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THE PROPERTIES OF RIGHT UNITS OF SEMIGROUPS BELONGING TO SOME CLASSES OF COMPLETE SEMIGROUPS OF BINARY RELATIONS

YA. DIASAMIDZE, SH. MAKHARADZE, N. ROKVA AND I. DIASAMIDZE

ABSTRACT. In the present work we consider the classes of complete semigroups of binary relations whose every element possesses the right unit. The largest right units of semigroups of the classes under consideration are found. The question whether the right units are the external or internal elements of the given group is elucidated.

რეზიუმე. ნაშრომში განხილულია ბინარულ მიმართებათა სრული ნახევარჯგუფების ისეთი კლასები, რომელთაგან აღებულ ნახევარჯგუფებს გააჩნიათ მარჯვენა ერთეულები. ნაპოვნია მათი უდიდესი მარჯვენა ერთეულები. ნაჩვენებია, თუ რომელი მოცემული ჯგუფისათვის არის მარჯვენა ერთეულები შიგა და რომლისათვის გარე.

1.0. Let X be an arbitrary nonempty set. D is a complete X-semilattice of unions, i.e., some nonempty set of subsets of X, closed with respect to the operations of set-theoretic union of elements from D. f is an arbitrary mapping of the set X in the set D. To every such mapping f we put into correspondence the binary relation α_j on the set X defined as follows: $\alpha_j = \bigcup\limits_{x\in X}\big(\{x\}\times f(x)\big)$. A set of such α_j $(f : X \to D)$ we denote by $B_X(D)$. It can be easily proved that $B_X(D)$ is a semigroup which is called a complete semigroup of binary relations defined by an X-semilattice of unions D.

Let $\alpha \in B_X(D)$, $Y \subseteq X$, $Y\alpha = \{x \in X \,|\, (y,x) \in \alpha \text{ for some } y \in Y\}$; $V(D,\alpha) = \{Y\alpha | Y \in D\}$; $\varnothing \neq D' \subseteq D$ and $N(D,D') = \{Z \in D | Z \subseteq Z' \text{ for every } Z' \in D'\}$. If $N(D,D') \neq \varnothing$, then $\cup N(D,D') \in D$ is an exact lower bound of the set D' in D. This element will be denoted by $\Lambda(D,D')$. Note that if there exists the element $\Lambda(D,D')$ in the semilattice D, then we will write $\Lambda(D,D') \in D$.

Definition 1. Let $t \in \breve{D} = \bigcup\limits_{Z\in D} Z$ and $D_t = \{Z \in D \,|\, t \in Z\}$. They say that the complete X-semilattice of unions D is the XI-semilattice, if it satisfies the following two relations:

(a) $\Lambda(D,D_t) \in D$ for every $t \in \breve{D}$;

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(b) $Z=\bigcup_{t\in Z}\Lambda(D,D_t)$ for every nonempty element Z of the semilattice D.

Definition 2. They say that the element α of the semigroup S is external, if for every elements $\alpha_1,\alpha_2\in S\setminus\{\alpha\}$ there exists $\alpha_1\cdot\alpha_2\neq\alpha$.

Definition 3. They say that the element α of the semigroup S is internal, if there exist the elements $\alpha_1,\alpha_2\in S\setminus\{\alpha\}$ such that $\alpha_1\cdot\alpha_2=\alpha$.

Definition 4. The one-to-one mapping φ of the complete X-semilattice of unions D onto itself is said to be a complete automorphism, if for every nonempty subset D' of the semilattice D the condition $\varphi(\cup D')=\bigcup_{T'\in D'}\varphi(T')$ is fulfilled.

Statements 1,2,3 and 5 can be found in [1] and [2].

Theorem 1. *The semigroup $B_X(D)$ possesses the right unit if and only if D is the XI-semilattice of unions.*

In this case, the binary relation ε can be represented in the form

$$\varepsilon=\varepsilon(D,f)=\bigcup_{t\in\breve{D}}\left(\{t\}\times\Lambda(D,D_t)\right)\cup\bigcup_{t'\in X\setminus\breve{D}}\left(\{t'\}\times f(t')\right),$$

where f is an arbitrary mapping of the set $X\setminus\breve{D}$ in the semilattice D which is always the right unit of the semigroup $B_X(D)$.

Theorem 2. *Let D be the complete X-semilattice of unions. If the binary relation ε of the form $\varepsilon=\varepsilon(D)=\bigcup_{t\in\breve{D}}\left(\{t\}\times\Lambda(D,D_t)\right)\cup\left((X\setminus\breve{D})\times\breve{D}\right)$ is the right unit of the semigroup $B_X(D)$, then it is the largest right unit of the given semigroup.*

Theorem 3. *The binary relation $\varepsilon\in B_X(D)$ is the right unit of the given semigroup if and only if ε is idempotent and $D=V(D,\varepsilon)$.*

Theorem 4. *For external elements of the semigroup S the following statements are valid:*

(a) *if any right unit of the semigroup S is external, then all right units of the semigroup S are external ones;*
(b) *if S' is some set of external elements of the semigroup S, then the set $S''=S\setminus S'$ is the subsemigroup of the given semigroup S;*
(c) *if S' is a set of all external elements of the semigroup S, and A is a generating set for S, then $S'\subseteq A$;*
(d) *if φ is an isomorphic mapping of the semigroup S onto S_1, and S' is a set of all external elements of the semigroup S, then $\varphi(S')$ will likewise be a set of all external elements of the semigroup S_1.*

Proof. We prove only Statement (a). Indeed, let the right unit ε' of the semigroup S be an external element of the semigroup S. If ε'' is another right unit of the semigroup S having decomposition $\varepsilon''=\alpha_1\cdot\alpha_2$ $(\alpha_1,\alpha_2\in S\setminus\{\varepsilon''\})$, then the equality

$\varepsilon' \cdot \varepsilon'' = \varepsilon'$ yields $(\varepsilon' \cdot \alpha_1) \cdot \alpha_2 = \varepsilon'$. Note that the condition $\alpha_2 = \varepsilon'$ implies that $\varepsilon'' = \alpha_1 \cdot \alpha_2 = \alpha_1 \cdot \varepsilon' = \alpha_1$. However, the latter equality contradicts the assumption that $\alpha_1 \in S \backslash \{\varepsilon''\}$. Hence $\alpha_2 \neq \varepsilon'$. If now $\varepsilon' \cdot \alpha_1 = \varepsilon'$, then the following equalities are valid: $\varepsilon' \cdot \alpha_2 = \varepsilon'$, $\varepsilon'' = \varepsilon'' \cdot \varepsilon' = \varepsilon'' \cdot (\varepsilon' \cdot \alpha_2) = (\varepsilon'' \cdot \varepsilon') \cdot \alpha_2 = \varepsilon'' \cdot \alpha_2$. Thus we obtain $\varepsilon'' = \varepsilon'' \cdot \varepsilon' = \varepsilon'' \cdot (\varepsilon' \cdot \alpha_2) = (\alpha_1 \cdot \varepsilon'') \cdot \alpha_2 = \alpha_1 \cdot \alpha_2 = \varepsilon''$, which contradicts the condition $\alpha_1 \in S \backslash \{\varepsilon''\}$. Hence $\varepsilon' \cdot \alpha_1 \neq \varepsilon'$. We get $\varepsilon' \cdot \alpha_1, \alpha_2 \in S \backslash \{\varepsilon'\}$ and $(\varepsilon' \cdot \alpha_1) \cdot \alpha_2 = \varepsilon'$. However, this contradicts the assumption that ε' is the external element of the semi-group S.

Thus Statement (a) is complete. □

Theorem 5. *Let $D = \{\breve{D}, T_{m-1}, \ldots, T_2, T_1\}$ be some finite X-semilattice of unions and $C(D) = \{P_0, P_{m-1}, \ldots, P_2, P_1\}$ be a family of sets of pairwise nonintersecting subsets of the set X. If χ is the mapping of the semilattice D on the family of sets $C(D)$, satisfying the condition $\chi(\breve{D}) = P_0$ and $\chi(Z_i) = P_i$ for every $i = 1, 2, \ldots, m-1$ and $\widehat{D}_Z = D \backslash \{T \in D \,|\, Z \subseteq T\}$, then the equalities*

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1}, \quad Z_i = P_0 \cup \bigcup_{T \in \widehat{D}_{Z_i}} \chi(T) \tag{1}$$

are valid.

In what follows, the above-given equalities will be called formal.

It is proved that when representing elements of the semilattice D in the form (1), among the parameters P_i $(i = 0, 1, 2, \ldots, m-1)$ there exist ones which for the given semilattice D cannot be empty sets. Such sets P_i $(0 < i \leq m-1)$ are called base sources, whereas the sets P_j $(0 \leq j \leq m-1)$ which can be empty are called the sources of completeness.

