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SOLVING GAMES BY DIFFERENTIAL EQUATIONS IN FINITE TIME

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Abstract. New method to solve symmetric matrix games by differential equations is described. It allows us to find out optimal mixed strategy in finite time, whilst the ODE-based method, belonging to Brown and von Neumann, in general case consideres ODE on $[0,\infty)$ and gives only the value of game.

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1. Introduction

Symmetric matrix game can be considered as one of the forms of general linear programming (LP) problem. Reduction a canonical problem of LP to a symmetric matrix games and vice versa can be performed easily. The first methods of solving these problems, including the well-known simplex algorithm were created by the researchers in the 50s (see [1], [2], [3]). The above mentioned simplex algorithm is of high theoretical and practical importance, since LP is widely used in resource distribution, production planning, portfolio investment, military strategy and many other fields. It is assumed that applying LP in economics, the costs decreased by 20%. This fact made to believe the importance of using mathematical methods those managers and producers which were earlier based on experience and intuition in their actions. Nowadays, it is an important tool in algorithm design. Many problems in integer programming, graph theory and other fields can be efficiently solved with LP.

At present, LP problems of large dimensions can be solved due to two main reasons: both the computing power and the theory of algorithms in

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LP are in progress. It is widely accepted, that the latter one is the most important. One can also see the actuality of this topic on many commercial LP software packages on the market of program products.

The idea of using differential equation to solve symmetric matrix games belongs to Brown and von Neumann (see [3]).

In the present paper a new algorithm based on the application of ordinary differential equations is considered. Except of the equations offered in [3], we use a well known scheme of using differential equations in the numerical methods of optimization (see [4]). Below we prove convergence of the offered algorithm and give the results of some tests of C-program constructed with the aim to realize the algorithm in practice. The first results obtained show that it is possible to abruptly improve this method.

Further, we shall consider everywhere a symmetrical matrix game with skew-symmetrical (i.e. $a_i^j = -a_j^i$) matrix $A = \left(a_i^j\right)_{i,j=1}^n$. We denote mixed strategies of player 2 by $y = (y^1, \dots, y^n)$, and a set of mixed strategies by Y.

2. Idea of the Brown-Von Neumann'S Method and its Some Characteristics

According to the Brown-von Neumann's method a mixed strategy $y(t)$ of player 2 is a function satisfying the system of differential equations:

$$\dot{y}^j = \varphi\left(u_j\left(t\right)\right) - \varphi\left(y\left(t\right)\right) y^j\left(t\right), \quad j \in \{1, \dots, n\}, \tag{1}$$

with the initial condition $y\left(0\right) = y_0$, where y_0 is an arbitrary mixed strategy,

$$u_i\left(y\left(t\right)\right) = \sum_{j=1}^n a_i^j y^j\left(t\right), \qquad \varphi\left(u_i\right) = \max\left\{0, u_i\right\}, \qquad \varphi\left(y\right) = \sum_{i=1}^n u_i. \tag{2}$$

About (1) it is sufficient for us to know the following: the solution is definite on the $[0, \infty)$ interval; as $t \to \infty$, an arbitrary limiting point of $y(t)$ is the optimal mixed strategy; the estimation takes place,

$$\varphi\left(y\left(t\right)\right) \leq \frac{C_1}{C_2 + t} \tag{3}$$

where C_1, C_2 are the constants depending on the problem data.

The mentioned method has the following characteristics:

1. If a solution is not unique, the method determines a price of game only;
2. Obtaining of game price demands an infinite time interval $[0, \infty)$;
3. Solving differential equations on $[0, \infty)$ are followed by some specific difficulties.

Because of the noted characteristics it is assumed only that application of differential equations in the matrix games is of interest for the theoretical point of view.

3. Idea of the Presented Method and its Some Characteristics.

We construct a system of differential equations

$$\dot{y} = v\left(y\right). \tag{4}$$

as a partial alternative to the system (1). It has a simple structure, is easily solvable and its trajectory is a piecewise linear function defined in the finite time interval.

