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ON COSHAPE INVARIANT EXTENSIONS OF FUNCTORS

V. BALADZE

Abstract. In this paper the coshape invariant and continuous extensions of group–valued covariant (contravariant) functors defined on the category of pairs of spaces having the homotopy type of a pair of finite CW - complexes, are constructed. With each pair of topological spaces the (co)homology and homotopy inj-groups and pro-groups, and their long exact sequences are associated. Is proved that any continuous map of topological spaces induce the long exact sequences relating the homotopy and (co)homology inj-groups and pro-groups of spaces and map. These groups are also used for expression of the classical relative Hurewicz theorem in the pointed coshape category.

რეზიუმე. შრომაში აგებულია სასრული CW კომპლექსების წყვილის ჰომოტოპიური ტიპის მქონე სივრცეთა წყვილების ქვეკატეგორიაზე განსაზღვრული, ჯგუფთა კატეგორიაში მნიშვნელობების მქონე ფუნქტორების კოშეიპურად ინვარიანტული და უწყვეტი გაგრძელებანი.

ტოპოლოგიურ სივრცეთა ყოველ წყვილთან ასოცირებულია (კო)ჰომოლოგიის და ჰომოტოპიის inj–ჯგუფები, pro–ჯგუფები და მათი გრძელი ზუსტი მიმდევრობები. დამტკიცებულია, რომ ტოპოლოგიურ სივრცეთა ნებისმიერი უწყვეტი ასახვა ინდუცირებს გრძელ ზუსტ მიმდევრობას, რომელიც ერთმანეთთან აკავშირებს სივრცეების და ასახვის ჰომოტოპიურ, (კო)ჰომოლოგიურ inj–ჯგუფებს და pro–ჯგუფებს. ეს ჯგუფები აგრეთვე გამოყენებულია მონიშნულწერტილიან სივრცეთა კოშეიპურ კატეგორიაში კლასიკური ჰურევიჩის ფარდობითი თეორემის გადმოსაცემად.

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Contents

Introduction

The coshape theory, as a shape theory ([7], [24]) is a spectral homotopy theory. The notion of coshape of a space was introduced by T. Porter [28]. The alternative definitions of coshape are given in the papers of A. Deliany and P. Hilton [11], Yu. T. Lisica [21] and the author ([3], [4]). The coshape theory is closely connected with the extensions of (co)homotopy and (co)homology functors from the category of spaces having the homotopy type of polyhedras to the category of all topological spaces. In particular, the spectral (co)homotopy groups [21] and the spectral singular (co)homology groups [8] of spaces are invariant functors of coshape theory. Besides, the (co)homotopy and (co)homology inj-groups and pro-groups of spaces ([5], [21], [30]) also induce coshape invariant functors. Note that the inj-groups and pro-groups are important coshape invariants because they contain much more information about the direct and inverse systems than their limits, even if these limits exist (see [24], [30], [32]).

The problem of extension of functors from the subcategory of spaces having the homotopy type of "good" spaces to the category of general topological spaces is one of the important problems of algebraic topology ([7], [8], [12], [16], [23], [24], [31]). The achievements in the solution of this problem have interesting applications in different branches of modern topology and algebra. For example, the shape and coshape functors of topological spaces, which are meaningful exstensions of homotopy functor of spaces having the homotopy type poluhedras, CW-complexes or ANR-spaces, play important roles in topology ([7], [18], [21], [23], [24]), dynamical systems [17], C^*–algebras ([6], [9], [14], [15]) and K–theory [10].

The present paper studies this problem. In Section 1 we give the preliminaries. Here we formulate some basic notions, and some facts of theory of semisimplicial complexes. Section 2 is devoted to the category of direct systems and its quotient category. It contains results which are playing essential roles in the construction of coshape category and in the whole paper. Sections 3 and 4 deal to the foundations of abstract and topological coshape categories. In Section 5 we describe the method of extending a group–valued covariant (contravariant) functor on the category of pairs of spaces having the homotopy type of a pair of finite CW-complexes and homotopy classes of maps to a group–valued covariant (contravariant) functor on the category of pairs of general topological spaces and homotopy classes of maps. More preciesely, we construct the coshape invariant and continuous extensions of covariant (contravariant) functors (cf. [2], [5], [12], [20], [21], [24], [35]). Section 6 is dedicated to the study of exact sequences of inj-groups and pro-groups. Here we prove the existence of long exact sequences of inj-homology, pro-cohomology and inj-homotopy groups of pairs of spaces. The purpose of Section 7 is to give a concept of the coshape of continuous maps. The geometric realizations of semisimplicial maps yield the functor from the category of maps of topological spaces to the inj–category of approriate homotopy category of maps of CW-complexes [13] the applications of which include the constructions of functors from the caregory of maps to the category of long exact sequences of inj–groups and pro–groups. Section 8 is devoted to the study of the classical relative Hurewicz Theorem [31] in the coshape theory. It is known that this theorem has interesting analogies in the shape theory and pro–homotopy theory ([1], [19], [24], [25], [26], [27], [29], [32], [33], [34]). We will prove an analog of the relative Hurewicz theorem in the pointed coshape theory. Our result is expressed in terms of homotopy and homology inj–groups.

Finally, we note that some results of this paper were announced in [4] without of proofs.

CHAPTER I
COSHAPE THEORY

1. Preliminaries

In present paper we use the notation of ([4], [21], [22], [24],[30], [32]). A space and map considered here mean a topological space and continuous map, respectively.

Let **Top** ($\mathbf{Top}_*$) denotes the category of spaces (pointed spaces) and maps (pointed maps). By $\mathbf{CW}_{\boldsymbol{f}}$ ($\mathbf{CW}_{\boldsymbol{f}^*}$) we denote the full subcategory of **Top** ($\mathbf{Top}_*$) consisting of all finite CW–complexes (pointed finite CW–complexes). We write **HTop** ($\mathbf{HTop}_*$) for the homotopy (pointed homotopy) category of the category **Top** ($\mathbf{Top}_*$). The symbol $\mathbf{HCW}_{\boldsymbol{f}}$ ($\mathbf{HCW}_{\boldsymbol{f}^*}$) denote the full subcategory of **HTop** ($\mathbf{HTop}_*$) whose objects are all spaces homotopy equivalent to a finite CW–complex (pointed finite CW–complex).

Let $\mathbf{Top}^2$ ($\mathbf{Top}^2_*$) be the category of pairs (pointed pairs) of spaces. By $\mathbf{CW}^2_{\boldsymbol{f}}$ ($\mathbf{CW}^2_{\boldsymbol{f}^*}$) we denote the full subcategory of $\mathbf{Top}^2$ ($\mathbf{Top}^2_*$) consisting of pairs (pointed pairs) of finite CW– complexes. We write $\mathbf{HTop}^2$ ($\mathbf{HTop}^2_*$) for the homotopy (pointed homotopy) category of the category $\mathbf{Top}^2$ ($\mathbf{Top}^2_*$). Let $\mathbf{HCW}^2_{\boldsymbol{f}}$) ($\mathbf{HCW}^2_{\boldsymbol{f}^*}$) denote the full subcategory of $\mathbf{HTop}^2$ ($\mathbf{HTop}^2_*$) whose objects are pairs (pointed pairs) of spaces homotopy equivalent to a pair (pointed pair) of finite CW–complexes.

We also write **CW** ($\mathbf{CW}_*$) for the category of CW–complexes (pointed CW–complexes). By **HCW** ($\mathbf{HCW}_*$) denote the full subcategory of **HTop** ($\mathbf{HTop}_*$) whose objects are all spaces homotopy equivalent to a CW–complex (pointed CW–complex). Similarly for $\mathbf{CW}^2$ ($\mathbf{CW}^2_*$) and $\mathbf{HCW}^2$ ($\mathbf{HCW}^2_*$).

Let **Gr** (**Ab**) denote the category of groups (abelian groups) and homomorphisms, and let $\mathbf{Set}_*$ denote the category of pointed sets and pointed functions.

Finally, note that the symbol $\mathbb{Z}$ denotes the group of integer numbers.