We prove that a number of overlapping elements of the base source preimage under the mapping χ is always equal to unit, while a number of overlapping elements of the preimage of the source of completeness under the mapping χ either do not exist, or are always more than unit.

Note that the set P_0 is always assumed to be the source of completeness.

2.0.

Theorem 1. *Let D be a finite XI-semilattice of unions. The right units of the semigroup $B_X(D)$ are the external elements of the given semigroup if and only if D does not possess a complete automorphism, except an identity automorphism.*

Proof. Suppose that all units of the semigroup $B_X(D)$ are the external elements of the semigroup $B_X(D)$, and the binary relation ε' is a right unit of the semigroup $B_X(D)$. By $G_X(D, \varepsilon')$ we denote a maximal subgroup of the semigroup $B_X(D)$, having as its own unit the binary relation ε'. If $|G_X(D, \varepsilon)| \geq 2$, then in the group $G_X(D, \varepsilon)$ there exist the elements α' and β' (not necessarily different), such that $\alpha' \neq \varepsilon'$, $\beta' \neq \varepsilon'$ and $\alpha' \circ \beta' = \varepsilon'$.

Hence the conditions $\alpha', \beta' \in B_X(D) \setminus \{\varepsilon'\}$ and $\alpha' \circ \beta' = \varepsilon'$ are fulfilled. Therefore the right unit ε' is the external element of the semigroup $B_X(D)$. The obtained contradiction shows that $|G_X(D, \varepsilon')| = 1$. It is known (see [3]) that the group $G_X(D, \varepsilon')$ is automorphic to the group Φ of all complete automorphisms of the semilattice $V(D, \varepsilon')$. Therefore $|\Phi| = 1$.

Thus we have obtained that the semilattice $V(D, \varepsilon')$ does not possess a complete automorphism, except an identity automorphism. By Theorem 1.3, the equality $V(D, \varepsilon') = D$ is valid. This implies that the semilattice D does not possess a complete automorphism, except an identical automorphism.

Assume now that the semilattice D does not possess a complete automorphism, except an identical one and prove that all units of the semigroup $B_X(D)$ are the external elements of the semigroup $B_X(D)$.

Indeed, let ε be the right unit of the semigroup $B_X(D)$, and $\alpha \circ \beta = \varepsilon$ for some $\alpha, \beta \in B_X(D) \setminus \{\varepsilon\}$. Then the inclusion $V(X^*, \varepsilon) \subseteq V(D, \beta)$ is valid. On the other hand, the conditions $V(D, \varepsilon) = D$ and

$$V(D, \varepsilon) = \begin{cases} V(X^*, \varepsilon) & \text{for} \quad \varnothing \notin D, \\ V(X^*, \varepsilon) \cup \{\varnothing\} & \text{for} \quad \varnothing \in D \end{cases}$$

are fulfilled. Therefore the equality $V(D, \beta) = D$ is valid. Consider now the mapping $\varphi : D \to V(D, \beta)$ satisfying the condition $\varphi(T) = T\beta$ for every $T \in D$. Show that the mapping φ is the automorphism of the semilattice D. Indeed, if $Y \in V(D, \beta)$, then $Y = T'\beta$ for some element $T' \in D$. Hence $\varphi(T') = T'\beta = Y$. Thus we obtain that φ is the mapping of the semilattice D on the semilattice $V(D, \beta)$. This implies that the mapping φ is one-to-one, since $V(D, \beta) = D$ is the finite set.

Next, if $\varnothing \neq D' \subseteq D$, then

$$\varphi(\cup D') = \varphi\Big(\bigcup_{T' \in D'} T'\Big) = \Big(\bigcup_{T' \in D'} T'\Big)\beta = \bigcup_{T' \in D'} T'\beta = \bigcup_{T' \in D'} \varphi(T').$$

Thus we obtain that φ is the complete automorphism of the semilattice D onto itself. By our assumption, the semilattice D does not possess a complete automorphism, except an identity one. Therefore φ is the identity automorphism. Hence the condition $\varphi(T) = T$, i.e., $T\beta = T$ for every $T \in D$, is fulfilled. If now $\beta = \bigcup_{T \in D} (Y_T^\beta \times T)$, then

$$\beta \circ \beta = \Big(\bigcup_{T \in D} \big(Y_T^\beta \times T\big)\Big) \circ \beta = \bigcup_{T \in D} \big(Y_T^\beta \times T\beta\big) = \bigcup_{T \in D} \big(Y_T^\beta \times T\big) = \beta.$$

By Theorem 1.3, from the conditions $\beta \circ \beta = \beta$ and $V(D, \beta) = D$ it follows that the binary relation β is the right unit of the semigroup $B_X(D)$. Taking this into account, from the equality $\alpha \circ \beta = \varepsilon$ we find that $\alpha \circ \beta = \alpha$, i.e., $\alpha = \varepsilon$. However, the last equality contradicts the assumption that $\alpha \in B_X(D) \setminus \{\varepsilon\}$. The obtained contradiction shows that the representation of the right unit ε in the form $\alpha \circ \beta = \varepsilon$, where $\alpha, \beta \in B_X(D) \setminus \{\varepsilon\}$, is impossible.

Consequently, the right unit ε is the external element of the semigroup $B_X(D)$. Thus taking into account Theorem 1.4, we find that all right units of the semigroup $B_X(D)$ are external. □

Corollary 1. *Let D be the finite XI-semilattice of unions. The right units of the semigroup $B_X(D)$ are the internal elements of the given semigroup if and only if D possesses the complete automorphism, different from the identity automorphism.*

Note that when studying idempotents of complete semi-groups of binary relations, more frequently are encountered the semigroups which are defined by the XI-semilattices of unions D_i $(i = 1, 2, \ldots, 7)$ presented in Fig. 1. Further, for the semigroups $B_X(D)$ we find the right units and some of their properties. It is known that of the above-given semilattices, D_1 and D_3 are always the XI-semilattices. The semilattices D_2, D_5 and D_6 are the XI-semilattices if and only if the minimal elements of semilattices under consideration are pairwise nonintersecting (see [4]). The semilattice D_4 is the XI-semilattice if and only if $T_1 \cap T_4 = \varnothing$ (see [4]).

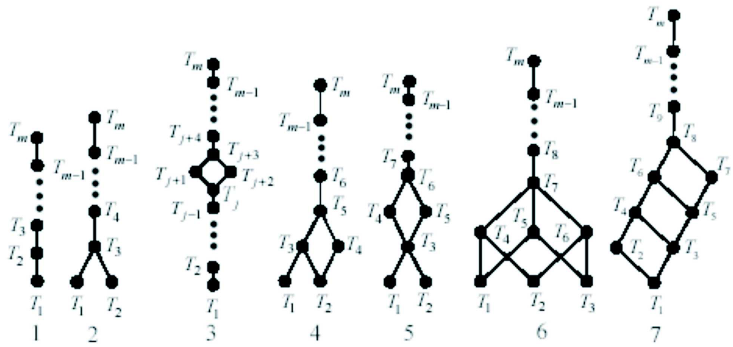

Fig. 1

2.1. Consider the semigroup $B_X(D_1)$. By the assumption, we have $T_1 \subset T_2 \subset \cdots \subset T_{m-1} \subset T_m$. In this case, the formal equalities of the semilattice D_1 and the exact lower bounds $\Lambda(D_1, D_{1t})$ of the sets $D_{1t} = \{t \in T \mid T \in D_1\}$ $(t \in \breve{D}_1)$ in the semilattice D_1 can be represented in the form

$$
\begin{aligned}
&T_m = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},\\
&T_{m-1} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-2},\\
&T_{m-2} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-3},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&T_2 = P_0 \cup P_1,\\
&T_1 = P_0,
\end{aligned}
$$

$$\Lambda(D_1, D_{1t}) = \begin{cases} T_m & \text{for} \quad t \in P_{m-1}, \\ T_{m-1} & \text{for} \quad t \in P_{m-2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ T_2 & \text{for} \quad t \in P_1, \\ T_1 & \text{for} \quad t \in P_0, \end{cases}$$

where $|P_0| \geq 0$, $|P_i| \geq 1$ for any $i = 1, 2, \ldots, m-1$.