When a field of ODE (4) is defined globally on the set of mixed strategies, then the end of the trajectory is an optimal mixed strategy.

For most games a field of ODE (4) is not defined for every mixed strategy, therefore, when a trajectory of ODE (4) goes into the deadlock (i.e. the end of the trajectory is a mixed strategy for which a field is not defined), to disturb the deadlock we have to apply the system (1). Then we turn again to solving ODE (4) and so on in turns until we go to that deadlock which is the optimal mixed strategy. So, in the general case it is necessary to use both systems. Applying them alternatively, we construct mixed strategy $y(t)$ depended on time and defined on the finite interval. The value of $y(t)$ at the last moment is the optimal mixed strategy, i.e. the solution.

Thus, it is possible to solve a matrix game in the finite time (not only to find its value) by using differential equations.

According to this approach, the first two shortcomings of Brown-von Neumann's method diminish completely, and the third one is weakened significantly. It should be noted that this method is improved not only theoretically. The C-Program constructed by us presents the first soft attempt of programming implementation of the method. It works effectively and safely for randomly chosen data when a dimension of matrix is of several tens order.

4. Geometrical Interpretation of the Method.

We call the value $\max_i \{u_i\left(y\right)\}$ (see designations (2)) the best response (of player 1) against mixed strategy y. Denote by $I\left(y\right)$ a set of indices giving the best response against y. If y is optimal, then the best response against it is 0. It is evident that the best response presents a function from a set of mixed strategies in R.

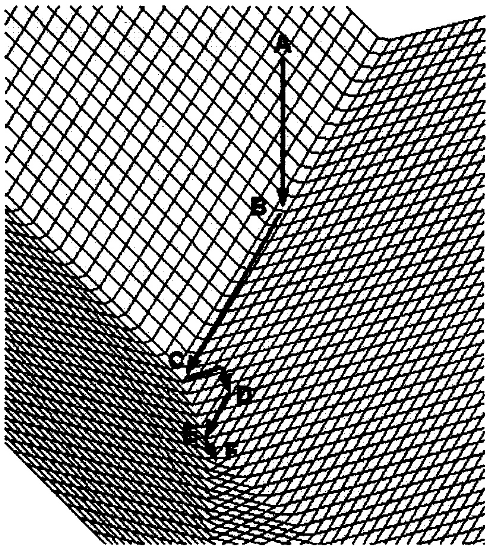

Fig. 1

Let us suppose that a graph fragment of the best response mapping has the form given in the Fig. 1 and $A = \left(y_0, \max_i \{u_i\,(y_0)\}\right)$. In this case the best response will be reached for one index, and solving system (4) with the initial value y_0 means to move from y_0 to the antigradient direction until reaching the point B on the graph; here the best response will be reached for two indices, therefore from this moment the segment BC of the graph of the best response corresponds to the solution of (4). The field of the system (4) is not defined at the point C because it is impossible to define a direction of the steepest descent for all responding surfaces simultaneously by the method described below. Starting from the time moment s corresponding to the point C, we solve the differential equation of Brown-von Neumann with the initial walue $y(s)$ until reaching a point s_1 such that $y(s_1)$ will not be a deadlock point and the best response against $y(s_1)$ (the value of function at the point D) is less than that at the point C. On Fig. 1, CD is not a segment yet. It corresponds to the solution of differential equation and can be a curve of a rather complex form. From the point D we can again define the trajectory of the system (4). For the strategy corresponding to the point F on the graph, the field is not defined, so from this moment we return to the Brown-von Neumann's equation, and so on until the best response becomes 0.

5. Definition of a Differential Equation. Construction of Solution

Let us denote a set of k-dimensional Boolean vector-rows by B_k. If $b_1 \in B_k$ and $b_2 \in B_l$ then (b_1, b_2) is a vector obtained by concatenation of these vectors and belongs to B_{k+l} .