Now consider the category of semisimplicial complexes and give some basic notions and facts from [22]. A semisimplicial complex (ssc) consists of a sequence $X = \{X_n \,|\, n = 0, 1, 2, \dots\}$ of disjoint sets together with a collection of maps

$$d_i^X : X_{n+1} \longrightarrow X_n, \quad i = 0, 1, \dots, n+1$$

and

$$s_j^X : X_n \longrightarrow X_{n+1}, \quad j = 0, 1, \dots, n$$

which are called the i–th face operator and j–th degeneracy operator, respectively, and satisfy the conditions:

$$\begin{aligned} d_i^X \cdot d_j^X &= d_{j-1}^X \cdot d_i^X, && i < j,\\ d_i^X \cdot s_j^X &= s_{j-1}^X \cdot d_i^X, && i < j,\\ d_i^X \cdot s_j^X &= 1, && i = j, j+1,\\ d_i^X \cdot s_j^X &= s_j^X \cdot d_{i-1}^X, && i > j+1,\\ s_i^X \cdot s_j^X &= s_{j+1}^X \cdot s_i^X, && i \le j. \end{aligned}$$

The elements of X_n are called the n–simplexes of X. Let Y be ssc. A semisimlicial map (ssm) is a map $f : X \longrightarrow Y = \{Y_n \,|\, n = 0, 1, 2, \dots\}$ mapping X_n to Y_n for each n and satisfies the conditions

$$d_i^Y f(x) = f(d_i^X x), \quad s_j^Y f(x) = f(s_j^X x)$$

for each simplex x of X and each maps d_i^X and s_j^X defined on x.

A ssc $X_0 = \{X_{0n} | \, n = 0, 1, 2, \dots\}$ is called a subcomplex of ssc $X = \{X_n | \, n = 0, 1, 2, \dots\}$ if $X_{0n} \subset X_n$ for each n and X_{0n} is closed under all permissible face and degeneracy operators. A semisimplicial pair is a pair of semisimplicial complexes (X, X_0) such that X_0 is a subcomplex of a ssc X. A ssm $f : (X, X_0) \longrightarrow (Y, Y_0)$ of pairs of ssc's is a ssm $f : X \longrightarrow Y$ such that $f(X_0) \subseteq Y_0$. By $\mathbf{Ssc}^2$ denote the category of pairs of ssc's and ssm's.

Let $S_n(X)$ be the collection of all continuous maps $\sigma : \Delta^n \longrightarrow X$ of standard n–simplex Δ^n into topological space X. Let $S(X) = \{S_n(X) \,|\, n = 0, 1, 2, \dots\}$ and let $d_i^* : \Delta^n \longrightarrow \Delta^{n+1}$ and $s_j^* : \Delta^n \longrightarrow \Delta^{n-1}$ be the maps given by formulas:

$$\begin{aligned} &d_i^*(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n), \;\; (t_0, \dots, t_n) \in \Delta^n,\\ &s_j^*(t_0, \dots, t_n) = (t_0, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots, t_n), \;\; (t_0, \dots, t_n) \in \Delta^n. \end{aligned}$$

Let $d_i : S_n(X) \longrightarrow S_{n-1}(X)$ and $s_j : S_n(X) \longrightarrow S_{n+1}(X)$ be the maps sending $x_n \in S_n(X)$ into $x_n d_i^*$ and $x_n s_j^*$, respectively. It is clear that

$S(X)$ is ssc. If $f : X \longrightarrow Y$ is a continuous map, then it induces a ssm $S(f) : S(X) \longrightarrow S(Y)$. By definition, $S(f)(\sigma) = f \cdot \sigma$, $\sigma : \Delta^n \longrightarrow X$. If $f : (X, X_0) \longrightarrow (Y, Y_0)$ is a continuous map of pairs of topological spaces, then $S(f)$ is the ssm $S(f) : (S(X), S(X_0)) \longrightarrow (S(Y), S(Y_0))$ of pairs of ssc's.

Now we associate to given ssc X and ssm $f : X \longrightarrow Y$ their geometric realizations, a CW–complex $|X|$ and a continuous map $|f| : |X| \longrightarrow |Y|$, respectively.

Let $M(X)$ be the topologized disjoint union of all copies (Δ^n, x_n), $x_n \in X_n$, i.e. $M(X) = \overset{\infty}{\underset{n=1}{\oplus}} \Delta^n \times X_n$. Let E be a equivalence relation on $M(X)$ given be the following conditions

$$(d_i^* t, x_n)\, E\, (t, d_i x_n), \quad t \in \Delta^{n-1},$$
$$(s_j^* t, x_n)\, E\, (t, s_j x_n), \quad t \in \Delta^{n+1}.$$

We say that the pairs (t, x) and (u, y) of $M(X)$ are E equivalent, (t, x) $E(u, y)$, if there exists a finite chain of such type equivalences begining at (t, x) and ending at (u, y). Let $|X| = M(X)/E$ and $\eta : M(X) \longrightarrow |X|$ be the quotient map given by formula:

$$\eta\big((t, x)\big) = \big[(t, x)\big], \quad (t, x) \in M(X).$$

Each ssm $f : X \longrightarrow Y$ induces a map $M(f) : M(X) \longrightarrow M(Y)$. By definition,

$$M(f)(t, x_n) = \big(t, f(x_n)\big), \quad x \in X_n, \quad t \in \Delta^n.$$

There exists a continuous map $|f| : |X| \longrightarrow |Y|$ defined by formula

$$|f|\big([(t, x_n)]\big) = \big[(t, f(x_n))\big], \quad x_n \in X_n, \quad t \in \Delta^n.$$

Note that the semisimplicial subcomplexes of the ssc X are one-to-one corespodence with the subcomplexes of the CW-complex $|X|$ (see [22], Ch. III, Sec. 4, Lemma 4.10).

Let $\mathrm{S} : \mathbf{Top}^2 \longrightarrow \mathbf{Ssc}^2$ and $\mathrm{R} : \mathbf{Ssc}^2 \longrightarrow \mathbf{Top}^2$ be the singular functor and the geometric realization functor given by formulas:

$$\mathrm{S}\big((X, X_0)\big) = \big(S(X), S(X_0)\big), \ (X, X_0) \in \mathbf{Top}^2,$$
$$\mathrm{S}(f) : \big(S(X), S(X_0)\big) \longrightarrow \big(S(Y), S(Y_0)\big), \big(f : (X, X_0) \longrightarrow (Y, Y_0)\big) \in \mathbf{Top}^2,$$
$$\mathrm{R}\big((X, X_0)\big) = \big(|X|, |X_0|\big), \ (X, X_0) \in \mathbf{Ssc}^2,$$
$$\mathrm{R}(f) = |f| : \big(|X|, |X_0|\big) \longrightarrow \big(|Y|, |Y_0|\big), \big(f : (X, X_0) \longrightarrow (Y, Y_0)\big) \in \mathbf{Ssc}^2.$$

For each pair $(X, X_0) \in \mathbf{Top}^2$ define a map

$$j_{(X, X_0)} : \big(|S(X)|, |S(X_0)|\big) \longrightarrow (X, X_0).$$

By definition,

$$j_{(X, X_0)}\big([(t, \sigma)]\big) = \sigma(t), \quad t \in \Delta^2, \quad \sigma : \Delta^n \longrightarrow X.$$

Let $f : (X, X_0) \longrightarrow (Y, Y_0)$ be a continuous map of pairs of spaces. The following diagram is commutative

$$\begin{array}{ccc}
\big(|S(X)|, |S(X_0)|\big) & \xrightarrow{\ |S(f)|\ } & \big(|S(Y)|, |S(Y_0)|\big) \\
\Big\downarrow{\scriptstyle j_{(X,X_0)}} & & \Big\downarrow{\scriptstyle j_{(Y,Y_0)}} \\
(X, X_0) & \xrightarrow{\quad f \quad} & (Y, Y_0)\ .
\end{array}$$

Consequently, $j = \{j_{(X,X_0)} \,|\, (X, X_0) \in \mathbf{Top}^2\}$ is a natural transformation of the composition R·S of singular and geometric realization functors to the identity functor $1_{\mathbf{Top}^2} : \mathbf{Top}^2 \longrightarrow \mathbf{Top}^2$. We have the following proposition (cf. [22], Ch. III, Sec. 4, Proposition 4.12).