Consider the binary relation ε_1 having the representation of the type

$$\varepsilon_1 = (P_{m-1} \times T_m) \cup (P_{m-2} \times T_{m-1}) \cup \cdots \cup (P_1 \times T_2) \cup (P_0 \times T_1) \cup \big((X \setminus T_m) \times T_m\big).$$

Clearly, $\varepsilon_1 \in B_X(D_1)$. Next, by virtue of the formal equalities of the semilattice D_1, we have

$$T_m \setminus T_{m-1} = P_{m-1},\ T_{m-1} \setminus T_{m-2} = P_{m-2}, \ldots, T_2 \setminus T_1 = P_1,\ T_1 = P_0.$$

Consequently, the binary relation ε_1 can be represented in the form

$$\begin{aligned} \varepsilon_1 = & \big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \\ & \cdots \cup \big((T_2 \setminus T_1) \times T_2\big) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big). \end{aligned} \tag{1}$$

Theorem 2.2. *For the semigroup $B_X(D_1)$ the following statements are valid:*
(a) *The binary relation ε_1 is its the largest right unit;*
(b) *all right units of the semigroup are its external elements.*

Proof. First, we prove Statement (a) of the given theorem. Indeed, it can be easily verified that $T_1\varepsilon_1 = T_1$ and $T_i\varepsilon_1 = T_i$ for any $i = 2, 3, \ldots, m$, since $(T_i \setminus T_{i-1}) \cap T_i \neq \varnothing$ and $T_1 \subset T_2 \subset \cdots \subset T_i$. This implies that the equalities $V(D_1, \varepsilon_1) = D_1$ and

$$\begin{aligned} \varepsilon_1 \circ \varepsilon_1 = & \Big(\big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big)\Big) \cup \cdots \cup \\ & \cup \big((T_2 \setminus T_1) \times T_2\big) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big) \circ \varepsilon_1 = \\ = & \big((T_m \setminus T_{m-1}) \times T_m\varepsilon_1\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\varepsilon_1\big) \cup \cdots \cup \\ & \cup \big((T_2 \setminus T_1) \times T_2\varepsilon_1\big) \cup (T_1 \times T_1\varepsilon_1) \cup \big((X \setminus T_m) \times T_m\varepsilon_1\big) = \\ = & \varepsilon_1 = \big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \cup \\ & \cup \big((T_2 \setminus T_1) \times T_2\big) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big) = \varepsilon_1. \end{aligned}$$

are valid. Taking now into account Theorem 1.3, we find that the binary relation ε_1 is the right unit of the semigroup $B_X(D_1)$. By Theorem 1.2, the relation ε_1 is the largest right unit of the semigroup $B_X(D_1)$.

Thus Statement (a) of the above theorem is proved.

It is clear that the semilattice D_1 does not possess the complete automorphism, except the identity automorphism. Hence taking into account Theorem 2.1, we find that all right units of the semigroup $B_X(D_1)$ are the external elements of the semigroup of the given semigroup. Statement (b) of the above theorem is proved. □

2.2. Consider the semigroup $B_X(D_2)$. By the assumption, we have $m \geq 3$, $T_1 \subset T_3 \subset \cdots \subset T_{m-1} \subset T_m$, $T_2 \subset T_3 \subset \cdots \subset T_{m-1} \subset T_m$. In this case, the formal equalities of the semilattice D_2 and the exact lower bounds $\Lambda(D_2, D_{2t})$ of the set $D_{2t} = \{t \in T \mid T \in D_2\}$ $(t \in \breve{D}_2)$ in the semilattice D_2 can be represented in the form

$$\begin{aligned}
&T_m = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},\\
&T_{m-1} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-2},\\
&T_{m-2} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&T_4 = P_0 \cup P_1 \cup P_2 \cup P_3,\\
&T_3 = P_0 \cup P_1 \cup P_2,\\
&T_2 = P_0 \cup P_1,\\
&T_1 = P_0 \cup P_2,
\end{aligned}$$

$$\Lambda(D_1, D_{2t}) = \begin{cases} T_m & \text{for} \quad t \in P_{m-1},\\ T_{m-1} & \text{for} \quad t \in P_{m-2},\\ T_{m-2} & \text{for} \quad t \in P_{m-3},\\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ T_4 & \text{for} \quad t \in P_3,\\ T_1 & \text{for} \quad t \in P_2,\\ T_2 & \text{for} \quad t \in P_1, \end{cases}$$

where $|P_i| \geq 1$ for any $i = 1, 2, \ldots, m-1$.

In this case, it is known that the semilattice D_2 is the XI-semilattice if and only if $T_1 \cap T_2 = \varnothing$. Therefore $P_0 = \varnothing$, since $T_1 \cap T_2 = (P_0 \cup P_2) \cap (P_0 \cup P_1) = P_0 = \varnothing$. Denote now by the symbol ε_2 the binary relation having the representation of the type

$$\begin{aligned}
\varepsilon_2 = (P_{m-1} \times T_m) \cup (P_{m-2} \times T_{m-1}) \cup \cdots \cup (P_3 \times T_4) \cup (P_1 \times T_2) \cup\\
\cup (P_2 \times T_1) \cup \big((X \setminus T_m) \times T_m\big).
\end{aligned}$$

Obviously, $\varepsilon_2 \in B_X(D_2)$. Moreover, from the formal equalities of the semilattice D_2 it follows that

$$T_m \setminus T_{m-1} = P_{m-1},\ T_{m-1} \setminus T_{m-2} = P_{m-2}, \ldots, T_4 \setminus T_3 = P_3,\ T_2 = P_1,\ T_1 = P_2.$$

Hence the equality

$$\begin{aligned}
\varepsilon_2 = &\big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \cup\\
&\cup \big((T_4 \setminus T_3) \times T_4\big) \cup (T_2 \times T_2) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big).
\end{aligned}$$

is valid.

Theorem 2. *Let $T_2 \cap T_1 = \varnothing$. Then for the semigroup $B_X(D_2)$ the following statements are valid.*

(a) *The binary relation ε_2 is its largest right unit;*

(b) *all right units of the given semigroup are its internal elements.*

Proof. Let us prove that $V(D_2,\varepsilon_2) = D_2$ and $\varepsilon_2 \circ \varepsilon_2 = \varepsilon_2$. Indeed, it is not difficult to verify that the equalities $T_1\varepsilon_2 = T_1$, $T_2\varepsilon_2 = T_2$ and

$$T_3\varepsilon_2 = T_3\big((T_2 \times T_2) \cup (T_1 \times T_1)\big) = T_2 \cup T_1 = T_3$$

are valid by the definition of the semilattice D_2, and $T_i\varepsilon_1 = T_i$, since $(T_i \setminus T_{i-1}) \cap T_i \neq \varnothing$, $T_1 \subset T_3 \subset \cdots \subset T_i$, $T_2 \subset T_3 \subset \cdots \subset T_i$ for every $i = 4,5,\ldots,m$. This implies that the equalities $V(D_2,\varepsilon_2) = D_2$ and

$$\begin{aligned}
\varepsilon_2 \circ \varepsilon_2 = \Big(&\big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \cup \\
&\cup \big((T_4 \setminus T_3) \times T_4\big) \cup (T_2 \times T_2) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big)\Big) \circ \varepsilon_1 = \\
= &\big((T_m \setminus T_{m-1}) \times T_m\varepsilon_2\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\varepsilon_2\big) \cup \cdots \cup \\
&\cup \big((T_4 \setminus T_3) \times T_4\varepsilon_2\big) \cup (T_2 \times T_2\varepsilon_2) \cup (T_1 \times T_1\varepsilon_2) \cup \big((X \setminus T_m) \times T_m\varepsilon_2\big) = \\
= &\big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \cup \\
&\cup \big((T_4 \setminus T_3) \times T_4\big) \cup (T_2 \times T_2) \cup (T_1 \times T_1 \cup \big((X \setminus T_m) \times T_m\big) = \varepsilon_2
\end{aligned}$$

are valid. Taking now into account Theorem 1.3, we find that the binary relation ε_2 is the right unit of the semigroup $B_X(D_2)$. By virtue of Theorem 1.2, the relation ε_2 is the largest right unit of the semigroup $B_X(D_2)$.

Thus Statement (a) of the given theorem is proved.

Note that the mapping having the form

$$\varphi = \begin{pmatrix} T_m & T_{m-1} & \cdots & T_3 & T_2 & T_1 \\ T_m & T_{m-1} & \cdots & T_3 & T_1 & T_2 \end{pmatrix}$$

is the automorphism of the semilattice D_2, different from the identity automorphism. Hence taking into account Corollary 2.1, we obtain that all right units of the semigroup $B_X(D_2)$ are internal ones.

Statement (b) of the given theorem is proved. □

Corollary 2. *For the largest right unit ε_2 of the semigroup $B_X(D_2)$ the equalities*

$$\varepsilon_2 = \begin{cases} (T_2 \times T_2) \cup (T_1 \times T_1) \cup \big((X \setminus T_3) \times T_3\big) & \textit{for} \quad m = 3, \\ (X \times T_3) \times T_4 \cup (T_2 \times T_2) \cup (T_1 \times T_1) & \textit{for} \quad m = 4 \end{cases}$$

are valid.

