_{-a_2}^{-a_1} \frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1^2 - a_1^2)},$$

$$M = \int\limits_{-\infty}^{-a_2} \frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1^2 - a_1^2)} - \int\limits_{-a_1}^{a_1} \frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1^2 - a_1^2)} + \int\limits_{a_1}^{\infty} \frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1^2 - a_1^2)}.$$

Then (51) takes the form

$$4aK^0 = +\frac{M}{L}\,p, \quad \text{i.e.,} \quad K^0 = \frac{L+M}{4aL}\,p.$$

It is not difficult to show that $L > 0$, $M > 0$ and consequently, $K^0 \| P$, $C \| P$ and $|P|^2 - 4aL^0P < 0$. Moreover, the conditions $PK^0 > 0$ and $|p|^2 - 4aPK^0 < 0$ should be fulfilled.

Having found the vector functions $\Phi_0(\zeta)$ and $F(\zeta)$, using formulas (43), we can find $\omega(\zeta)$ and $\psi_0(\zeta)$, and hence the stressed state of the body and of an unknown part of the boundary of the doubly connected domain.

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