

TRUNCATION ERRORS OF THE LOGALITHMIC POTENTIAL OPERATORS

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ABSTRACT. In the present paper we consider the error estimate of the logalithmic potential operators. We are mainly interested in L^2 - $W^{s,2}$ estimates.

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1. INTRODUCTION

This paper concerns with the regularization of the logarithmic potential operator

$$If(x) := \int_{\Omega} f(y) \log|x-y| dy, f \in L^2(\Omega)$$

where Ω is a bounded domain in \mathbf{R}^n with $n \geq 2$. By regularization we mean the operator I_{δ} given by

$$I_{\delta}f(x) := \int_{\Omega} f(y) \log(|x-y| + \delta) dy, f \in L^2(\Omega)$$

for $\delta > 0$.

There is a vast amount of researches on If . The following is an example of the boundedness of I_{δ} . By the Hölder inequality we have the following.

Theorem 1.1. *There exists a constant $c = c_{\Omega} > 0$ such that*

$$\|I_{\delta}\|_2 \leq c \|f\|_2$$

for all $f \in L^2$. More precisely, c can be taken as

$$c = \int_{\Omega \ominus \Omega} |\log|z|| dz,$$

where $\Omega \ominus \Omega := \{x-y : x, y \in \Omega\}$.

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Recently it turned out to be numerically effective to adopt the regularization of the singular integral operators. In [2] we considered a regularization of the fractional integral operator

$$f \mapsto \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

in connection with the following formula which is interesting of its own right. Let $n = 2$. Then consider

$$\begin{aligned} y_1 &= f_1(x) = f_1(x_1, x_2) \\ y_2 &= f_2(x) = f_2(x_1, x_2) \end{aligned} \quad (1)$$

and the inverse mapping f^{-1} of f

$$\begin{aligned} x_1 &= (f^{-1})_1(y) = (f^{-1})_1(y_1, y_2) \\ x_2 &= (f^{-1})_2(y) = (f^{-1})_2(y_1, y_2), \end{aligned} \quad (2)$$

where f is an invertible function in C^1 whose Jacobian matrix never vanishes. Then, we can represent

$$(f^{-1})_1(y^*)(f^{-1})_2(y^*) \quad (3)$$

in terms of the direct mapping (1.5).

Proposition 1.2. [6] *For the mappings (1), (2) with (3), we obtain the representation, for any $y^* = (y_1^*, y_2^*) \in f(D)$,*

$$\begin{aligned} \begin{pmatrix} (f^{-1})_1(y^*) \\ (f^{-1})_2(y^*) \end{pmatrix} &= \frac{1}{2\pi} \oint_{\partial D} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d\text{Arctan} \frac{f_2(x) - y_2^*}{f_1(x) - y_1^*} \\ &\quad - \frac{1}{2\pi} \int_D \frac{1}{|f(x) - y^*|^2} \text{adj}J(x) \begin{pmatrix} f_1(x) - y_1^* \\ f_2(x) - y_2^* \end{pmatrix} dx_1 dx_2. \end{aligned}$$

In [1] this type of regularization was considered from the viewpoint of numerical analysis. They considered the L^∞ -estimates.

In the present paper we shall investigate L^2 - $W^{s,2}$ estimates with $s > \frac{n}{2}$. We are interested in the results on Hilbert spaces because the above formula was obtained in connection with the Hilbert spaces.

2. BOUNDEDNESS OF I_δ , $\delta > 0$

Let us investigate the boundedness of I_δ with $0 \leq \delta \leq 1$.

Below, taking into account the fact that Ω is bounded, we shall investigate the $L^2(\mathbf{R}^n)$ - $W^{m,2}(\mathbf{R}^n)$ boundedness of

$$I_\delta^* f(x) := \int_{\mathbf{R}^n} \varphi(x-y) \log(|x-y| + \delta) \cdot f(y) dy,$$

where $\varphi \in \mathcal{S}$ is a smooth function that equals 1 near the origin and vanishes outside a compact set. Now that we have dealt with the L^2 boundedness, it is convenient to consider the derivative

$$\frac{\partial}{\partial x_j} I^* f(x) = \int_{\mathbf{R}^n} \frac{\partial}{\partial x_j} [\varphi(x-y) \log|x-y| + \delta] \cdot f(y) dy.$$

In view of the fact that

$$f \in L^2 \mapsto \int_{\mathbf{R}^n} \partial_{j,x} \varphi(x-y) \cdot \log(|x-y| + \delta) \cdot f(y) dy \in L^2$$

is just a smoothing operator, it is sufficient to deal with

$$T_{j,\delta} f(x) : \int_{\mathbf{R}^n} \varphi(x-y) \frac{x_j - y_j}{|x-y|(|x-y| + \delta)} f(y) dy.$$

Thus, our concern goes to the convolution operator whose integral kernel is given by

$$K_{j,\delta}(x) := \frac{x_j \varphi(x)}{|x|(|x| + \delta)}.$$

Our next claim is the following.

