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THE PROBLEM OF FINDING OPTIMAL HOLES IN AN ELASTIC SQUARE

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ABSTRACT. An elastic square weakened by four unknown equal holes is considered. The normal displacement and tangential stress on the outer boundary of the square are equal to zero, and the boundary of unknown holes is under the action of normal constant stresses. Unknown hole boundaries are found for the condition that the tangential normal stress on those boundaries takes one and the same constant value.

რეზიუმე. განხილულია ოთხი უცნობი ტოლი სიდიდის ხვრელით შესუსტებული დრეკადი კვადრატი, რომლის გარე საზღვარზე ნორმალური გადაადგილება და მხები ძაბვა ნულის ტოლია, ხოლო უცნობი ხვრელების საზღვარზე მოქმედებს მუდმივი სიდიდის ნორმალური ძაბვა.

მოძებნილია ხვრელების უცნობი საზღვარი იმ პირობით, რომ მასზე ტანგენციალური ნორმალური ძაბვა ღებულობდეს ერთი და იგივე მუდმივ მნიშვნელობას.

The axisymmetric problems of the plane theory of elasticity and bending of a plate with a partially unknown boundary have been studied in [1,2,3,4 and 5]. In the present paper we consider the problem of elastic equilibrium for a stationary rigid body from which a square is cut out and an elastic square is imbedded. Before deformation, the contour of the latter coincided with that of the square cut out from the rigid body. The elastic square is weakened by four unknown, coinciding under the axial symmetry, equal holes which intersect the square diagonals and are symmetric with respect to these diagonals and to the segments connecting the middle points of opposite square sides. Since the problem is axisymmetric, we consider the shaded part of the square, i.e., the curvilinear pentagon $A_1A_2A_3A_4A_5$ (Fig. 1).

Introduce the notation: $A_kA_{k+1} = \Gamma_k$ $(k = 1, 2, 3)$, $\Gamma_4 = A_5A_1$, $\Gamma = \bigcup\limits_{k=1}^{4}\Gamma_k$, the arc A_4A_5 we denote by Γ_5 and the domain occupied by the curvilinear pentagon by S.

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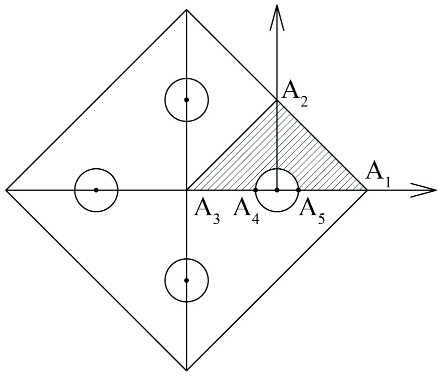

Fig. 1

Assume that the boundaries of unknown holes are under the action of equal normal stresses. Suppose also that the body surfaces are absolutely smooth and hence the friction forces will be neglected.

The problem is formulated as follows: Find unknown holes and the stressed state of the body for the condition that the normal tangential stress on the hole boundaries takes constant value, i.e., $\sigma_s = k = \text{const}$. On the given part of the square boundary the tangential stress $\tau_{ns} = v_n = 0$, and on an unknown part of the body $\sigma_n = p$, $\tau_{ns} = 0$.

Using the formulas of Kolosov-Muskhelishvili [6], the boundary conditions can be written as follows:

$$\operatorname{Re} e^{-i\alpha(t)}\big(\varkappa\,\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)}\,\big) = 2\mu\, u_n = 0, \quad t \in \Gamma, \tag{1}$$

$$\operatorname{Re} e^{-i\alpha(t)}\big(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\,\big) = b(t), \quad t \in \Gamma, \tag{2}$$

$$\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} = p\,t + c_0, \quad t \in \Gamma_5, \tag{3}$$

$$\operatorname{Re} \varphi'(t) = \sigma_n + \sigma_s = \frac{k+p}{4}\,, \quad t \in \Gamma_5, \tag{4}$$

where $\mu, \varkappa$ are the elastic constants, $\alpha(t)$ is the angle made by the normal and the Ox-axis, c_0 is an arbitrary constant,

$$b(t) = \int\limits_{A_1}^{t} \sigma_n(t_0) \sin\big(\alpha(t) - \alpha(t_0)\big)\, ds_0, \quad t \in \Gamma.$$

Obviously, $\alpha(t)$ and $b(t)$ are the piecewise constant functions.