Let us divide the set Y in disjoint sets using elements of a space B_{2n}. $Y_{(b_1,b_2)} \subset Y$ corresponds to each vector $(b_1, b_2) \in B_{2n} \quad (b_1, b_2 \in B_n)$ in the following way: $y \in Y_{(b_1,b_2)}$ if

1. $i \in I(y) \ \Leftrightarrow \ b_1^i = 1$,
2. $y^j > 0 \ \Leftrightarrow \ b_2^j = 1$.

In other words, a component of the vector $b_1 = \left(b_1^1, \ldots, b_1^n\right)$ equals to 1 if and only if its index is a number of a row giving the best response; a component of the vector $b_2 = \left(b_2^1, \ldots, b_2^n\right)$ equals to 1 if and only if its index is an index of some positive component of the vector $y \in Y_{(b_1,b_2)}$.

It is clear that some $Y_{(b_1,b_2)} = \emptyset$, but in the general case

$$\left(\tilde{b}_1, \tilde{b}_2\right) \neq (b_1, b_2) \quad \Rightarrow \quad Y_{\left(\tilde{b}_1, \tilde{b}_2\right)} \cap Y_{(b_1,b_2)} = \emptyset.$$

Afterwards, an arbitrarily given vector $h = \left(h^1, \ldots, h^n\right)$ on the surface of the unit sphere in R_n will be called the direction. Consider a minimization problem for each $y \in Y$:

$$\begin{cases} a_{i_1} h^T \to \min, \\ a_{i_1} h^T = a_{i_2} h^T, \\ \quad . \quad . \quad . \\ a_{i_1} h^T = a_{i_k} h^T, \\ h h^T = 1, \\ \sum_j h^j = 0, \\ \left(y^j = 0\right) \Rightarrow \left(h^j = 0\right), \end{cases} \tag{5}$$

where $I(y) = \{i_1, \ldots, i_k\}$. The following result is obvious.

Proposition 5.1. *The same minimization problem of the kind* (5) *corresponds to each point of an arbitrary taken non-empty* $Y_{(b_1,b_2)}$.

Because B_{2n} is a finite set, we have also

Corollary 5.2. *Number of minimization problems of the form* (5) *is finite.*

Definition 5.3. If problem (5) corresponding to $y_0 \in Y_{(b_1,b_2)} \neq \emptyset$ has a unique solution, say $\tilde{h}$, then for each $y \in Y_{(b_1,b_2)}$ we take $v(y) = \tilde{h}$. Otherwise, we suppose that the field is not definite, $Y_{(b_1,b_2)}$ is a deadlock and any $y \in Y_{(b_1,b_2)}$ is a deadlock point.

If admissible set in the minimization problem (5) is nonempty, then by virtue of Weierstrass's theorem this problem has a solution.

Corollary 5.4. *The range of the field v of system* (4) *is a finite set.*

Corollary 5.5. *Number of deadlocks is finite.*

Theorem 5.6. *If a mixed strategy $y_0 \in Y$ is not a deadlock point, then in the sufficiently small neighborhood of a point $t = 0$:*

- *Function $y(t) = y_0 + tv(y_0)$ presents the solution of system* (4) *with the initial condition $y(0) = y_0$;*
- *The best response $\max_i \{u_i(y(t))\}$ is a decreasing function of a variable t.*

Proof. Let $y_0 \in Y_{(b_1,b_2)}$. Let us take arbitrarily such indices i_1 and i that $b_1^{i_1} = 1, \quad b_1^i = 0$. Then

$$a_{i_1} y_0^T \equiv u_{i_1}(y_0) = \max_k \{u_k(y_0)\},$$

$$(a_{i_1} - a_i) y_0^T > 0.$$

Because

$$t \mapsto (a_{i_1} - a_i)(y_0 + tv(y_0))^T$$

is continuous, therefore it preserves the same sign in the sufficiently small neighborhood of the point $t = 0$, i.e.

$$I(y_0) = I(y_0 + tv(y_0))$$

in the same neighborhood. If $b_2^j = 0$ then $(y_0 + tv(y_0))^j = 0, \quad \forall t$. If $b_2^j = 1$, then $(y_0 + tv(y_0))^j > 0$ for sufficiently small t.