Proof. The validity of the above corollary follows directly from Theorem 2.2, from the inclusion $T_3 \subset T_4$ and from the equality $\big((X \setminus T_4) \times T_4\big) \cup \big((T_4 \setminus T_3) \times T_4\big) = (X \setminus T_3) \times T_4$. □

2.3. Consider the semigroup $B_X(D_3)$. By the assumption, we have $1 \le j \le m-3$ and

$$T_1 \subset T_2 \subset \cdots \subset T_j \subset T_{j+1} \subset T_{j+3} \subset \cdots \subset T_m,$$
$$T_1 \subset T_2 \subset \cdots \subset T_j \subset T_{j+2} \subset T_{j+3} \subset \cdots \subset T_m,$$
$$T_{j+1} \backslash T_{j+2} \neq \varnothing, \quad T_{j+2} \backslash T_{j+1} \neq \varnothing, \quad T_{j+1} \cup T_{j+2} = T_{j+3}.$$

In this case, the formal equalities of the semilattice D_3 and the exact lower bounds $\Lambda(D_3, D_{3t})$ of the set $D_{3t} = \{t \in T \mid T \in D_3\}$ $(t \in \breve{D}_3)$ in the semilattice D_3 can be represented in the form

$$\begin{aligned}
&T_m = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{j+2} \cup P_{j+3} \cup \cdots \cup P_{m-1},\\
&T_{m-1} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{j+2} \cup P_{j+3} \cup \cdots \cup P_{m-2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&T_{j+4} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{j+2} \cup P_{j+3},\\
&T_{j+3} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{j+2},\\
&T_{j+2} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1} \cup P_j \cup P_{j+1},\\
&T_{j+1} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1} \cup P_j \cup P_{j+2},\\
&T_j = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1},\\
&T_{j-1} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{j-2},\\
&\cdots\cdots\cdots\cdots\cdots\\
&T_3 = P_0 \cup P_1 \cup P_2,\\
&T_2 = P_0 \cup P_1,\\
&T_1 = P_0,
\end{aligned}$$

$$\Lambda(D_3, D_{3t}) = \begin{cases}
T_m & \text{for} \quad t \in P_{m-1},\\
T_{m-1} & \text{for} \quad t \in P_{m-2},\\
\cdots\cdots\cdots\cdots & \\
T_{j+4} & \text{for} \quad t \in P_{j+3},\\
T_{j+1} & \text{for} \quad t \in P_{j+2},\\
T_{j+2} & \text{for} \quad t \in P_{j+1},\\
T_j & \text{for} \quad t \in P_j,\\
T_j & \text{for} \quad t \in P_{j-1},\\
T_{j-1} & \text{for} \quad t \in P_{j-2},\\
\cdots\cdots\cdots\cdots & \\
T_2 & \text{for} \quad t \in P_1,\\
T_1 & \text{for} \quad t \in P_0,
\end{cases}$$

where $|P_0| \ge 0$, $|P_j| \ge 0$ and $|P_i| \ge 1$ for any $i = 0, 1, 2, \ldots, j-1, j+1, \ldots, m-1$. Denote now by the symbol ε_3 the binary relation having the form

$$\begin{aligned}\varepsilon_3 =&(P_{m-1}\times T_m)\cup(P_{m-2}\times T_{m-1})\cup\cdots\cup(P_{j+3}\times T_{j+4})\cup\\ &\cup(P_{j+1}\times T_{j+2})\cup(P_{j+2}\times T_{j+1})\cup\big((P_j\cup P_{j-1})\times T_j\big)\cup\\ &\cup(P_{j-2}\times T_{j-1})\cup\cdots\cup(P_1\times T_2)\cup(P_0\times T_1)\cup\big((X\setminus T_m)\times T_m\big).\end{aligned}$$

By virtue of the formal equalities of the semilattice D_3, we have

$$\begin{aligned}&T_m\setminus T_{m-1}=P_{m-1},\ T_{m-1}\setminus T_{m-2}=P_{m-2},\dots,T_{j+4}\setminus T_{j+3}=P_{j+3},\\ &T_{j+2}\setminus T_{j+1}=P_{j+1},\ T_{j+1}\setminus T_{j+2}=P_{j+2},\ (T_{j+2}\cap T_{j+1})\setminus T_{j-1}=P_j\cup P_{j-1},\\ &T_{j-1}\setminus T_{j-2}=P_{j-2},\dots,T_3\setminus T_2=P_2,\ T_2\setminus T_1=P_1,\ T_1=P_0.\end{aligned}$$

Therefore the equality

$$\begin{aligned}\varepsilon_3 =&\big((T_m\setminus T_{m-1})\times T_m\big)\cup\big((T_{m-1}\setminus T_{m-2})\times T_{m-1}\big)\cup\cdots\cup\\ &\cup\big((T_{j+4}\setminus T_{j+3})\times T_{j+4}\big)\cup\big((T_{j+2}\setminus T_{j+1})\times T_{j+2}\big)\cup\\ &\cup\big((T_{j+1}\setminus T_{j+2})\times T_{j+1}\big)\Big(\big((T_{j+2}\cap T_{j+1})\setminus T_{j-1}\big)\times T_j\Big)\cup\\ &\cup\big((T_{j-1}\setminus T_{j-2})\times T_{j-1}\big)\cup\cdots\cup\big((T_2\setminus T_1)\times T_2\big)\cup(T_1\times T_1)\cup\\ &\cup\big((X\setminus T_m)\times T_m\big)\end{aligned}$$

is valid. (In the above formulas, the elements T_j and P_j are assumed to be empty symbols if $j<1$, or $j>m$).

Theorem 3. *For the semigroup $B_X(D_3)$ the following statements are valid.*
(a) *The binary relation ε_3 is its largest right unit;*
(b) *all right units of the semigroup are its internal elements.*

Proof. Indeed, the equalities $T_1\varepsilon_3=T_1$ and

$$\begin{aligned}T_{j+3}\varepsilon_3 =& T_{j+3}\big((T_{j+2}\setminus T_{j+1})\times T_{j+2}\big)\cup T_{j+3}\big((T_{j+1}\setminus T_{j+2})\times T_{j+1}\big)\cup\\ &\cup T_{j+3}\Big(\big((T_{j+2}\cap T_{j+1})\setminus T_{j-1}\big)\times T_j\Big)\cup T_{j+3}\big((T_{j-1}\setminus T_{j-2})\times T_{j-1}\big)\cup\\ &\cup\cdots\cup T_{j+3}\big((T_2\setminus T_1)\times T_2\big)\cup T_{j+3}(T_1\times T_1)=\\ &=T_{j+2}\cup T_{j+1}\cup\cdots\cup T_1=T_{j+2}\cup T_{j+1}=T_{j+3}\end{aligned}$$

are valid by the definition of the semilattice D_3 and $T_i\varepsilon_3=T_i$, since $(T_i\setminus T_{i-1})\cap T_i\neq\varnothing$, $T_1\subset\cdots\subset T_j\subset T_{j+1}\subset T_{j+3}\subset\cdots\subset T_i$, $T_1\subset\cdots\subset T_j\subset T_{j+2}\subset T_{j+3}\subset\cdots\subset T_i$ for any $i=2,\dots,j+2,j+4,\dots,m$. Hence the equalities $V(D_3,\varepsilon_3)=D_3$ and $\varepsilon_3\circ\varepsilon_3=\varepsilon_3$ are valid. Taking now into account Theorem 1.3, we find that the binary relation ε_3 is the right unit of the semigroup $B_X(D_3)$. By Theorem 1.2, the relation ε_3 is the largest right unit of the semigroup $B_X(D_3)$.

Thus Statement (a) of the given theorem is proved.

Note that the mapping having the form

$$\varphi=\begin{pmatrix}T_m & T_{m-1} & \cdots & T_{j+3} & T_{j+2} & T_{j+1} & T_j & \cdots & T_3 & T_2 & T_1\\ T_m & T_{m-1} & \cdots & T_{j+3} & T_{j+1} & T_{j+2} & T_j & \cdots & T_3 & T_2 & T_1\end{pmatrix}$$

is the automorphism of the semilattice D_3, different from the identity automorphism. Thus taking into account Corollary 2.1, we find that all right units of the semigroup $B_X(D_3)$ are internal ones.