Proposition 1. *Let (K, K_0) be a pair of ssc's. For each map $g : (|K|, |K_0|) \longrightarrow (X, X_0)$ of $(|K|, |K_0|)$ to pair $(X, X_0) \in \mathbf{Top}^2$ there exists a ssm $\overline{g} : (K, K_0) \longrightarrow (S(X), S(X_0))$ such that $g = j_{(X,X_0)} \cdot |\overline{g}|$.*

Proof. Indeed, let $\overline{g} : K \longrightarrow S(X)$ be a ssm defined in [22]. Let $|\sigma|$ be any n–cell of $|K|$ and $\varphi_n : \Delta^n \longrightarrow |\sigma|$ its characteristic map, the restriction of η to (Δ^n, σ) in $M(K)$. By definition of $\overline{g}$

$$\overline{g}(t) = \eta\big(\varphi_\sigma^{-1}(t), g\varphi_n\big) \in \big|S(X)\big|, \quad t \in |\sigma|.$$

It is easy to see that $\overline{g} : K \longrightarrow S(X)$ is well defined ssm, which induces the ssm of pairs $\overline{g} : (K, K_0) \longrightarrow (S(X), S(X_0))$ and satisfies the condition $g = j_{(X,X_0)} \cdot |\overline{g}|$. □

One special kind of category is the category $\mathbf{A}$ determined by set $(A, \leq)$, where $\leq$ is a binary relation on A. A pair $(A, \leq)$ is called a ordered set if $\leq$ is a relation with properties:

OR1) If $\alpha \leq \alpha'$ and $\alpha' \leq \alpha''$, then $\alpha \leq \alpha''$.

OR2) For each $\alpha \in A$, $\alpha \leq \alpha$.

OR3) If $\alpha \leq \alpha'$ and $\alpha' \leq \alpha$, then $\alpha = \alpha'$.

A set A with a relation $\leq$ having properties OR1) and OR2) is called a preordered set.

A relation $\leq$ on A is called antisymmetric if it has property OR3).

A set A with a relation $\leq$ is called a directed set if it is a preordered set and the relation $\leq$ has property:

For any two elements $\alpha, \alpha' \in A$ there exists an element $\alpha'' \in A$ such that $\alpha, \alpha \leq \alpha''$.

A subset A' of a preordered set A is cofinal in A if for each $\alpha \in A$ there exists an element $\alpha' \in A'$ such that $\alpha \leq \alpha'$.

We say that an element α of an ordered set A is a maximal element of A if $\alpha \leq \alpha' \in A$ implies that $\alpha = \alpha'$.

A function $\varphi : (A, \leq) \longrightarrow (B, \leq\ ')$ is called an increasing function if $f(\alpha) \leq\ ' f(\alpha')$ for each pair $\alpha \leq \alpha'$.

A set A with an order $\leq$ is said to be a cofinite provided for each element $\alpha \in A$ the subset $\{\alpha' \in A \,|\, \alpha' \leq \alpha\}$ has a finite cardinality [24]. In next we shall use the following lemma proved in [24].

Lemma 2. *Let $f : A \longrightarrow B$ be a function of a cofinite preordered set A to a directed set B. Then there exists an increasing function $\psi : A \longrightarrow B$ such that $\varphi \leq \psi$.*

Now give some notions and facts of the pro–category whose detailed description was given in [24].

An inverse system in $\boldsymbol{\mathcal{T}}$ is a contravariant functor $\boldsymbol{X}$ from the category $\mathbf{A}$ induced by directed set $(A, \leq)$ to the category $\boldsymbol{\mathcal{T}}$, i.e. inverse system $\boldsymbol{X}$ in $\boldsymbol{\mathcal{T}}$ is a family $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$, where X_α, $\alpha \in A$ is an object of $\boldsymbol{\mathcal{T}}$ and $p_{\alpha\alpha'} : X_{\alpha'} \longrightarrow X_\alpha$, $\alpha \leq \alpha'$ is a bonding morphism with properties $p_{\alpha\alpha} = 1_{X_\alpha} : X_\alpha \longrightarrow X_\alpha$, $\alpha \in A$ and $p_{\alpha\alpha''} = p_{\alpha\alpha'} \cdot p_{\alpha'\alpha''}$, $\alpha \leq \alpha' \leq \alpha''$.

A morphism $(f_\beta, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ of inverse system $\boldsymbol{X}$ to inverse system $\boldsymbol{Y}$ is a family (f_β, φ), where $\varphi : B \longrightarrow A$ is a function and $f_\beta : X_{\varphi(\beta)} \longrightarrow Y_\beta$, $\beta \in B$ is a morphism such that for each pair $\beta \leq \beta'$ there exists an index $\alpha \geq \varphi(\beta), \varphi(\beta')$ with $f_\beta \cdot p_{\varphi(\beta)\alpha} = q_{\beta\beta'} \cdot f_{\beta'} \cdot p_{\varphi(\beta')\alpha}$.

The composition of morphisms $(f_\beta, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $(g_\gamma, \psi) : \boldsymbol{Y} \longrightarrow \boldsymbol{Z} = (Z_\gamma, r_{\gamma\gamma'}, C)$ is a family $(h_\gamma, \chi) : \boldsymbol{X} \longrightarrow \boldsymbol{Z}$, where $h_\gamma = f_{\psi(\gamma)} \cdot g_\gamma$ and $\chi = \varphi \cdot \psi : C \longrightarrow A$.

The identity morphism of inverse system $\boldsymbol{X}$ inself is a family $(1_{X_\alpha}, 1_A) : \boldsymbol{X} \longrightarrow \boldsymbol{X}$ consisting of identity morphisms $1_{X_\alpha} : X_\alpha \longrightarrow X_\alpha$, $\alpha \in A$ and the identity function $1_A : A \longrightarrow A$.

By **inv**–$\boldsymbol{\mathcal{T}}$ denote the category whose objects are inverse systems in $\boldsymbol{\mathcal{T}}$ and whose morphisms are morphisms of inverse systems in $\boldsymbol{\mathcal{T}}$.

Two morphisms $(f_\beta, \varphi), (g_\beta, \psi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ in **inv**–$\boldsymbol{\mathcal{T}}$ are called equivalent, $(f_\beta, \varphi) \sim (g_\beta, \psi)$, if for each index $\beta \in B$ there exists an index $\alpha \geq \varphi(\beta), \psi(\beta)$ such that $f_\beta \cdot p_{\varphi(\beta)\alpha} = g_\beta \cdot p_{\psi(\beta)\alpha}$.

The equivalence class of morphism $(f_\beta, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ denote by $\boldsymbol{f} = [(f_\beta, \varphi)] : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$. A composition of equivalence classes $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{g} = [(g_\gamma, \psi)] : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ is defined as the class

$$\boldsymbol{g} \cdot \boldsymbol{f} = \big[(g_\gamma, \psi) \cdot (f_\beta, \varphi)\big].$$

Let $\mathbf{1}_{\boldsymbol{X}} = [(1_{X_\alpha}, 1_A)]$. For each morphisms $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Z} \longrightarrow \boldsymbol{W}$ hold equalities $\mathbf{1}_{\boldsymbol{Y}} \cdot \boldsymbol{f} = \boldsymbol{f} = \boldsymbol{f} \cdot \mathbf{1}_{\boldsymbol{X}}$ and $\boldsymbol{h} \cdot (\boldsymbol{g} \cdot \boldsymbol{f}) = (\boldsymbol{h} \cdot \boldsymbol{g}) \cdot \boldsymbol{f}$

Thus, we have obtained a factor-category

$$\mathbf{pro}\text{–}\boldsymbol{\mathcal{T}} = \mathbf{inv}\text{–}\boldsymbol{\mathcal{T}}/_\sim$$

whose objects are inverse systems in $\boldsymbol{\mathcal{T}}$ and whose morphisms are equivalence classes of morphisms in **inv**–$\boldsymbol{\mathcal{T}}$.

An inverse limit of inverse system $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ is a pair $(X, \boldsymbol{p})$, where $\boldsymbol{p} : X \longrightarrow \boldsymbol{X}$ is a morphism in **pro**–$\boldsymbol{\mathcal{T}}$ satisfying the following condition:

For each morphism $\boldsymbol{g} : Y \longrightarrow \boldsymbol{X}$ there exists a unique morphism $g : Y \longrightarrow X$ such that $\boldsymbol{p} \cdot g = \boldsymbol{g}$.

In this case we write $\varprojlim \boldsymbol{X} = X$.

Let $f, g : X' \longrightarrow X$ be morphisms of $\boldsymbol{\mathcal{T}}$. A morphism $\varphi : Y \longrightarrow X'$ for which $f \cdot \varphi = g \cdot \varphi$ us called an equalizer of morphisms f and g, if for each morphism $\varphi' : Y' \longrightarrow X'$ with $f \cdot \varphi' = g \cdot \varphi'$ there exists a morphism $\psi : Y' \longrightarrow Y$ such that $\varphi \cdot \psi = \varphi'$.