Since the whole number of these functions is finite, therefore there always is a common time interval in which all functions preserve signs. Hence, in the same interval

$$y_0 + tv(y_0) \in Y_{(b_1,b_2)},$$

i.e.

$$v(y_0 + tv(y_0)) = v(y_0).$$

The first part of the Theorem is proved, because $\frac{d}{dt}(y_0 + tv(y_0)) = v(y_0)$.

Since a solution of (5) is unique for given $y_0 \in Y$ therefore $a_{i_1}(v(y_0))^T < 0$ that completes the proof. □

It follows directly from the proved theorem that one of the deadlock points is the optimal mixed strategy itself. Otherwise it will be possible to lessen the best response, changing optimal strategy in the direction of solution of the system (5).

It is easy to determine the length of that maximal segment on which a solution of system (4) with the initial condition $y(0) = y_0$ is a linear function. For this it is sufficient to find out two moments of time, - when at moving from y_0 in direction $v(y_0)$ a set $I\left(y_0 + tv\left(y_0\right)\right)$ will get one more index and when one more component of $y_0 + tv\left(y_0\right)$ becomes 0. While the first (minimal) of them is not reached, a field of differential equation is constant and, respectively, the solution remains a linear.

Denote

$$\alpha_1 = \begin{cases} \min\left\{ -\frac{(a_{i_1} - a_i) y_0^T}{(a_{i_1} - a_i)[v(y_0)]^T} \middle| i \notin I\left(y\right),\ \left(a_{i_1} - a_i\right)\left[v\left(y_0\right)\right]^T < 0 \right\}, \\ \infty, \text{if there are no indices with the properties } i \notin I\left(y\right), \\ \quad \left(a_{i_1} - a_i\right)\left[v\left(y_0\right)\right]^T < 0, \end{cases} \tag{6}$$

$$\alpha_2 = \begin{cases} \min\left\{ -\frac{y_0^j}{[v(y_0)]^j} \middle| \ j \in \{1, \ldots, n\}, \quad \left[v\left(y_0\right)\right]^j < 0 \right\}, \\ \infty, \text{if there are no indices with the property } \left[v\left(y_0\right)\right]^j < 0. \end{cases} \tag{7}$$

Now it is clear that the following is valid.

Proposition 5.7. *If a mixed strategy $y_0 \in Y$ is not a deadlock point and $\alpha = \min\{\alpha_1, \alpha_2\}$ then a linear function*

$$y_0 + tv\left(y_0\right)$$

is a solution of system (4) *with an initial condition $y(0) = y_0$ when $t \in [0, \alpha]$, but it is not a solution when $t > \alpha$.*

6. Description of Algorithm. Verification of Corectness.

Now, when we can solve both the (1) and (4) systems, let us describe the algorithm solving symmetrical matrix game using differential equations. For this let us bring a pseudocode of algorithm using standard statements whose sense is clear. In the beginning we assume $t_0 = 0$, y_0 is arbitrary.

```
1  while ( max_i {u_i(y_0)} > 0 ){
2          if( y_0 is a deadlock point) {
3                  while( max_i {u_i(y_0)} > max_i {u_i(y(t))} ){
4                          to solve system (1) with y(t_0) = y_0;
5                  }
6          }
7          else{
8                  while ( y(t) is not a deadlock point ){
9                          to solve system (4) with the y(t_0) = y_0;
10                 }
11         }
12         y_0 denotes the end of trajectory,
13         t_0 denotes the final moment of time;
```

According to the row 1 algorithm acts until the best response against y_0 becomes 0. In rows 2-6 the case is considered when y_0 is deadlock. Then the Brown-von Neumann system is being solved until the best response against the end of trajectory becomes smaller than against y_0. By virtue of the estimation (3), for arbitrary datas the rows 2-6 will be fulfilled in finite time. Rows 7-11 correspond to the case when y_0 is not a deadlock. Then we solve the (4) until the end of trajectory becomes a deadlock point. According to Corollary 5.4 the field v of system (4) has finite range. According to Proposition 5.7 the solution of (4) is piecewise linear function and according to construction different segments of the solution correspond to different $Y_{(b_1,b_2)}$. Number of such cases is finite. Thus, rows 5-7 will be also fulfilled in finite time.