Statement (b) is proved. $\square$

Corollary 3. *For the largest right unit* ε_3 *of the semigroup* $B_X(D_3)$ *the following equalities are valid:*

$$\varepsilon_3 = \begin{cases} \big((T_3\setminus T_2)\times T_3\big)\cup\big((T_2\setminus T_3)\times T_2\big)\cup\big((T_2\cap T_3)\times T_1\big)\cup \\ \qquad \cup\big((X\setminus T_4)\times T_4\big) \ \textit{for} \ j=1,\ m=4, \\ \big((T_4\setminus T_3)\times T_4\big)\cup\big((T_3\setminus T_4)\times T_3\big)\cup\Big(\big((T_4\cap T_3)\setminus T_1\big)\times T_2\Big)\cup \\ \qquad \cup(T_1\times T_1)\cup\big((X\setminus T_5)\times T_5\big) \ \textit{for} \ j=2,\ m=5, \\ \big((X\setminus T_4)\times T_5\big)\cup\big((T_3\setminus T_2)\times T_3\big)\cup\big((T_2\setminus T_3)\times T_2\big)\cup \\ \qquad \cup\big((T_2\cap T_3)\times T_1\big)\cup \ \textit{for} \ j=1,\ m=5, \\ \big((X\setminus T_5)\times T_6\big)\cup\big((T_4\setminus T_3)\times T_4\big)\cup\big((T_3\setminus T_4)\times T_3\big)\cup \\ \qquad \cup\Big(\big((T_3\cap T_4)\setminus T_1\big)\times T_2\Big)\cup(T_1\times T_1) \ \textit{for} \ j=2,\ m=6. \end{cases}$$

Proof. The validity of the above Corollary follows directly from Theorem 2.3, from the inclusion $T_4\subset T_5\subset T_6$ and from the equalities

$$\big((X\setminus T_5)\times T_5\big)\cup\big((T_5\setminus T_4)\times T_5\big)=(X\setminus T_4)\times T_5,$$
$$\big((X\setminus T_6)\times T_6\big)\cup\big((T_6\setminus T_5)\times T_6\big)=(X\setminus T_5)\times T_6. \qquad \square$$

2.4. Consider the semigroup $B_X(D_4)$. By the definition, we have $m\geq 5$,

$$\begin{aligned} &T_1\subset T_3\subset T_5\subset\cdots\subset T_m,\ T_2\subset T_3\subset T_5\subset\cdots\subset T_m,\\ &T_2\subset T_4\subset T_5\subset\cdots\subset T_m,\\ &T_1\setminus T_2\neq\varnothing,\ T_2\setminus T_1\neq\varnothing,\ T_3\setminus T_4\neq\varnothing,\ T_4\setminus T_3\neq\varnothing,\\ &T_1\cup T_2=T_3,\ T_3\cup T_4=T_5. \end{aligned}$$

In this case, the formal equalities of the semilattice D_4 and the exact lower bounds $\Lambda(D_4,D_{4t})$ of the set $D_{4t}=\{t\in T\mid T\in D_4\}$ $(t\in \breve{D}_4)$ in the semilattice D_4 can be represented in the form

$$\begin{aligned} &T_m=P_0\cup P_1\cup P_2\cup\cdots\cup P_{m-1},\\ &T_{m-1}=P_0\cup P_1\cup P_2\cup\cdots\cup P_{m-2},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &T_6=P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5,\\ &T_5=P_0\cup P_1\cup P_2\cup P_3\cup P_4,\\ &T_4=P_0\cup P_1\cup P_2\cup P_3,\\ &T_3=P_0\cup P_1\cup P_2\cup P_4,\\ &T_2=P_0\cup P_1, \end{aligned}$$

$$T_1 = P_0 \cup P_2 \cup P_4,$$

$$\Lambda(D_4, D_{4t}) = \begin{cases} T_m & \text{for} \quad t \in P_{m-1}, \\ T_{m-1} & \text{for} \quad t \in P_{m-2}, \\ \dots\dots\dots\dots\dots\dots \\ T_1 & \text{for} \quad t \in P_4, \\ T_4 & \text{for} \quad t \in P_3, \\ T_2 & \text{for} \quad t \in P_1, \end{cases}$$

where $|P_0| \geq 0$, $|P_2| \geq 0$ and $|P_i| \geq 1$ for any $i = 1, 3, \dots, m-1$. We assume that $T_1 \cap T_4 = \varnothing$. Then the equalities $T_1 \cap T_4 = (P_0 \cup P_2 \cup P_4) \cap (P_0 \cup P_1 \cup P_2 \cup P_3) = P_0 \cup P_2 = \varnothing$ are valid, and hence $P_0 = P_2 = \varnothing$.

Denote now by the symbol ε_4 the binary relation having the representation of the type

$$\varepsilon_4 = (P_{m-1} \times T_m) \cup (P_{m-2} \times T_{m-1}) \cup \dots \cup (P_5 \times T_6) \cup (P_3 \times T_4) \cup \\ \cup (P_1 \times T_2) \cup (P_4 \times T_1) \cup \big((X \setminus T_m) \times T_m\big).$$

By the formal equalities of the semilattice D_4, we have

$$T_m \setminus T_{m-1} = P_{m-1},\ T_{m-1} \setminus T_{m-2} = P_{m-2}, \dots, T_6 \setminus T_5 = P_5, \dots, T_1 = P_4, \\ T_4 \setminus T_3 = P_3,\ T_2 = P_1,$$

since $T_1 = P_0 \cup P_2 \cup P_4$ and $P_0 = P_2 = \varnothing$. Therefore the equality

$$\varepsilon_4 = \big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \dots \cup \big((T_6 \setminus T_5) \times T_6\big) \cup \\ \cup \big((T_4 \setminus T_3) \times T_4\big) \cup (T_2 \times T_2) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big)$$

is valid.

Theorem 4. *Let $T_4 \cap T_1 = \varnothing$. Then for the semigroup $B_X(D_4)$ the following statements are valid.*

(a) *The binary relation ε_4 is its largest right unit;*

(b) *all right units of the given semigroup are its external elements.*

Proof. Indeed, the equalities $T_1\varepsilon_3 = T_1$, $T_2\varepsilon_3 = T_2$,

$$T_3\varepsilon_4 = T_3\Big((T_2 \times T_2) \cup \big((T_3 \setminus T_4) \times T_1\big)\Big) = \\ = T_3(T_2 \times T_2) \cup T_3\big((T_3 \setminus T_4) \times T_1\big) = T_2 \cup T_1 = T_3,$$

$$T_5\varepsilon_4 = T_5\Big(\big((T_4 \setminus T_3) \times T_4\big) \cup (T_2 \times T_2) \cup \big((T_3 \setminus T_4) \times T_1\big)\Big) = \\ = T_5\big((T_4 \setminus T_3) \times T_4\big) \cup T_5(T_2 \times T_2) \cup T_5\big((T_3 \setminus T_4) \times T_1\big) = \\ = T_4 \cup T_2 \cup T_1 = T_5$$

are valid by the definition of the semilattice D_4 and $T_i\varepsilon_4 = T_i$, since $(T_i \setminus T_{i-1}) \cap T_i \neq \varnothing$, $T_1 \subset T_3 \subset T_5 \subset \dots \subset T_i$, $T_2 \subset T_3 \subset T_5 \subset \dots \subset T_i$, $T_2 \subset T_4 \subset T_5 \subset \dots \subset T_i$ for any $i = 4, 6, \dots, m$. Thus the equalities $V(D_4, \varepsilon_4) = D_4$ and $\varepsilon_4 \circ \varepsilon_4 = \varepsilon_4$ are valid.

Taking now into account Theorem 1.3, we find that the binary relation ε_4 is the right unit of the semigroup $B_X(D_4)$. By Theorem 1.2, the relation ε_4 is the largest right unit of the semigroup $B_X(D_4)$.

Thus Statement (a) of the given theorem is proved.

Note that the semilattice D_4 does not possess automorphism, except the identity automorphism. Hence by Theorem 2.1, it follows that all right units of the semigroup $B_X(D_4)$ are its external elements.

Statement (b) of the given theorem is proved. □

Corollary 4. *For the largest right unit ε_4 of the semigroup $B_X(D_4)$ the equalities*

$$\varepsilon_4 = \begin{cases} \big((T_4\setminus T_3)\times T_4\big)\cup(T_2\times T_2)\cup(T_1\times T_1)\cup \\ \qquad \cup\big((X\setminus T_5)\times T_5\big) \ \ for\ \ m=5, \\ \big((X\setminus T_5)\times T_6\big)\cup\big((T_4\setminus T_3)\times T_4\big)\cup(T_2\times T_2)\cup \\ \qquad \cup(T_1\times T_1) \ \ for\ \ m=6 \end{cases}$$

are valid.