Let $\boldsymbol{\mathcal{T}}$ be a category with the properties:

IL1) Every family of objects of $\boldsymbol{\mathcal{T}}$ has a product in $\boldsymbol{\mathcal{T}}$.

IL2) For each two morphisms of $\boldsymbol{\mathcal{T}}$ there exists an equalizer morphism.

Each inverse system in the category $\boldsymbol{\mathcal{T}}$ with properties IL1) and IL2) has unverse limit in $\boldsymbol{\mathcal{T}}$. Note that there exist inverse limits in the categories **Set**, $\mathbf{Set}_*$, **Ab**, **Gr**, **Top**, $\mathbf{Top}_*$.

A morphism $(f_\beta, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ in **inv**–$\boldsymbol{\mathcal{T}}$ induces a unique morphism $f : \varprojlim \boldsymbol{X} \longrightarrow \varprojlim \boldsymbol{Y}$ for which holds equality

$$f_\beta \cdot p_{\varphi(\beta)} = q_\beta \cdot f, \quad \beta \in B.$$

Indeed, the morphisms $g_\beta = f_\beta \cdot p_{\varphi(\beta)} : X \longrightarrow Y_\beta$, $\beta \in B$ induce a morphism $\boldsymbol{g} : X \longrightarrow \boldsymbol{Y}$ and by definition of the inverse limit there exists a unique morphism $f : X \longrightarrow Y$ such that $\boldsymbol{g} = \boldsymbol{q} \cdot f$.

Note that if $(f'_\beta, \varphi') \sim (f_\beta, \varphi)$, then $f_\beta \cdot p_{\varphi(\beta)} = f'_\beta \cdot p_{\varphi'(\beta)}$. Consequently, f depends only on the morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ given by $(f_\beta, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$. The morphism f denote by $\varprojlim \boldsymbol{f}$. It is clear that $\varprojlim(\boldsymbol{g} \cdot \boldsymbol{f}) = \varprojlim \boldsymbol{g} \cdot \varprojlim \boldsymbol{f}$ and $\varprojlim(\mathbf{1}_{\boldsymbol{X}}) = 1_{\varprojlim \boldsymbol{X}}$.

Thus, if $\boldsymbol{\mathcal{T}}$ is a category with inverse limits then there exists a functor

$$\varprojlim : \mathbf{pro}\text{–}\boldsymbol{\mathcal{T}} \longrightarrow \boldsymbol{\mathcal{T}}.$$

Let A' be a cofinal subject of a directed set A. Then inverse subsystem $\boldsymbol{X}' = (X_\alpha, p_{\alpha\alpha'}, A)$ of inverse system $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ in $\boldsymbol{\mathcal{T}}$ is isomorphic to $\boldsymbol{X}$ in **inj**–$\boldsymbol{\mathcal{T}}$. Hence, $\varprojlim \boldsymbol{X}' = \varprojlim \boldsymbol{X}$.

2. Inj–$\boldsymbol{\mathcal{T}}$ Category

Let $\boldsymbol{\mathcal{T}}$ be a arbitrary category. A direct system in $\boldsymbol{\mathcal{T}}$ is a covariant functor $\boldsymbol{X}$ from the category $\mathbf{A}$ determined by directed set $(A, \leq)$ to the category $\boldsymbol{\mathcal{T}}$, i.e. direct system $\boldsymbol{X}$ in $\boldsymbol{\mathcal{T}}$ is a family $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$, where X_α, $\alpha \in A$ is an object of $\boldsymbol{\mathcal{T}}$ and $p_{\alpha\alpha'} : X_\alpha \longrightarrow X_{\alpha'}$, $\alpha \leq \alpha'$ is a bonding morphism with properties $p_{\alpha\alpha} = 1_{X_\alpha} : X_\alpha \longrightarrow X_\alpha$, $\alpha \in A$ and $p_{\alpha'\alpha''} \cdot p_{\alpha\alpha'} = p_{\alpha\alpha''}$,

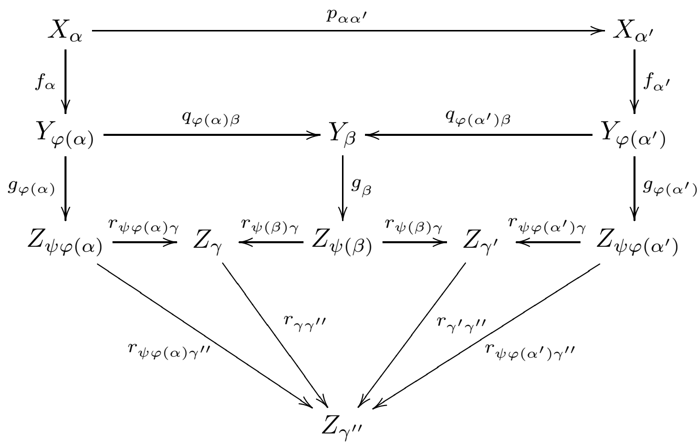
$\alpha \leq \alpha' \leq \alpha''$. For every object $X \in \boldsymbol{\mathcal{T}}$ by (X) we denote direct system indexed by a singleton and having only one term X.

A morphism $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ of **dir**–$\boldsymbol{\mathcal{T}}$, called a mapping of direct systems, consists of a function $\varphi : A \longrightarrow B$ and of a collection of morphisms $f_\alpha : X_\alpha \longrightarrow Y_{\varphi(\alpha)}$, $\alpha \in A$, such that for each pair $\alpha \leq \alpha'$ there is an index $\beta \geq \varphi(\alpha)$, $\varphi(\alpha')$ with $q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\varphi(\alpha')\beta} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'}$.

For each pair $\alpha \leq \alpha'$ of the set A, $p_{\alpha\alpha'} \cdot 1_{X_\alpha} = 1_{X_{\alpha'}} \cdot p_{\alpha\alpha'}$. Hence, the family $(1_{X_\alpha}, 1_A)$ is a mapping of direct system $\boldsymbol{X}$ inself.

The composition (h_α, ζ) of morphisms $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $(g_\beta, \psi) : \boldsymbol{Y} \longrightarrow \boldsymbol{Z} = (Z_\gamma, r_{\gamma\gamma'}, C)$ is defined in usual manner. The mapping of direct systems $(h_\alpha, \zeta) : \boldsymbol{X} \longrightarrow \boldsymbol{Z}$ consists of the function $\zeta = \psi \cdot \varphi$ and of the collection of morphisms $h_\alpha = g_{\varphi(\alpha)} \cdot f_\alpha : X_\alpha \longrightarrow Z_{h(\alpha)}$.

Indeed, for each pair $\alpha \leq \alpha'$ there exist indexes $\beta \geq \varphi(\alpha), \varphi(\alpha')$, $\gamma \geq \psi\varphi(\alpha), \beta$, $\gamma' \geq \psi\varphi(\alpha'), \beta'$ and $\gamma'' > \gamma, \gamma'$, such that the following diagram commutes



Consequently, we have

$$
\begin{aligned}
r_{h(\alpha)\gamma''} \cdot h_\alpha &= r_{\psi\varphi(\alpha)\gamma''} \cdot g_{\varphi(\alpha)} \cdot f_\alpha = r_{\gamma\gamma''} \cdot r_{\psi\varphi(\alpha)\gamma} \cdot g_{\varphi(\alpha)} \cdot f_\alpha \\
&= r_{\gamma\gamma''} \cdot r_{\psi(\beta)\gamma} \cdot g_\beta \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha = r_{\gamma'\gamma''} \cdot r_{\psi(\beta)\gamma'} \cdot g_\beta \cdot g_{\varphi(\alpha)\beta} \cdot f_\alpha \\
&= r_{\gamma'\gamma''} \cdot r_{\psi(\beta)\gamma} \cdot g_\beta \cdot q_{\varphi(\alpha')\beta} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'} \\
&= r_{\gamma'\gamma''} \cdot r_{\psi(\beta)\gamma'} \cdot g_\beta \cdot q_{\varphi(\alpha')\beta} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'} \\
&= r_{\gamma'\gamma''} \cdot r_{\psi\varphi(\alpha')\gamma'} \cdot g_{\varphi(\alpha')} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'} \\
&= r_{\psi\varphi(\alpha')\gamma''} \cdot g_{\varphi(\alpha')} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'} = r_{h(\alpha')\gamma''} \cdot h_{\alpha'} \cdot p_{\alpha\alpha'}.
\end{aligned}
$$

For each three morphisms $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, $(g_\beta, \psi) : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ and $(h_\gamma, \chi) : \boldsymbol{Z} \longrightarrow \boldsymbol{W}$ we have

$$(h_\gamma, \chi) \cdot \big((g_\beta, \psi) \cdot (f_\alpha, \varphi)\big) = \big((h_\gamma, \chi) \cdot (g_\beta, \psi)\big) \cdot (f_\alpha, \varphi),$$

because $\chi \cdot (\psi \cdot \varphi) = (\chi \cdot \psi) \cdot \varphi$ and $h_{\psi\varphi(\alpha)} \cdot (g_{\varphi_\alpha} \cdot f_\alpha) = (h_{\psi\varphi(\alpha)} \cdot g_{\varphi(\alpha)}) \cdot f_\alpha$ for each $\alpha \in A$.