Finally, let us show that the statement **while** of row (1) receives control finitly many times. The proof of this fact is based on the following

Proposition 6.1. *If $Y_{(b_1,b_2)}$ is a deadlock then the restriction of the best response function on $Y_{(b_1,b_2)}$ is a constant function.*

Proof. When $Y_{(b_1,b_2)}$ is one-pointed then the resume of the Proposition is obvious.

Let us suppose $Y_{(b_1,b_2)}$ is not one-pointed and the restriction of the best response function on the $Y_{(b_1,b_2)}$ is not a constant function. The same problem of kind (5) corresponds to each element of $Y_{(b_1,b_2)}$. The feasible set is compact and that is why, according to Weierstrass's theorem, there exists a global minimal in (5). Because $Y_{(b_1,b_2)}$ is deadlock, therefore the minimal is not unique. Hence system (5) has at least two different minima: $h_1 \neq h_2$. Since we assume that a restriction of the best response function on $Y_{(b_1,b_2)}$ is not a constant function, then we have

$$a_i h_1^T = a_i h_2^T < 0 \tag{8}$$

for $i \in I(y)$. Let i_0 be one of such indices. It is evident

$$\frac{1}{\|h_1 + h_2\|} (h_1 + h_2) \tag{9}$$

is also feasible vector. As $h_1 \neq h_2$, then $\|h_1 + h_2\| < 2$. Hence,

$$\frac{1}{\|h_1 + h_2\|} a_i (h_1 + h_2)^T < \frac{1}{2} 2 a_i h_1^T = a_i h_1^T.$$

That means the best response is smaller against the (9) than against the minimal, i.e. two minima cannot exist, a field of differential equation is definite and $Y_{(b_1,b_2)}$ is not deadlock. The obtained contradiction proves the Proposition. □

The statement **while** takes a parameter y_0 at each repetition and y_0 gets a new value at the end of the statement body. The algorithm is performed in such a manner that the best response is smaller against the changed y_0.

Without the loss of generality we assume that from the very start the statement **while** gets a parameter y_0 which is a point of deadlock (otherwise we begin to argue after the statement **while** is first performed).

Always, the statement **while** fulfilled two times in succession returns necessarily such y_0 at least once which is again deadlock point, but by virtue of Proposition 6.1 belongs to another deadlock, because the best response is smaller. As the whole number of deadlocks is finite then after finite number of steps we achieve the deadlock the points of which are the solutions of our problem.

7. Some Tests

Let us consider some symmetric games of different dimensions as tests of using our method.