Proof. The validity of the given corollary follows directly from Theorem 2.4, inclusions $T_5\subset T_6$ and from the equality $\big((X\setminus T_6)\times T_6\big)\cup\big((T_6\setminus T_5)\times T_6\big)=(X\setminus T_5)\times T_6$. □

2.5. Consider the semigroup $B_X(D_5)$. By the assumption, we have $m\geq 6$,

$$\begin{aligned} &T_1\subset T_3\subset T_4\subset T_6\subset\cdots\subset T_m,\ \ T_1\subset T_3\subset T_5\subset T_6\subset\cdots\subset T_m,\\ &T_2\subset T_3\subset T_4\subset T_6\subset\cdots\subset T_m,\ \ T_2\subset T_3\subset T_5\subset T_6\subset\cdots\subset T_m,\\ &T_1\setminus T_2\neq\varnothing,\ \ T_2\setminus T_1\neq\varnothing,\ \ T_4\setminus T_5\neq\varnothing,\ \ T_5\setminus T_4\neq\varnothing,\\ &T_1\cup T_2=T_3,\ \ T_4\cup T_5=T_6. \end{aligned}$$

In this case, the formal equalities of the semilattice D_5 and the exact lower bounds $\Lambda(D_5,D_{5t})$ of the set $D_{5t}=\{t\in T\mid T\in D_5\}$ $(t\in\breve{D}_5)$ in the semilattice D_5 can be represented in the form

$$\begin{aligned} &T_m=P_0\cup P_1\cup P_2\cup\cdots\cup P_{m-1},\\ &T_{m-1}=P_0\cup P_1\cup P_2\cup\cdots\cup P_{m-2},\\ &T_{m-2}=P_0\cup P_1\cup P_2\cup\cdots\cup P_{m-3},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &T_7=P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6,\\ &T_6=P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5,\\ &T_5=P_0\cup P_1\cup P_2\cup P_3\cup P_4,\\ &T_4=P_0\cup P_1\cup P_2\cup P_3\cup P_5,\\ &T_3=P_0\cup P_1\cup P_2,\\ &T_2=P_0\cup P_1, \end{aligned}$$

$$T_1 = P_0 \cup P_2,$$

$$\Lambda(D_5, D_{5t}) = \begin{cases} T_m & \text{for} \quad t \in P_{m-1}, \\ T_{m-1} & \text{for} \quad t \in P_{m-2}, \\ \dots\dots\dots\dots\dots\dots & \\ T_7 & \text{for} \quad t \in P_6, \\ T_4 & \text{for} \quad t \in P_5, \\ T_5 & \text{for} \quad t \in P_4, \\ T_3 & \text{for} \quad t \in P_3, \\ T_1 & \text{for} \quad t \in P_2, \\ T_2 & \text{for} \quad t \in P_1, \end{cases}$$

where $|P_0| \geq 0$, $|P_6| \geq 0$ and $|P_i| \geq 1$ for any $i = 1,2,3,4,5,7,\dots,m-1$. Assuming that $T_1 \cap T_2 = \varnothing$, we have $P_0 = \varnothing$. Denote now by the symbol ε_5 the binary relation having the representation of the type

$$\begin{aligned} \varepsilon_5 = & (P_{m-1} \times T_m) \cup (P_{m-2} \times T_{m-1}) \cup \cdots \cup (P_6 \times T_7) \cup (P_4 \times T_5) \cup \\ & \cup (P_5 \times T_4) \cup (P_3 \times T_3) \cup (P_1 \times T_2) \cup (P_2 \times T_1) \cup \big((X \setminus T_m) \times T_m\big). \end{aligned}$$

By the formal equalities of the semilattice D_5, we have

$$\begin{aligned} & T_m \setminus T_{m-1} = P_{m-1},\ T_{m-1} \setminus T_{m-2} = P_{m-2}, \dots, T_7 \setminus T_6 = P_6, \\ & T_5 \setminus T_4 = P_5,\ T_4 \setminus T_5 = P_4,\ (T_5 \cap T_4) \setminus T_3 = P_3,\ T_2 = P_1,\ T_1 = P_2, \end{aligned}$$

since $T_2 = P_0 \cup P_1$, $T_1 = P_0 \cup P_2$ and $P_0 = \varnothing$. Thus the equality

$$\begin{aligned} \varepsilon_5 = & \big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \cup \big((T_7 \setminus T_6) \times T_7\big) \cup \\ & \cup \big((T_5 \setminus T_4) \times T_5\big) \cup \big((T_4 \times T_5) \times T_4\big) \cup \Big(\big((T_5 \cap T_4) \setminus T_3\big) \times T_3\Big) \cup \\ & \cup (T_2 \times T_2) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big) \end{aligned}$$

is valid.

Theorem 5. *Let $T_2 \cap T_1 = \varnothing$. Then for the semigroup $B_X(D_5)$ the following statements are valid.*

(a) *The binary relation ε_5 is its largest right unit;*

(b) *all right units of the given semigroup are its internal elements.*

Proof. Indeed, the equalities $T_1\varepsilon_5 = T_1$, $T_2\varepsilon_5 = T_2$

$$\begin{aligned} T_3\varepsilon_5 = & T_3\big((T_2 \times T_2) \cup (T_1 \times T_1)\big) = T_3(T_2 \times T_2) \cup T_3(T_1 \times T_1) = \\ & = T_2 \cup T_1 = T_3, \end{aligned}$$

$$\begin{aligned} T_6\varepsilon_5 = & T_6\Big(\big((T_5 \setminus T_4) \times T_5\big) \cup \big((T_4 \setminus T_5) \times T_4\big) \cup \Big(\big((T_5 \cap T_4) \setminus T_3\big) \times T_3\Big) \cup \\ & \cup (T_2 \times T_2) \cup (T_1 \times T_1)\Big) = T_6\big((T_5 \setminus T_4) \times T_5\big) \cup T_6\big((T_4 \setminus T_5) \times T_4\big) \cup \end{aligned}$$

$$\cup T_6\Big(\big((T_5\cap T_4)\setminus T_3\big)\times T_4\Big)\cup T_6(T_2\times T_2)\cup T_6(T_1\times T_1)=$$
$$=T_5\cup T_4\cup T_3\cup T_2\cup T_1=T_6$$

are valid by the definition of the semilattice D_5 and $T_i\varepsilon_5=T_i$, since $(T_i\setminus T_{i-1})\cap T_i\neq\varnothing$, $T_1\subset T_3\subset T_4\subset T_6\subset\cdots\subset T_m$, $T_1\subset T_3\subset T_5\subset T_6\subset\cdots\subset T_m$, $T_2\subset T_3\subset T_4\subset T_6\subset\cdots\subset T_m$, $T_2\subset T_3\subset T_5\subset T_6\subset\cdots\subset T_m$ for any $i=5,7,\dots,m$. This implies that the equalities $V(D_5,\varepsilon_5)=D_5$ and $\varepsilon_5\circ\varepsilon_5=\varepsilon_5$ are valid. Taking now into account Theorem 1.3, we find that the binary relation ε_5 is the right unit of the semigroup $B_X(D_5)$. By Theorem 1.2, the relation ε_5 is the largest right unit of the semigroup $B_X(D_5)$.

Thus Statement (a) of the given theorem is proved.

Note that the mappings

$$\varphi_1=\begin{pmatrix}T_m&\dots&T_6&T_5&T_4&T_3&T_2&T_1\\T_m&\dots&T_6&T_5&T_4&T_3&T_2&T_1\end{pmatrix},\ \varphi_2=\begin{pmatrix}T_m&\dots&T_6&T_5&T_4&T_3&T_2&T_1\\T_m&\dots&T_6&T_4&T_5&T_3&T_2&T_1\end{pmatrix},$$
$$\varphi_3=\begin{pmatrix}T_m&\dots&T_6&T_5&T_4&T_3&T_2&T_1\\T_m&\dots&T_6&T_5&T_4&T_3&T_1&T_2\end{pmatrix},\ \varphi_4=\begin{pmatrix}T_m&\dots&T_6&T_5&T_4&T_3&T_2&T_1\\T_m&\dots&T_6&T_4&T_5&T_3&T_1&T_2\end{pmatrix},$$

are the automorphisms of the semilattice D_5. Hence taking into account Theorem 2.1, we obtain that all right units of the semigroup $B_X(D_5)$ are its internal elements.

Statement (b) is proved. □

Corollary 5. *For the largest right unit ε_5 of the semigroup $B_X(D_5)$ the following equalities are valid:*

$$\varepsilon_5=\begin{cases}\big((T_5\setminus T_4)\times T_5\big)\cup\big((T_4\setminus T_5)\times T_4\big)\cup\Big(\big((T_5\cap T_4)\setminus T_3\big)\times T_3\Big)\cup\\ \quad\cup(T_2\times T_2)\cup(T_1\times T_1)\cup\big((X\setminus T_6)\times T_6\big)\ \ \textit{for}\ \ m=6,\\ \big((X\setminus T_6)\times T_7\big)\cup\big((T_5\setminus T_4)\times T_5\big)\cup\big((T_4\setminus T_5)\times T_4\big)\cup\\ \quad\cup\Big(\big((T_5\cap T_4)\setminus T_3\big)\times T_3\Big)\cup(T_2\times T_2)\cup(T_1\times T_1)\ \ \textit{for}\ \ m=7.\end{cases}$$

Proof. The validity of the above corollary follows directly from Theorem 2.5, from the inclusions $T_6\subset T_7$ and from the equality $\big((X\setminus T_7)\times T_7\big)\cup\big((T_7\setminus T_6)\times T_7\big)=(X\setminus T_6)\times T_7$. □