It is clear that

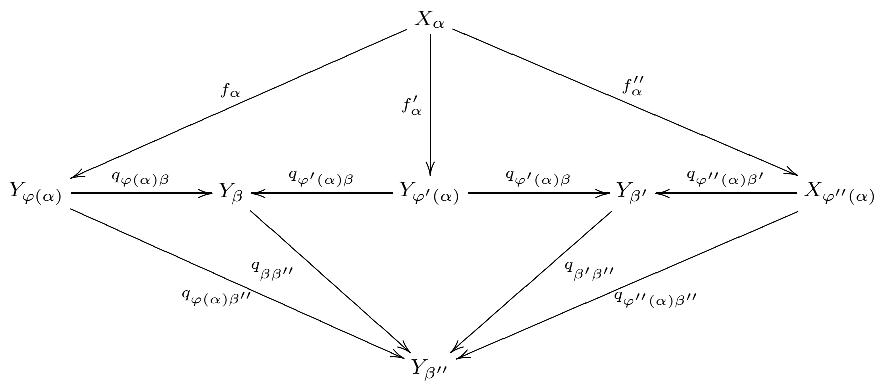
$$(1_{Y_\beta}, 1_B) \cdot (f_\alpha, \varphi) = (f_\alpha, \varphi) = (f_\alpha, \varphi) \cdot (1_{X_\alpha}, 1_A).$$

Thus, the direct systems of the category $\boldsymbol{\mathcal{T}}$ and their morphisms form a category **dir**–$\boldsymbol{\mathcal{T}}$.

Two mappings of direct systems (f_α, φ), $(g_\alpha, \psi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ are said to be equivalent, $(f_\alpha, \varphi) \sim (g_\alpha, \psi)$, if for each index $\alpha \in A$ there is an index $\beta \geq \varphi(\alpha)$, $\psi(\alpha)$ such that $q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\psi(\alpha)\beta} \cdot g_\alpha$.

The relation $\sim$ is an equivalence relation on the set of morphisms of $\boldsymbol{X}$ to $\boldsymbol{Y}$. The facts $(f_\alpha, \varphi) \sim (f_\alpha, \varphi)$ and $(f_\alpha, \varphi) \sim (f'_\alpha, \varphi')$ implies $(f'_\alpha, \varphi') \sim (f_\alpha, \varphi)$ are obviously.

Let $(f_\alpha, \varphi) \sim (f'_\alpha, \varphi')$ and $(f'_\alpha, \varphi') \sim (f''_\alpha, \varphi'')$. There exist indexes $\beta \geq \varphi(\alpha), \varphi'(\alpha)$, $\beta' \geq \varphi'(\alpha), \varphi''(\alpha)$ and $\beta'' > \beta, \beta'$ such that the following diagram commutes



Consequently,

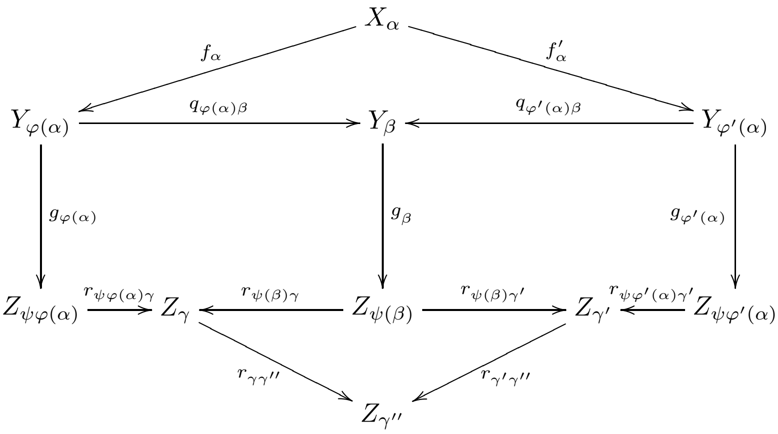
$$\begin{aligned} q_{\varphi(\alpha)\beta''} \cdot f_\alpha &= q_{\beta\beta''} \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\beta\beta''} \cdot q_{\varphi'(\alpha)\beta} \cdot f'_\alpha \\ &= q_{\beta'\beta''} \cdot q_{\varphi'(\alpha)\beta} \cdot f'_\alpha = q_{\beta'\beta''} \cdot q_{\varphi''(\alpha)\beta'} \cdot f''_\alpha = q_{\varphi''(\alpha)\beta'} \cdot f''_\alpha. \end{aligned}$$

Thus, $(f_\alpha, \varphi) \sim (f'_\alpha, \varphi')$.

Proposition 3. *Let* $(f_\alpha, \varphi), (f'_\alpha, \varphi') : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ *and* $(g_\beta, \psi), (g'_\beta, \psi') : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ *be morphisms of the category* **dir**–$\boldsymbol{\mathcal{T}}$*. If* $(f_\alpha, \varphi) \sim (f'_\alpha, \varphi')$ *and* $(g_\beta, \psi) \sim (g'_\beta, \psi')$*, then* $(g_\beta, \psi) \circ (f_\alpha, \varphi) \sim (g'_\beta, \psi') \cdot (f'_\alpha, \varphi')$.

Proof. To achieve this we first shall prove that if $(f_\alpha, \varphi) \sim (f'_\alpha, \varphi')$, then $(g_\beta, \psi) \cdot (f_\alpha, \varphi) \sim (g_\beta, \psi) \cdot (f'_\alpha, \varphi')$. Indeed, for each index $\alpha \in A$ there exist indexes $\beta \geq \varphi(\alpha), \varphi(\alpha')$, $\gamma \geq \psi\varphi(\alpha), \psi(\beta)$, $\gamma' \geq \psi(\beta), \psi\varphi'(\alpha)$ and $\gamma'' \geq \gamma, \gamma'$

such that the following diagram commutes

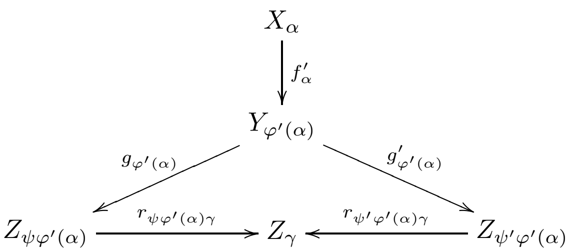


Consequently,

$$\begin{aligned}
r_{\psi\varphi(\alpha)\gamma''} \cdot g_{\varphi(\alpha)} \cdot f_\alpha &= r_{\gamma\gamma''} \cdot r_{\psi\varphi(\alpha)\gamma} \cdot g_{\varphi(\alpha)} \cdot f_\alpha \\
&= r_{\gamma\gamma''} \cdot r_{\psi(\beta)\gamma} \cdot g_\beta \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha \\
&= r_{\gamma\gamma''} \cdot r_{\psi(\beta)\gamma} \cdot g_\beta \cdot q_{\varphi'(\alpha)\beta} \cdot f'_\alpha = r_{\gamma'\gamma''} \cdot r_{\psi(\beta)\gamma'} \cdot g_\beta \cdot q_{\varphi'(\alpha)\beta} \cdot f'_\alpha \\
&= r_{\gamma'\gamma''} \cdot r_{\psi\varphi'(\alpha)\gamma'} \cdot g_{\varphi'(\alpha)} \cdot f'_\alpha = r_{\psi\varphi'(\alpha)\gamma''} \cdot g_{\varphi'(\alpha)} \cdot f'_\alpha,
\end{aligned}$$

i.e. $r_{\psi\varphi(\alpha)\gamma''} \cdot g_{\varphi(\alpha)} \cdot f_\alpha = r_{\psi\varphi'(\alpha)\gamma''} \cdot g_{\varphi'(\alpha)} \cdot f'_\alpha$. Hence we get the equivalence relation $(g_\beta, \psi) \cdot (f_\alpha, \varphi) \sim (g_\beta, \psi) \cdot (f'_\alpha, \varphi')$.