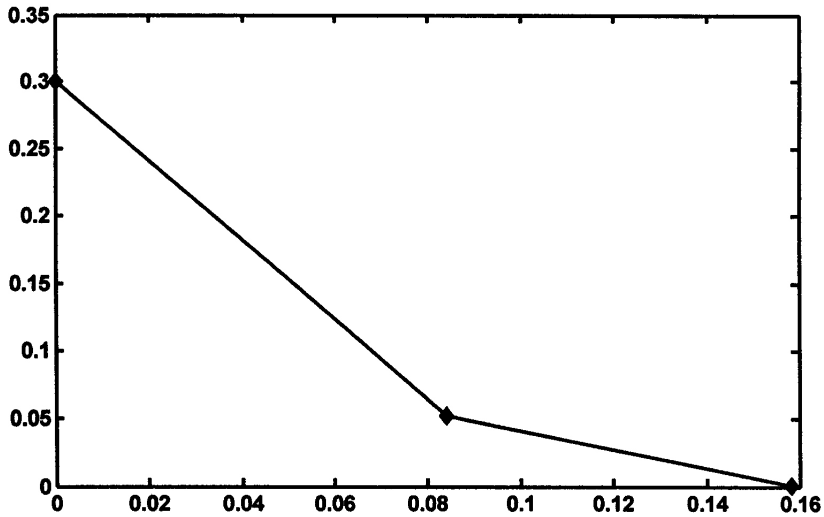

Fig. 2

Example 1. Let us have

$$A = \begin{pmatrix} 0 & 1 & -2 & 3 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & -1 \\ -3 & 0 & 1 & 0 \end{pmatrix}, \quad y_0^T = \begin{pmatrix} 0.1 \\ 0. \\ 0.6 \\ 0.3 \end{pmatrix}.$$

An optimal mixed strategy y ≈ (0.1667; 0.0000; 0.5000; 0.3333) is obtained as a result of solving system (4) without using the system of Brown-von Neumann. The Fig. 2 shows a change of the best response corresponding to the time-dependent mixed strategy (from y_0 to the solution).

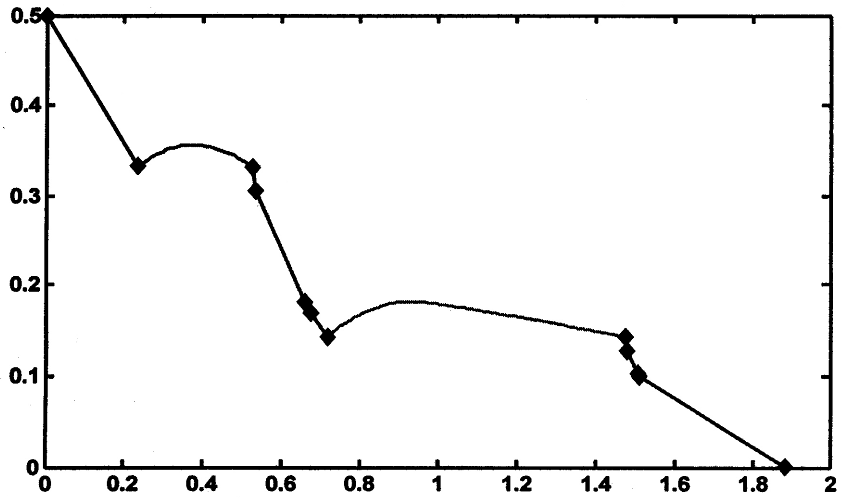

Fig. 3

Example 2. Let us have

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & -3 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}, \quad y_0^T = \begin{pmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{pmatrix}.$$

An optimal mixed strategy y≈ (0.3333; 0.0000; 0.0000; 0.3333; 0.3333) will be got by using alternatively systems (4) and (1). The Fig. 3 shows a graph of the best response function. Fragments of trajectories corresponding to system (4) are monotonically decreasing and convex. Two fragments of trajectory (first begins at 0.2, the second at nearly 0.8) correspond to Brown-Neumann's system. For comparison we can see how mixed strategies and the best response function $v(t)$ is changed in the sufficient large interval of time in the case of using Brown-Neumann's system only. In the Fig. 4 the lower graph corresponds to the best response and the others describe a strategy change:

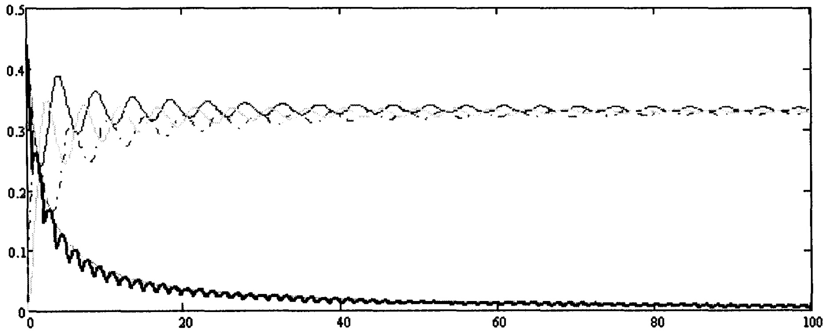

Fig. 4

Note, our method solves completely the given matrix game in the rather small time interval.