2.6. Consider the semigroup $B_X(D_6)$. By the assumption, we have $m\geq 7$,

$$T_1\subset T_4\subset T_7\subset T_8\subset\cdots\subset T_m,\ \ T_1\subset T_5\subset T_7\subset T_8\subset\cdots\subset T_m,$$
$$T_2\subset T_4\subset T_7\subset T_8\subset\cdots\subset T_m,\ \ T_2\subset T_6\subset T_7\subset T_8\subset\cdots\subset T_m,$$
$$T_3\subset T_5\subset T_7\subset T_8\subset\cdots\subset T_m,\ \ T_3\subset T_6\subset T_7\subset T_8\subset\cdots\subset T_m,$$
$$T_1\setminus T_2\neq\varnothing,\ \ T_2\setminus T_1\neq\varnothing,\ \ T_3\setminus T_1\neq\varnothing,\ \ T_1\setminus T_3\neq\varnothing,$$
$$T_2\setminus T_3\neq\varnothing,\ \ T_3\setminus T_2\neq\varnothing,\ \ T_4\setminus T_5\neq\varnothing,\ \ T_5\setminus T_4\neq\varnothing,$$
$$T_4\setminus T_6\neq\varnothing,\ \ T_6\setminus T_4\neq\varnothing,\ \ T_5\setminus T_6\neq\varnothing,\ \ T_6\setminus T_5\neq\varnothing,$$
$$T_1\cup T_2=T_4,\ \ T_1\cup T_3=T_5,\ \ T_2\cup T_3=T_6,$$

$$T_4 \cup T_5 = T_4 \cup T_6 = T_5 \cup T_6 = T_7,\ \ T_1 \cup T_6 = T_2 \cup T_5 = T_3 \cup T_4 = T_7.$$

In this case, the formal equalities of the semilattice D_6 and the exact lower bounds $\Lambda(D_6, D_{6t})$ of the set $D_{6t} = \{t \in T \mid T \in D_6\}$ $(t \in \breve{D}_6)$ in the semilattice D_6 can be represented in the form

$$\begin{aligned}
&T_m = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},\\
&T_{m-1} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&T_7 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6,\\
&T_6 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5,\\
&T_5 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_6,\\
&T_4 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_5 \cup P_6,\\
&T_3 = P_0 \cup P_1 \cup P_2 \cup P_4,\\
&T_2 = P_0 \cup P_1 \cup P_3 \cup P_5,\\
&T_1 = P_0 \cup P_2 \cup P_3 \cup P_6,
\end{aligned}$$

$$\Lambda(D_6, D_{6t}) = \begin{cases}
T_3 & \text{for} \quad t \in P_4,\\
T_2 & \text{for} \quad t \in P_5,\\
T_1 & \text{for} \quad t \in P_6,\\
T_8 & \text{for} \quad t \in P_7,\\
\dots\dots\dots\dots\\
T_{m-1} & \text{for} \quad t \in P_{m-2},\\
T_m & \text{for} \quad t \in P_{m-1},
\end{cases}$$

where $|P_0| \geq 0$, $|P_1| \geq 0$, $|P_3| \geq 0$ and $|P_i| \geq 1$ for any $i = 4, 5, 6, 7, \ldots, m-1$. We assume that $T_1 \cap T_2 = \varnothing$, $T_1 \cap T_3 = \varnothing$ and $T_2 \cap T_3 = \varnothing$. Then $P_0 = P_1 = P_2 = P_3 = \varnothing$. By the symbol ε_6 we denote the binary relation having the representation of the type

$$\begin{aligned}
\varepsilon_6 = &(P_{m-1} \times T_m) \cup (P_{m-2} \times T_{m-1}) \cup \cdots \cup (P_7 \times T_8) \cup (P_4 \times T_3) \cup\\
&\cup (P_5 \times T_2) \cup (P_6 \times T_1) \cup \big((X \setminus T_m) \times T_m\big).
\end{aligned}$$

Owing to the formal equalities of the semi-lattice D_6, we find that

$$\begin{aligned}
&T_m \setminus T_{m-1} = P_{m-1},\ T_{m-1} \setminus T_{m-2} = P_{m-2}, \ldots, T_8 \setminus T_7 = P_7,\\
&T_3 = P_4,\ T_2 = P_5,\ T_1 = P_6,
\end{aligned}$$

since $T_3 = P_0 \cup P_1 \cup P_2 \cup P_4$, $T_2 = P_0 \cup P_1 \cup P_3 \cup P_5$, $T_1 = P_0 \cup P_2 \cup P_3 \cup P_6$ and $P_0 = P_1 = P_2 = P_3 = \varnothing$. Therefore the equality

$$\begin{aligned}
\varepsilon_6 = &\big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \cup \big((T_8 \setminus T_7) \times T_8\big) \cup\\
&\cup (T_3 \times T_3) \cup (T_2 \times T_2) \cup (T_1 \times T_1) \cup \big((X \setminus T_m) \times T_m\big)
\end{aligned}$$

is valid.

Theorem 6. *Let the elements of the set* $\{T_3, T_2, T_1\}$ *be pairwise nonintersecting. Then for the semigroup* $B_X(D_6)$ *the following statements are valid.*

(a) *The binary relation* ε_6 *is its largest right unit;*

(b) *all right units of the given semigroup are its internal elements.*

Proof. Indeed, the equalities $T_1\varepsilon_6 = T_1$, $T_2\varepsilon_6 = T_2$, $T_3\varepsilon_6 = T_3$,

$$\begin{aligned}
&T_4\varepsilon_6 = T_4\big((T_2\times T_2)\cup(T_1\times T_1)\big) = T_2\cup T_1 = T_4,\\
&T_5\varepsilon_6 = T_5\big((T_3\times T_3)\cup(T_1\times T_1)\big) = T_3\cup T_1 = T_5,\\
&T_6\varepsilon_6 = T_6\big((T_3\times T_3)\cup(T_2\times T_2)\big) = T_3\cup T_2 = T_6,\\
&T_7\varepsilon_6 = T_4\big((T_3\times T_3)\cup(T_2\times T_2)\cup(T_1\times T_1)\big) =\\
&\qquad = T_3\cup T_2\cup T_1 = T_6\cup T_1 = T_7
\end{aligned}$$

are valid due to the condition that the elements of the set $\{T_3, T_2, T_1\}$ are nonintersecting and to the definition of the semilattice D_6. Moreover, $T_i\varepsilon_5 = T_i$, since $(T_i\setminus T_{i-1})\cap T_i \neq \varnothing$ and

$$\begin{aligned}
&T_1\subset T_4\subset T_7\subset T_8\subset\cdots\subset T_i,\ \ T_1\subset T_5\subset T_7\subset T_8\subset\cdots\subset T_i,\\
&T_2\subset T_4\subset T_7\subset T_8\subset\cdots\subset T_i,\ \ T_2\subset T_6\subset T_7\subset T_8\subset\cdots\subset T_i,\\
&T_3\subset T_5\subset T_7\subset T_8\subset\cdots\subset T_i,\ \ T_3\subset T_6\subset T_7\subset T_8\subset\cdots\subset T_i,
\end{aligned}$$

for any $i = 8, 9, \ldots, m$. Thus the equalities $V(D_6, \varepsilon_6) = D_6$ and $\varepsilon_6\circ\varepsilon_6 = \varepsilon_6$ are valid. Taking now into account Theorem 1.3, we find that the binary relation ε_6 is the right unit of the semigroup $B_X(D_6)$. By Theorem 1.2, the relation ε_6 is the largest right unit of the semigroup $B_X(D_6)$.

Thus Statement (a) of the given theorem is proved.

Note that the mappings

$$\begin{aligned}
&\varphi_1 = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7\ldots T_m\\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7\ldots T_m\end{pmatrix},\ \varphi_2 = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7\ldots T_m\\ T_2 & T_1 & T_3 & T_4 & T_6 & T_5 & T_7\ldots T_m\end{pmatrix},\\
&\varphi_3 = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7\ldots T_m\\ T_3 & T_2 & T_1 & T_6 & T_5 & T_4 & T_7\ldots T_m\end{pmatrix},\ \varphi_4 = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7\ldots T_m\\ T_1 & T_3 & T_2 & T_5 & T_4 & T_6 & T_7\ldots T_m\end{pmatrix},\\
&\varphi_5 = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7\ldots T_m\\ T_2 & T_3 & T_1 & T_6 & T_4 & T_5 & T_7\ldots T_m\end{pmatrix},\ \varphi_6 = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7\ldots T_m\\ T_3 & T_1 & T_2 & T_5 & T_6 & T_4 & T_7\ldots T_m\end{pmatrix}.
\end{aligned}$$

are the automorphisms of the semilattice D_6. Hence taking into account Theorem 2.1, we obtain that all right units of the semigroup $B_X(D_6)$ are its internal elements.