Proposition 2.1. $K_{j,\delta}$, $j = 1, 2, \dots, n$, $0 \leq \delta \leq 1$ is contained in a bounded set in $B_{1\infty}^{n-1}$.

Proof. We decompose

$$K_{j,\delta}(x) = \sum_{\nu=1}^{\infty} \frac{x_j (\varphi(2^{-\nu+1}x) - \varphi(2^{-\nu}x))}{|x|(|x| + \delta)} = \sum_{\nu=1}^{\infty} K_{j,\delta,\nu}(x).$$

It is not hard to see that

$$|\partial^\alpha K_{j,\delta,\nu}(x)| \leq c_\alpha 2^{\nu-|\alpha|}.$$

Thus, $K_{j,\delta,\nu}$ can be seen as an atom (for $B_{1\infty}^{n-1}$) centered at $Q_{\nu 0}$. The assumption that $n \geq 2$ allows us not to postulate the moment condition on atoms. For details of the atomic decomposition we refer to [4]. As a result, we obtain

$$\|K_{j,\delta} : B_{1\infty}^{n-1}\| \leq c$$

with $c > 0$ independent of δ . Thus, the proof is now complete. □

The following is a slight but useful extension of the Riemann-Lebesgue theorem.

Proposition 2.2. *There exists a constant $c > 0$ such that*

$$\|\mathcal{F}f\|_\infty \leq c \|f : B_{1\infty}^0\|$$

for all $f \in B_{1\infty}^0$.

For the proof we refer to [3, Theorem 2.4]. This proposition gives

$$\sup_{0 \leq \delta \leq 1} \|\mathcal{F}[\partial^\alpha K_{j,\delta}]\|_\infty < \infty$$

for all α with length less than $n - 1$.

If we invoke the Plancherel theorem, we obtain the following.

Theorem 2.3. *Let $n \geq 2$. Then*

$$\|I_\delta^* f : W^{2,n}\| \leq c \|f\|_2$$

for all $f \in L^2$.

3. TRUNCATION ERROR

Now let us consider the truncation error. If $f \in \mathcal{S}$, then we have

$$I^* f(x) - I_\delta^* f(x) = \int_{\mathbf{R}^n} \left(\int_0^\delta \frac{\varphi(x-y) dt}{|x-y|+t} \right) f(y) dx - \int_0^\delta \int_{\mathbf{R}^n} \left(\frac{\varphi(x-y)f(y) dx}{|x-y|+t} \right) dt.$$

Thus, we are in the position of applying what we have obtained. One of the key observations is that we have only to pass to the first derivative when we consider the truncation error.

Theorem 3.1. *Let $0 \leq \delta \leq 1$. Then we have*

$$\|(I^* - I_\delta^*)f : W^{n-1,2}\| \leq c \delta \|f\|_2$$

for all $f \in L^2$.

4. APPLICATIONS

If $n \geq 3$, then Theorem 3.1 is sufficient for our present purpose because $W^{n-1,2} \subset BUC$. However, if $n = 2$, the embedding $W^{n-1,2} \subset BUC$ fails. This is a fact known as the Gilberg-Trudinger theorem. However, if we sacrifice the truncation estimate somehow, we can obtain a nice estimate for $n = 2$.

Theorem 4.1. *Suppose that $0 < \varepsilon < 1$. Then we have*

$$\|(I^* - I_\delta^*)f : W^{1+\varepsilon,2}\| \leq c \delta^{1-\varepsilon} \|f\|_2.$$

$W^{1+\varepsilon}$ being embedded into BUC , we can say that I_δ approximates I very well.

Proof. Just interpolate

$$\|(I^* - I_\delta^*)f : W^{1,2}\| \leq c \delta \|f\|_2$$

$$\|(I^* - I_\delta^*)f : W^{2,2}\| \leq c \|f\|_2,$$

which are obtained in Theorems 3.1 and 2.3 respectively. \square

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