Adding the boundary conditions (1) and (2), differentiating the obtained relation and taking into account that $\alpha(t)$ and $b(t)$ are the piecewise constant functions, we obtain

$$\operatorname{Im} \varphi'(t) = 0, \quad t \in \Gamma. \tag{5}$$

The conditions (4) and (5) is the Kolosov-Muskhelishvili's problem for the domain S:

$$\begin{aligned} &\operatorname{Re}\,\varphi'(t) = \frac{p+k}{4}, \quad t \in \Gamma_5, \\ &\operatorname{Im}\,\varphi'(t) = 0, \qquad\quad t \in \Gamma, \end{aligned} \tag{6}$$

which in our case has the unique solution

$$\varphi(z) = \frac{p+k}{4}\, z + \ell, \tag{7}$$

where ℓ is an arbitrary constant.

Using formula (7), the boundary conditions (1), (3) take the form

$$\operatorname{Re}\, e^{-i\alpha(t)}\big((\varkappa-1)mt - \overline{\psi(t)}\big) = -\varkappa \operatorname{Re}\big(e^{-i\alpha(t)}\cdot \ell\big), \ \ t \in \Gamma, \tag{8}$$

$$(2m-p)t + \overline{\psi(t)} + c = 0, \quad t \in \Gamma_5, \tag{9}$$

where $m = \frac{k+p}{4}$, $c = -c_0 + \ell$ is the constant.

If $t \in \Gamma_k$ $(k = \overline{1,4})$, then

$$\operatorname{Re}\big(e^{-i\alpha_k}\cdot t\big) = \operatorname{Re}\big(e^{-i\alpha_k}\cdot A_k\big), \ \ t \in \Gamma_k, \ \ k = \overline{1,4}, \tag{10}$$

$$\alpha_1 = \frac{\pi}{4}\,, \ \ \alpha_2 = \frac{3\pi}{4}\,, \ \ \alpha_3 = \alpha_4 = \frac{3\pi}{2}\,. \tag{11}$$

The relations (8) and (10) yield

$$\begin{aligned} \operatorname{Re} e^{-i\alpha_k}\overline{\psi(t)} &= (\varkappa-1)m \operatorname{Re}\big(e^{-i\alpha_k}\cdot A_k\big) - \\ &- \varkappa \operatorname{Re}\big(e^{-i\alpha_k}\cdot \ell\big), \ \ t \in \Gamma_k, \ \ k = \overline{1,4}. \end{aligned} \tag{12}$$

Thus our problem is reduced to the boundary value problem (9), (10), (12).

By virtue of the formula

$$z = \omega(\zeta) \tag{13}$$

we map conformally the domain S onto the semi-circle $|\zeta| < 1$, $\operatorname{Im}\zeta > 0$ of unit radius, of a complex variable ζ. By means of the above transformation the arc A_4A_5 turns into the diameter $\zeta \in (-1,1)$; $A_4 \to a_4 = -1$, $A_5 \to a_5 = 1$, $A_2 \to a_2 = i$. A_4, A_5, A_2 are the fixed points.

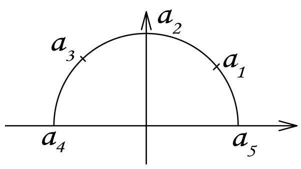

Fig. 2

We map two points A_1 and A_3 onto the unknown points a_1 and a_3 (Fig. 2).

Transformation of formula (13) allows us to write the boundary conditions (9), (10), (12) in the form

$$(2m-p)\,\omega(\sigma) + \overline{\psi_0(\sigma)} = c, \ \ \sigma \in (-1,1), \tag{14}$$

$$\begin{aligned} \operatorname{Re}\big(e^{-i\alpha_k}\,\overline{\psi_0(\sigma)}\big) &= (\varkappa-1)\,m \operatorname{Re}\big(e^{-i\alpha_k}\cdot A_k\big) - \\ &- \varkappa \operatorname{Re}\big(e^{-i\alpha_k}\cdot \ell\big), \ \ \sigma \in \gamma, \end{aligned} \tag{15}$$

$$\operatorname{Re}\big(e^{-i\alpha_k}\,\omega(\sigma)\big) = \operatorname{Re}\big(e^{-i\alpha_k}\cdot A_k\big),\quad \sigma\in\gamma, \tag{16}$$

where $\psi_0(\zeta)=\psi(\omega(\zeta))$, γ is the mapping of Γ, i.e., is the semi-circle.