Statement (b) of the given theorem is proved. □

Corollary 6. *For the largest right unit* ε_6 *of the semigroup* $B_X(D_6)$ *the equalities*

$$\varepsilon_6 = \begin{cases} \big((X\setminus T_7)\times T_7\big)\cup(T_3\times T_3)\cup(T_2\times T_2)\cup(T_1\times T_1) & \textit{for}\ \ m=7,\\ \big((X\setminus T_7)\setminus T_8\big)\cup(T_3\times T_3)\cup(T_2\times T_2)\cup(T_1\times T_1) & \textit{for}\ \ m=8\end{cases}$$

are valid.

Proof. The validity of the above corollary follows directly from Theorem 2.6. □

2.7. Consider the semigroup $B_X(D_6)$. By the assumption, we have $m \geq 8$,

$$T_1 \subset T_2,\ T_1 \subset T_3,\ T_2 \cup T_3 = T_4,\ T_4 \cup T_5 = T_6,\ T_6 \cup T_7 = T_8,$$
$$T_8 \subset T_9 \subset \cdots \subset T_{m-1} \subset T_m,\ T_2 \backslash T_3 \neq \varnothing,\ T_3 \backslash T_2 \neq \varnothing,$$
$$T_4 \backslash T_5 \neq \varnothing,\ T_5 \backslash T_4 \neq \varnothing,\ T_6 \backslash T_7 \neq \varnothing,\ T_7 \backslash T_6 \neq \varnothing,$$

In this case, the formal equalities of the semilattice D_7 and the exact lower bounds of the set $\Lambda(D_7, D_{7t})$ in the semilattice $D_{7t} = \{t \in T \mid T \in D_7\}$ can be represented in the form

$$\begin{aligned}
&T_m = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},\\
&T_{m-1} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&T_9 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8,\\
&T_8 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
&T_7 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6,\\
&T_6 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_7,\\
&T_5 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4,\\
&T_4 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_5 \cup P_7,\\
&T_3 = P_0 \cup P_1 \cup P_2,\\
&T_2 = P_0 \cup P_1 \cup P_3 \cup P_5 \cup P_7,\\
&T_1 = P_0,
\end{aligned}$$

$$\Lambda(D_7, D_{7t}) = \begin{cases}
T_1 & \text{for} \quad t \in P_0,\\
T_1 & \text{for} \quad t \in P_1,\\
T_3 & \text{for} \quad t \in P_2,\\
T_1 & \text{for} \quad t \in P_3,\\
T_5 & \text{for} \quad t \in P_4,\\
T_1 & \text{for} \quad t \in P_5,\\
T_7 & \text{for} \quad t \in P_6,\\
T_2 & \text{for} \quad t \in P_7,\\
\cdots\cdots\cdots & \\
T_{m-1} & \text{for} \quad t \in P_{m-2},\\
T_m & \text{for} \quad t \in P_{m-1},
\end{cases}$$

where $|P_0| \geq 0$, $|P_3| \geq 0$, $|P_5| \geq 0$ and $|P_i| \geq 1$ for any $i = 2, 4, 6, 7, \ldots, m-1$. Denote now by the symbol ε_7 the binary relation having the representation of the type

$$\varepsilon_7 = (P_{m-1} \times T_m) \cup (P_{m-2} \times T_{m-1}) \cup \cdots \cup (P_7 \times T_8) \cup (P_6 \times T_7) \cup$$

$$\cup (P_4 \times T_5) \cup (P_2 \times T_3) \cup (P_7 \times T_2) \cup$$
$$\cup \big((P_0 \cup P_1 \cup P_3 \cup P_5) \times T_1\big) \cup \big((X \setminus T_m) \times T_m\big).$$

By the formal equalities of the semilattice D_7, we have

$$T_m \setminus T_{m-1} = P_{m-1},\ T_{m-1} \setminus T_{m-2} = P_{m-2}, \dots, T_8 \setminus T_7 = P_7,\ T_7 \setminus T_6 = P_6,$$
$$T_5 \setminus T_4 = P_4,\ T_3 \setminus T_2 = P_2,\ T_6 \setminus T_7 = P_7,\ T_2 \cap T_7 = P_0 \cup P_1 \cup P_3 \cup P_5,$$

Therefore the equality

$$\varepsilon_7 = \big((T_m \setminus T_{m-1}) \times T_m\big) \cup \big((T_{m-1} \setminus T_{m-2}) \times T_{m-1}\big) \cup \cdots \cup \big((T_8 \setminus T_7) \times T_8\big) \cup$$
$$\cup \big((T_7 \setminus T_6) \times T_7\big) \cup \big((T_5 \setminus T_4) \times T_5\big) \cup \big((T_3 \setminus T_2) \times T_3\big) \cup$$
$$\cup \big((T_6 \setminus T_7) \times T_2\big) \cup \big((T_2 \cap T_7) \times T_1\big) \cup \big((X \setminus T_m) \times T_m\big)$$

is valid.

Theorem 7. *For the semigroup $B_X(D_7)$ the following statements are valid.*
(a) *The binary relation ε_7 is its largest right unit;*
(b) *all right units of the semigroup are its external elements.*

Proof. Indeed, the equalities

$$T_1\varepsilon_7 = T_1\big((T_2 \cap T_7) \times T_1\big) = T_1,$$
$$T_2\varepsilon_7 = T_2\Big(\big((T_6 \setminus T_7) \times T_2\big) \cup \big((T_2 \cap T_7) \times T_1\big)\Big) =$$
$$= T_2\big((T_6 \setminus T_7) \times T_2\big) \cup T_2\big((T_2 \cap T_7) \times T_1\big) = T_2 \cup T_1 = T_2,$$
$$T_4\varepsilon_7 = T_4\Big(\big((T_3 \setminus T_2) \times T_3\big) \cup \big((T_6 \setminus T_7) \times T_2\big) \cup \big((T_2 \cap T_7) \times T_1\big)\Big) =$$
$$= T_4\big((T_3 \setminus T_2) \times T_3\big) \cup T_4\big((T_6 \setminus T_7) \times T_2\big) \cup T_4\big((T_2 \cap T_7) \times T_1\big) =$$
$$= T_3 \cup T_2 \cup T_1 = T_3 \cup T_2 = T_4$$

are valid, since

$$T_1 \cap (T_1 \cap T_7) = T_1 \cap T_1 = T_1,$$
$$T_2 \cap (T_6 \setminus T_7) = (P_0 \cup P_1 \cup P_3 \cup P_5 \cup P_7) \cap \{P_7\} = P_7 \neq \varnothing,$$
$$T_4 \cap (T_3 \setminus T_2) = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_5 \cup P_7) \cap \{P_2\} = P_2 \neq \varnothing,$$
$$T_4 \cap (T_6 \setminus T_7) = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_5 \cup P_7) \cap \{P_7\} = P_7 \neq \varnothing,$$

by the definition of the semilattice D_7. Moreover, $T_i\varepsilon_5 = T_i$, since $(T_i \setminus T_{i-1}) \cap T_i \neq \varnothing$ for any $i = 5, 6, \dots, m$. This implies that the equalities $V(D_7, \varepsilon_7) = D_7$ and $\varepsilon_7 \circ \varepsilon_7 = \varepsilon_7$ are valid. Taking now into account Theorem 1.3, we find that the binary relation ε_7 is the largest right unit of the semigroup $B_X(D_7)$.

Thus Statement (a) of the above theorem is proved.

Note that the semilattice D_7 does not possess the automorphism, except identity automorphism. Hence by Theorem 2.1, it follows that all right units of the semilattice $B_X(D_7)$ are its external elements.

Statement (b) of the given theorem is proved. □

Corollary 7. *For the largest right unit* ε_7 *of the semigroup* $B_X(D_7)$ *the equalities*

$$\varepsilon_7 = \begin{cases} \big((T_7 \setminus T_6) \times T_7\big) \cup \big((T_5 \setminus T_4) \times T_5\big) \cup \big((T_3 \setminus T_2) \times T_3\big) \cup \\ \quad \cup \big((T_6 \setminus T_7) \times T_2\big) \cup \big((T_7 \setminus T_2) \times T_1\big) \cup \big((X \setminus T_8) \times T_8\big) \ \textit{for} \ m = 8, \\ \big((X \setminus T_8) \times T_9\big) \cup \big((T_7 \setminus T_6) \times T_7\big) \cup \big((T_5 \setminus T_4) \times T_5\big) \cup \\ \quad \cup \big((T_3 \setminus T_2) \times T_3\big) \cup \big((T_6 \setminus T_7) \times T_2\big) \cup \big((T_2 \cap T_7) \times T_1\big) \ \textit{for} \ m = 9. \end{cases}$$

are valid.

Proof. The validity of the above corollary follows directly from Theorem 2.7, from the inclusion $T_8 \subset T_9$ and from the equality $\big((X \setminus T_9) \times T_9\big) \cup \big((T_9 \setminus T_8) \times T_9\big) = (X \setminus T_8) \times T_9$. □

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Authors' addresses:

Department of Mathematics of Shota Rustavely
Batumi State University, 35, Ninoshvili st., Batumi 6010
Georgia