Now we prove that if $(g_\beta, \psi) \sim (g'_\beta, \psi')$, then $(g_\beta, \psi) \cdot (f'_\alpha, \varphi') \sim (g'_\beta, \psi') \cdot (f'_\alpha, \varphi')$. For each index $\alpha \in A$ there exists an index $\gamma \geq \psi\varphi'(\alpha), \psi'\varphi'(\alpha)$ such that the following diagram commutes



Indeed, the equality $r_{\psi\varphi'(\alpha)\gamma} \cdot g_{\varphi'(\alpha)} = r_{\psi'\varphi'(\alpha)\gamma} \cdot g'_{\varphi'(\alpha)}$ implies that $r_{\psi\varphi'(\alpha)\gamma} \cdot g_{\varphi'(\alpha)} \cdot f'_\alpha = r_{\psi'\varphi'(\alpha)\gamma} \cdot g'_{\varphi'(\alpha)} \cdot f'_\alpha$, i.e. $(g_\beta, \varphi) \cdot (f'_\alpha, \varphi') \sim (g'_\beta, \varphi') \cdot (f'_\alpha, \varphi')$.

Finally, we have obtained the following equivalence relation $(g_\beta, \varphi) \cdot (f_\alpha, \varphi) \sim (g'_\beta, \varphi') \cdot (f'_\alpha, \varphi')$. □

Let $\boldsymbol{f} = [(f_\alpha, \varphi)] : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{g} = [(g_\beta, \psi)] : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ be the equivalence classes of morphisms in **dir**–$\boldsymbol{\mathcal{T}}$. By Proposition 3 the composition

$\boldsymbol{g}\cdot\boldsymbol{f}$ is well defined:

$$\boldsymbol{g}\cdot\boldsymbol{f} = \big[(g_\beta,\psi\big]\cdot\big[(f_\alpha,\varphi)\big] = \big[(g_\beta,\psi)\big]\cdot\big[(f_\alpha,\varphi)\big].$$

Note that $\mathbf{1}_{\boldsymbol{Y}}\cdot\boldsymbol{f} = \boldsymbol{f} = \boldsymbol{f}\cdot\mathbf{1}_{\boldsymbol{X}}$ and $\boldsymbol{h}\cdot(\boldsymbol{g}\cdot\boldsymbol{f}) = (\boldsymbol{h}\cdot\boldsymbol{g})\cdot\boldsymbol{f}$ for each equivalence classes $\boldsymbol{f}:\boldsymbol{X}\longrightarrow\boldsymbol{Y}$, $\boldsymbol{g}:\boldsymbol{Y}\longrightarrow\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Z}\longrightarrow\boldsymbol{W}$.

Consequently, there is a quotient category

$$\mathbf{inj}\text{–}\boldsymbol{\mathcal{T}} = \mathbf{dir}-\boldsymbol{\mathcal{T}}/_\sim$$

whose objects are objects of $\mathbf{dir}$–$\boldsymbol{\mathcal{T}}$ and whose morphisms are equivalence classes $\boldsymbol{f} = [(f_\alpha,\varphi)]$ of morphisms (f_α,φ) from $\mathbf{dir}$–$\boldsymbol{\mathcal{T}}$. The category $\mathbf{inj}$–$\boldsymbol{\mathcal{T}}$ is dual to the pro-category $\mathbf{pro}$–$\boldsymbol{\mathcal{T}}$ [24].

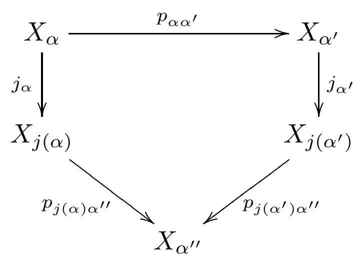
Theorem 4. *Let $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)\in\mathbf{inj}$–$\boldsymbol{\mathcal{T}}$ and let A' be a confinal subset of the set A. Then $\boldsymbol{X}$ and $\boldsymbol{X}' = (X_\alpha, p_{\alpha\alpha'}, A')$ are isomorphic objects of the category $\mathbf{inj}$–$\boldsymbol{\mathcal{T}}$.*

Proof. Indeed, there exist the functions $i : A'\longrightarrow A$ and $j : A\longrightarrow A'$ such that

$$i(\alpha) = \alpha,\quad \alpha\in A',$$
$$j(\alpha)\geq\alpha,\quad \alpha\in A.$$

Consider the families (i_α, i) and (j_α, j), where $i_\alpha = 1_{X_\alpha} : X_\alpha\longrightarrow X_\alpha$ for $\alpha\in A'$ and $j_\alpha = p_{\alpha j(\alpha)} : X_\alpha\longrightarrow X_{j(\alpha)}$ for $\alpha\in A$.

Let $\alpha\leq\alpha'$ be a pair of set A and $\alpha''\geq j(\alpha), j(\alpha')$. The following diagram commutes



Indeed,

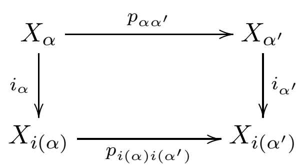
$$p_{j(\alpha)\alpha''}\cdot j_\alpha = p_{j(\alpha)\alpha''}\cdot p_{\alpha j(\alpha)} = p_{\alpha\alpha''}$$

and

$$p_{j(\alpha')\alpha''}\cdot j_{\alpha'}\cdot p_{\alpha\alpha'} = p_{j(\alpha')\alpha''}\cdot p_{\alpha' j(\alpha')}\cdot p_{\alpha\alpha'} = p_{\alpha\alpha''}.$$

Hence, $p_{j(\alpha)\alpha''}\cdot j_\alpha = p_{j(\alpha')\alpha''}\cdot j_{\alpha'}\cdot p_{\alpha\alpha'}$.

For each pair $\alpha\leq\alpha'$ of the subset A' the diagram

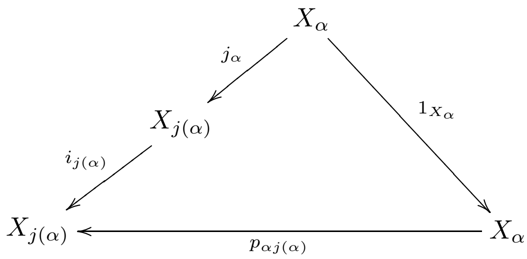


commutes, because

$$p_{i(\alpha)i(\alpha')} \cdot i_\alpha = p_{i(\alpha)i(\alpha')} \cdot 1_{X_\alpha} = p_{i(\alpha)i(\alpha')} = p_{\alpha\alpha'} = 1_{X_{\alpha'}} \cdot p_{\alpha\alpha'} = i_{\alpha'} \cdot p_{\alpha\alpha'}.$$

Consequently, we have morphisms $\boldsymbol{i} = [(i_\alpha, i)] : \boldsymbol{X}' \longrightarrow \boldsymbol{X}$ and $\boldsymbol{j} = [(j_\alpha, j)] : \boldsymbol{X} \longrightarrow \boldsymbol{X}'$.

For each index $\alpha \in A$ the following diagram also commutes

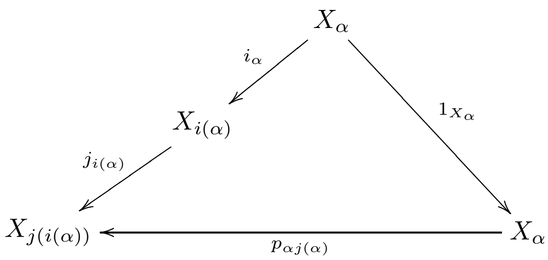


because

$$i_{j_\alpha} \cdot j_\alpha = p_{j(\alpha)j(\alpha)} \cdot p_{\alpha j(\alpha)} = p_{\alpha j(\alpha)} = p_{\alpha j(\alpha)} \cdot 1_{X_\alpha}.$$

Consequently, $(i_\alpha, i) \cdot (j_\alpha, j) \sim (1_{X_\alpha}, 1_A)$. Thus, $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{1_X}$.