Consider the function

$$W(\zeta) = \begin{cases} (2m-p)\,\omega(\zeta)-c, & \operatorname{Im}\zeta>0,\ \ |\zeta|<1,\\ -\overline{\psi_0(\overline{\zeta})}, & \operatorname{Im}\zeta<0,\ \ |\zeta|<1. \end{cases} \tag{17}$$

and show that the function $W(\zeta)$ is analytic in the circle $|\zeta|<1$. Its boundary values on the diameter $(-1,1)$ have the form

$$W^+(\xi) = (2m-p)\,\omega(\xi)-c,\quad \xi\in(-1.1),$$
$$W^-(\xi) = -\overline{\psi_0(\xi)},\qquad \xi\in(-1.1).$$

From the above formulas we find that

$$W^+(\xi)-W^-(\xi) = (2m-p)\,\omega(\xi)+\overline{\psi_0(\xi)}-c,$$

and using (14), we obtain

$$W^+(\xi)=W^-(\xi),$$

i.e., the function $W(\zeta)$ is analytic in the circle $|\zeta|<1$.

Applying now the relation (17), we can rewrite the boundary conditions (15) and (16) as follows:

$$\begin{aligned} \operatorname{Re}\big(e^{-i\alpha(\sigma)}W(\sigma)\big) &= (2m-p)\ \operatorname{Re}\big(e^{-i\alpha(\sigma)}A(\sigma)\big)-\\ &\quad -\operatorname{Re}\big(e^{-i\alpha(\sigma)}c\big),\quad \sigma\in\gamma,\\ \operatorname{Re}\big(e^{-i\alpha(\sigma)}W(\sigma)\big) &= -(\varkappa-1)m\ \operatorname{Re}\big(e^{-i\alpha(\sigma)}A(\sigma)\big)-\\ &\quad -\varkappa\ \operatorname{Re}\big(e^{-i\alpha(\sigma)}\ell\big),\quad \sigma\in\gamma_0, \end{aligned} \tag{18}$$

where γ_0 is the mirror image of γ with respect to the diameter $(-1,1)$.

The boundary conditions (18) is the Riemann-Hilbert problem for a circle which in an expanded form looks as

$$W(\sigma)+e^{2i\alpha(\sigma)}\overline{W(\sigma)} = \begin{cases} 2(2m-p)\ \operatorname{Re}\big(e^{-i\alpha(\sigma)}A(\sigma)\big)\cdot e^{i\alpha(\sigma)}-\\ \qquad -\operatorname{Re}\big(e^{-i\alpha(\sigma)}c\big),\quad \sigma\in\gamma,\\ -2(\varkappa-1)m\ \operatorname{Re}\big(e^{-i\alpha(\sigma)}A(\sigma)\big)\cdot e^{i\alpha(\sigma)}-\\ \qquad -\varkappa\ \operatorname{Re}\big(e^{-i\alpha(\sigma)}\ell\big),\quad \sigma\in\gamma_0. \end{cases} \tag{19}$$

A solution of the problem can be represented in the form [8]

$$W(\zeta)=\frac{X(\zeta)}{4\pi i}\int\limits_{\gamma\cup\gamma_0}\frac{\zeta+\sigma}{\sigma-\zeta}\cdot\frac{f(\sigma)\,e^{i\alpha(\sigma)}}{X(\sigma)\,\sigma}\,d\sigma, \tag{20}$$

where $f(\sigma)$ is the right-hand side of the relation (19),

$$X(\zeta)=\frac{1}{4\pi i}\int\limits_{\gamma+\gamma_0}\frac{\zeta+\sigma}{\sigma-\zeta}\cdot\frac{2\alpha\, i}{\sigma}\,d\sigma,\quad |\zeta|<1. \tag{21}$$