Example 3. In this case A is skew-symmetric 70x70 matrix filled with random numbers from the range [-5, 5] by using the function **rand()**. All components of y_0 are equal. Then we still get optimal mixed strategy (exact numerical values of which are unessential at the moment) by alternatively solving piecewise constant and Brown-Neumann's systems which is shown in Fig. 5.

The scales of ordinate axes are different in the different cells on the Table. The last value of the best response function in each graph is the starting value of the best response function in the following graph. The higher graphs present trajectories of the system (4) and the lowers present those of Brown-von Neumann's one.

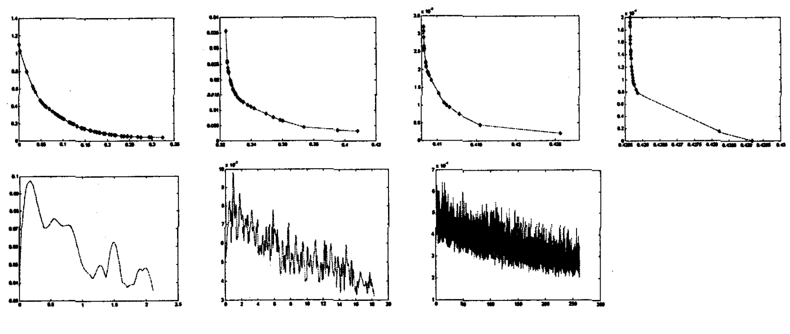

Fig. 5

Example 4. Despite its simplicity this example is of the most significance for us: a standard simplex algorithm is cyclic (see [5]) on the linear programming equivalent of this game:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -1/4 & -1/2 & 0 & -3/4 \\ 0 & 0 & 0 & 0 & -8 & 12 & 0 & 20 \\ 0 & 0 & 0 & 0 & 1 & 1/2 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & -9 & -3 & 0 & 6 \\ 1/4 & 8 & -1 & 9 & 0 & 0 & 0 & 0 \\ 1/2 & -12 & -1/2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 3/4 & -20 & 1/2 & -6 & 0 & 0 & 1 & 0 \end{pmatrix},$$

all components of y_0 are equal. The optimal mixed strategy is

$$\begin{pmatrix} 0.22235519387266497 \\ 0.01364289133556861 \\ 0.10914313068455081 \\ 0.00000000000000000 \\ 0.27285782671133046 \\ 0.00000000000000000 \\ 0.27285782671135300 \\ 0.10914313068453252 \end{pmatrix}.$$

Solving the system (4) without using the Brown-von Neumann's system, allows to get this strategy. This means that our method of solving this problem is maximally efficient. The following Figure shows the graph of the best response function in this problem.

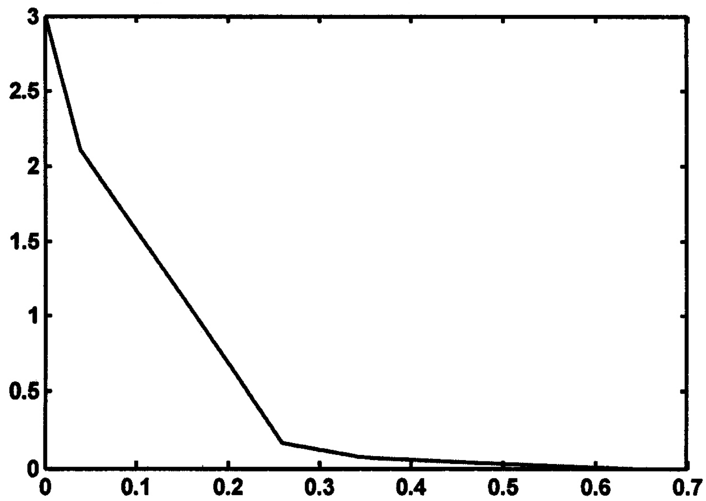

Fig. 6

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