For each index $\alpha \in A'$ we also have commutative diagram



because

$$j_{i(\alpha)} \cdot i_\alpha = j_\alpha \cdot 1_{X_\alpha} = p_{\alpha j(\alpha)} = p_{\alpha j(\alpha)} \cdot 1_{X_\alpha}.$$

Thus, $(j_\alpha, j) \cdot (i_\alpha, i) \sim (1_{X_\alpha}, 1_{A'})$. Consequently, $\boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{1_{X'}}$, and hence, $\boldsymbol{X}$ and $\boldsymbol{X}'$ are isomorphic objects of **inj**–$\boldsymbol{\mathcal{T}}$. □

By Remark 1 of ([24], Ch. I, 21.1) every direct system $\boldsymbol{X}$ contains an isomorphic direct subsystem indexed by a directed ordered set.

Let $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ be a direct system in $\boldsymbol{\mathcal{T}}$. A direct limit of $\boldsymbol{X}$ is a pair $(X, \boldsymbol{p})$ consisting of an object $X \in \boldsymbol{\mathcal{T}}$ and morphism $\boldsymbol{p} : \boldsymbol{X} \longrightarrow X$ in **inj**–$\boldsymbol{\mathcal{T}}$ with the following universal property:

For each morphism $\boldsymbol{g} : \boldsymbol{X} \longrightarrow Y$ in **inj**–$\boldsymbol{\mathcal{T}}$ there exists a unique morphism $g : X \longrightarrow Y$ such that $g \cdot \boldsymbol{p} = \boldsymbol{g}$.

A direct limit of $\boldsymbol{X}$ is unique up to a natural isomorphism. Indeed, let $(X', \boldsymbol{p}')$ be another direct limit of $\boldsymbol{X}$. Then there exist unique morphisms $i : X \longrightarrow X'$, $i\,' : X' \longrightarrow X$ for which $i \cdot \boldsymbol{p} = \boldsymbol{p}'$ and $i\,' \cdot \boldsymbol{p}' = \boldsymbol{p}$. Note that

$i \cdot i' \cdot \boldsymbol{p}' = \boldsymbol{p}$, $\quad 1_{X'} \cdot \boldsymbol{p}' = \boldsymbol{p}'$ and $i' \cdot i \cdot \boldsymbol{p} = \boldsymbol{p}$, $\quad 1_X \cdot \boldsymbol{p} = \boldsymbol{p}$. By uniqueness, $i \cdot i' = 1_{X'}$ and $i' \cdot i = 1_X$. Hence, $i : X \longrightarrow X'$ is an isomorphism in $\boldsymbol{\mathcal{T}}$.

If $(X, \boldsymbol{p})$ is a direct limit of direct system $\boldsymbol{X}$ then we write $\lim \boldsymbol{X} = X$.

Let $f, g : X \longrightarrow X'$ be morhisms of $\boldsymbol{\mathcal{T}}$. A morphism $\varphi : X' \longrightarrow Y$ for which $\varphi \cdot f = \varphi \cdot g$ is called a coequalizer of morphisms f and g, if for each morphism $\varphi' : X' \longrightarrow Y'$ with $\varphi' \cdot f = \varphi' \cdot g$ there exists a morphism $\psi : Y \longrightarrow Y'$ such that $\psi\varphi = \varphi'$.

Theorem 5. *Let $\boldsymbol{\mathcal{T}}$ be a category with properties:*

DL1) *For each family of objects in $\boldsymbol{\mathcal{T}}$ there exists a coproduct.*

DL2) *For each two morphisms there exists a coequalizer morphism.*

Then every direct system in $\boldsymbol{\mathcal{T}}$ has a direct limit.

The proof of Theorem 5 based on the following

Lemma 6. *Let $\boldsymbol{\mathcal{T}}$ be a category with properties* DL1) *and* DL2). *Then for each pair of morphisms f_α, $g_\alpha : X_\alpha \longrightarrow X$, $\alpha \in A$ in $\boldsymbol{\mathcal{T}}$, there exists a morphism $\varphi : X \longrightarrow Y$ in $\boldsymbol{\mathcal{T}}$ with properties:*

i) *$\varphi \cdot f_\alpha = \varphi \cdot g_\alpha$ for each $\alpha \in A$.*

ii) *For each morphism $\varphi' : X \longrightarrow Y'$ satisfying the condition* i) *there exists a unique morphism $\psi : Y \longrightarrow Y'$ such that $\psi \cdot \varphi = \varphi'$.*

Proof. It is clear that the family f_α, $\alpha \in A$ induce a morphism $f : \bigoplus_{\alpha \in A} X_\alpha \rightarrow X$ with $f_\alpha = f \cdot i_\alpha$, where $i_\alpha : X_\alpha \longrightarrow \bigoplus_{\alpha \in A} X_\alpha$ is the injection. The family g_α, $\alpha \in A$ also induce a morphism $g : \bigoplus_{\alpha \in A} X_\alpha \longrightarrow X$ for which $g \cdot i_\alpha = g_\alpha$. There exists a morphism $\varphi : X \longrightarrow Y$ with $\varphi \cdot f = \varphi \cdot g$. Note that $\varphi \cdot f \cdot i_\alpha = \varphi \cdot g \cdot i_\alpha$ for each $\alpha \in A$. Consequently, $\varphi \cdot f_\alpha = \varphi \cdot g_\alpha$, $\alpha \in A$.

Assume that for a morphism $\varphi' : X \longrightarrow Y'$ holds equality $\varphi' \cdot f_\alpha = \varphi' \cdot g_\alpha$, $\alpha \in A$. Then $\varphi' \cdot f_\alpha \cdot i_\alpha = \varphi' \cdot g_\alpha \cdot i_\alpha$, $\alpha \in A$, i.e. $\varphi' \cdot f = \varphi' \cdot g$. Hence, there exists a unique morphism $\psi : Y \longrightarrow Y'$ such that $\psi \cdot \varphi = \varphi'$. □

Proof of Theorem 5. Let $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ be a direct system in $\boldsymbol{\mathcal{T}}$. For each pair $\alpha \leq \alpha'$ consider morphisms $i_{\alpha'} \cdot p_{\alpha\alpha'}, i_\alpha : X_\alpha \longrightarrow \bigoplus_{\alpha \in A} X_\alpha$. By Lemma 6 there exists a morphism $\varphi : \bigoplus_{\alpha \in A} X_\alpha \longrightarrow X$ having properties i) and ii). Let $p_\alpha = \varphi \cdot i_\alpha : X_\alpha \longrightarrow X$, $\alpha \in A$. By condition i) for each pair $\alpha \leq \alpha'$ holds equality $p_{\alpha'} \cdot p_{\alpha\alpha'} = p_\alpha$. Consequently, the morphisms $p_\alpha : X_\alpha \longrightarrow X$, $\alpha \in A$ form a morphism $\boldsymbol{p} = [(p_\alpha)] : \boldsymbol{X} \longrightarrow (X)$. Now show that $\varinjlim \boldsymbol{X} = X$. Let $Y \in \boldsymbol{\mathcal{T}}$ and $\boldsymbol{g} = [(g_\alpha)] : \boldsymbol{X} \longrightarrow (Y)$ is a morphism given by morphisms $g_\alpha : X_\alpha \longrightarrow Y$, $\alpha \in A$. Then $g_{\alpha'} \cdot p_{\alpha\alpha'} = g_\alpha$ for each pair $\alpha \leq \alpha'$. Besides, there exists a morphism $\varphi' : \bigoplus_{\alpha \in A} X_\alpha \longrightarrow Y$ such that $\varphi' \cdot i_\alpha = g_\alpha$. Hence, we have $\varphi' \cdot i_\alpha = \varphi' \cdot i_{\alpha'} \cdot p_{\alpha\alpha'}$. By condition ii) there is a unique morphism $g : X \longrightarrow Y$ for which $g \cdot \varphi = \varphi'$. Thus, $g \cdot p_\alpha = g_\alpha$, i.e. $g \cdot \boldsymbol{p} = \boldsymbol{g}$. □

Note that there exist direct limits in the categories **Set**, $\mathbf{Set}_*$, **Ab**, **Gr**, **Top**, $\mathbf{Top}_*$.