In our case

$$\begin{aligned}
&\alpha(\sigma) = \frac{\pi}{4}, \quad f(\sigma) = \sqrt{2}\,d + 2\,\mathrm{Re}\left(e^{i\frac{\pi}{4}} \cdot c\right), \quad \sigma \in a_1a_2,\\
&\alpha(\sigma) = \frac{3\pi}{4}, \quad f(\sigma) = \sqrt{2}\,d - 2\,\mathrm{Im}\left(e^{i\frac{\pi}{4}} \cdot c\right), \quad \sigma \in a_2a_3,\\
&\alpha(\sigma) = \frac{3\pi}{2}, \quad f(\sigma) = \mathrm{Im}\ c, \quad \sigma \in a_3a_4 \cup a_5a_1,\\
&\alpha(\sigma) = \frac{3\pi}{2}, \quad f(\sigma) = -2\varkappa\ \mathrm{Im}\,\ell, \quad \sigma \in \overline{a}_4\,\overline{a}_3 \cup \overline{a}_1\,\overline{a}_5, \qquad (22)\\
&\alpha(\sigma) = \frac{3\pi}{4}, \quad f(\sigma) = -(\varkappa-1)\sqrt{2}\,d - \varkappa\,\mathrm{Im}\,\ell, \quad \sigma \in \overline{a}_3\,\overline{a}_2,\\
&\alpha(\sigma) = \frac{\pi}{4}, \quad f(\sigma) = -(\varkappa-1)\sqrt{2}\,d - \varkappa\,\mathrm{Re}\,\ell, \quad \sigma \in \overline{a}_2\,\overline{a}_1,\\
&\alpha(\sigma) = \frac{3\pi}{2}, \quad f(\sigma) = -\varkappa\ \mathrm{Im}\,\ell, \quad \sigma \in \overline{a}_1\,\overline{a}_5,
\end{aligned}$$

where d is the diagonal of the square.

Taking in formula (21) into account the values $\alpha(\sigma)$ appearing in formulas (22), after calculations we obtain

$$\begin{aligned}
X(\zeta) &= \sqrt[4]{\frac{\zeta-a}{\zeta-a_1}\cdot\left(\frac{\zeta-a_3}{\zeta-a_2}\right)^3\left(\frac{\zeta-\overline{a}_3}{\zeta-a_3}\right)^2\left(\frac{\zeta-\overline{a}_2}{\zeta-\overline{a}_3}\right)^3\frac{\zeta-\overline{a}_1}{\zeta-\overline{a}_2}\left(\frac{\zeta-a_1}{\zeta-\overline{a}_1}\right)^2}\cdot\\
&\quad\cdot e^{-\frac{1}{2\pi i}\int\limits_{\gamma+\gamma_0}\frac{\alpha(\sigma)\,d\sigma}{\sigma}}, \qquad (23)\\
X(\sigma) &= \left|X(\sigma)\right| e^{i\alpha}. \qquad (24)
\end{aligned}$$

Substituting in formula (20) the value $X(\zeta)$ appearing in (23), the value $X(\sigma)$ appearing in (24), and the value $f(\sigma)$ appearing in (22), we obtain the solution of our problem which is unbounded at the points $\zeta = a_2$, $\zeta = \overline{a}_3$, $\zeta = \overline{a}_1$. For the solution to be bounded, it is necessary and sufficient that the conditions

$$\begin{aligned}
&\int\limits_{\gamma+\gamma_0} \frac{a_2+\sigma}{\sigma-a_2}\cdot\frac{f(\sigma)}{\sigma|X(\sigma)|}\,d\sigma = 0,\\
&\int\limits_{\gamma+\gamma_0} \frac{\overline{a}_3+\sigma}{\sigma-\overline{a}_3}\cdot\frac{f(\sigma)}{\sigma|X(\sigma)|}\,d\sigma = 0,\\
&\int\limits_{\gamma+\gamma_0} \frac{\overline{a}_1+\sigma}{\sigma-\overline{a}_1}\cdot\frac{f(\sigma)}{\sigma|X(\sigma)|}\,d\sigma = 0.
\end{aligned}$$

be fulfilled.

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