Let $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ be a morphism of the category **dir**–$\boldsymbol{\mathcal{T}}$. Show that it induces a unique morphism of the category $\boldsymbol{\mathcal{T}}$, $f : \varinjlim \boldsymbol{X} \longrightarrow \varinjlim \boldsymbol{Y}$ such that for each index $\alpha \in A$, $q_{\varphi(\alpha)} \cdot f_\alpha = f \cdot \varphi_\alpha$. For each index $\alpha \in A$ consider morphisms $g_\alpha = q_{\varphi(\alpha)} \cdot f_\alpha : X_\alpha \longrightarrow Y$ and show that $g_{\alpha'} \cdot p_{\alpha\alpha'} = g_\alpha$ for each pair $\alpha \le \alpha'$. Indeed, there exists an index $\beta \ge \varphi(\alpha), \varphi(\alpha')$ such that

$$q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\varphi(\alpha')\beta} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'}, \;\; q_{\varphi(\alpha)} = q_\beta \cdot q_{\varphi(\alpha)\beta}, \;\; q_{\varphi(\alpha')} = q_\beta \cdot q_{\varphi(\alpha')\beta}.$$

Consequently,

$$\begin{aligned} g_\alpha =& q_{\varphi(\alpha)} \cdot f_\alpha = q_\beta \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_\beta \cdot q_{\varphi(\alpha')\beta} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'} = \\ &= q_{\varphi(\alpha')} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'} = q_{\alpha'} \cdot p_{\alpha\alpha'}. \end{aligned}$$

Thus, the morphisms $g_\alpha : X_\alpha \longrightarrow Y$, $\alpha \in A$ induce a morphism $\boldsymbol{g}$: $[(g_\alpha)] : \boldsymbol{X} \longrightarrow Y$. By definition of direct limit there exists a unique morphism $f : X \longrightarrow Y$ such that $f \cdot \boldsymbol{p} = \boldsymbol{g}$. For each $\alpha \in A$ we have $g_\alpha = f \cdot p_\alpha$, i.e. $q_{\varphi(\alpha)} \cdot f_\alpha = f \cdot p_\alpha, \quad \alpha \in A$.

The morphism f does not depends on the choose of representatives of morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$. Let $(f_\alpha, \varphi) \sim (f'_\alpha, \varphi')$. Then there exists an index $\beta \ge \varphi(\alpha), \varphi(\alpha')$ such that $q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\varphi'(\alpha)\beta} \cdot f_{\alpha'}$. Hence $q_\beta \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_\beta \cdot q_{\varphi'(\alpha)\beta} \cdot f'_\alpha$, i.e. $q_{\varphi(\alpha)} \cdot f_\alpha = q_{\varphi'(\alpha)} \cdot f'_\alpha, \quad \alpha \in A$. Thus, the morphism $f : X \longrightarrow Y$ depends only on the morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$. By $\varinjlim \boldsymbol{f}$ denote the defined morphism f. It is clear that $\varinjlim(\boldsymbol{g} \cdot \boldsymbol{f}) = \varinjlim(\boldsymbol{g}) \cdot \varinjlim(\boldsymbol{f})$ and $\varinjlim(1_{\boldsymbol{X}}) = 1_{\varinjlim \boldsymbol{X}}$.

Consequently, we have obtained the following proposition.

Proposition 7. *If $\boldsymbol{\mathcal{T}}$ is a category with direct limits then there exists a functor*

$$\varinjlim : \mathbf{inj}\text{–}\boldsymbol{\mathcal{T}} \longrightarrow \boldsymbol{\mathcal{T}}.$$

Corollary 8. *Let$\boldsymbol{X}'$ be a cofinal subsystem of direct system $\boldsymbol{X}$ in $\boldsymbol{\mathcal{T}}$. If there exists $\varinjlim \boldsymbol{X}$, then it is isomorphic to $\varinjlim \boldsymbol{X}'$.*

Proposition 9. *If a direct system $\boldsymbol{X} \in$ **inj**–$\boldsymbol{\mathcal{T}}$ is dominated in **inj**–$\boldsymbol{\mathcal{T}}$ by an object $Y \in \boldsymbol{\mathcal{T}}$, then every direct limit $\boldsymbol{p} : \boldsymbol{X} \longrightarrow X$ of X is an isomorphism.*

Proof. By condition of proposition there exist morphisms $\boldsymbol{f} : \boldsymbol{X} \longrightarrow Y$ and $\boldsymbol{g} : Y \longrightarrow \boldsymbol{X}$ such that $\boldsymbol{g} \cdot \boldsymbol{f} = 1_{\boldsymbol{X}}$. Besides, there exists a unique morphism $g : X \longrightarrow Y$ with $g \cdot \boldsymbol{p} = \boldsymbol{f}$. Let $\boldsymbol{q} = \boldsymbol{g} \cdot g : X \longrightarrow \boldsymbol{X}$. Note that $\boldsymbol{q} \cdot \boldsymbol{p} = \boldsymbol{g} \cdot g \cdot \boldsymbol{p} = \boldsymbol{g} \cdot \boldsymbol{f} = 1_{\boldsymbol{X}}$. We also have $(\boldsymbol{p} \cdot \boldsymbol{q}) \cdot \boldsymbol{p} = \boldsymbol{p} \cdot (\boldsymbol{q} \cdot \boldsymbol{p}) = \boldsymbol{p} \cdot 1_{\boldsymbol{X}} = \boldsymbol{p} = 1_X \cdot \boldsymbol{p}$. By uniqueness, $\boldsymbol{p} \cdot \boldsymbol{q} = 1_X$. □

Lemma 10. *Let $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ be a morphism of the category* **dir**–$\boldsymbol{\mathcal{T}}$ *and let A be a cofinite directed set. Then there exists a morphism $(g_\alpha, \psi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ of the category* **dir**–$\boldsymbol{\mathcal{T}}$ *such that $\psi : A \longrightarrow B$ is an increasing function, $g_{\alpha'} \cdot p_{\alpha\alpha'} = q_{\psi(\alpha)\psi(\alpha')} \cdot g_\alpha$ for each pair $\alpha \le \alpha'$ and $(f_\alpha, \varphi) \sim (g_\alpha, \psi)$.*

Proof. Let α be an arbitrary index of A. There exist finite indexes $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $\alpha_i \le \alpha$, $i = 1, 2, \dots, n$. For each index α_i, $i = 1, 2, \dots, n$ choose an index $\beta_i \ge \varphi(\alpha_i), \varphi(\alpha)$ for which holds the equality

$$q_{\varphi(\alpha_i)\beta_i} \cdot f_{\alpha_i} = q_{\varphi(\alpha)\beta_i} \cdot f_\alpha \cdot p_{\alpha_i\alpha}, \quad i = 1, 2, \dots, n.$$

Let $\beta \ge \beta_1, \beta_2, \dots, \beta_n$. It is clear that

$$q_{\beta_i\beta} \cdot q_{\varphi(\alpha_i)\beta_i} \cdot f_{\alpha_i} = q_{\beta_i\beta} \cdot q_{\varphi(\alpha)\beta_i} \cdot f_\alpha \cdot p_{\alpha_i\alpha},$$

i.e.

$$q_{\varphi(\alpha_i)\beta} \cdot f_{\alpha_i} = q_{\varphi(\alpha)\beta} \cdot f_\alpha \cdot p_{\alpha_i\alpha}, \quad i = 1, 2, \dots, n.$$

The correspodense $\alpha \longrightarrow \beta$ defines a function $\varphi' : A \longrightarrow B$. By Lemma 2 of section 1 there exists an increasing function $\psi : A \longrightarrow B$ such that $\varphi'(\alpha) \le \psi(\alpha)$ for each $\alpha \in A$.

Consequently, for each pair $\alpha \le \alpha'$ holds equality

$$q_{\varphi(\alpha)\varphi'(\alpha')} \cdot f_\alpha = q_{\varphi(\alpha')\varphi'(\alpha')} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'}.$$

Since $\psi(\alpha') \ge \varphi'(\alpha')$ we have

$$q_{\varphi(\alpha)\psi(\alpha')} \cdot f_\alpha = q_{\varphi(\alpha')\psi(\alpha')} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'}, \quad \alpha \le \alpha'.$$

For each pair $\alpha \le \alpha'$ we have $\varphi(\alpha) \le \varphi'(\alpha) \le \psi(\alpha) \le \psi(\alpha')$. Clearly,

$$q_{\varphi(\alpha)\psi(\alpha')} = q_{\psi(\alpha)\psi(\alpha')} \cdot q_{\varphi(\alpha)\psi(\alpha)}.$$

Hence,

$$q_{\psi(\alpha)\psi(\alpha')} \cdot q_{\varphi(\alpha)\psi(\alpha)} \cdot f_\alpha = q_{\varphi(\alpha')\psi(\alpha')} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'}.$$

Let $g_\alpha = q_{\varphi(\alpha)\psi(\alpha)} \cdot f_\alpha$, $\alpha \in A$. It now follows that for each pair $\alpha \le \alpha'$

$$q_{\psi(\alpha)\psi(\alpha')} \cdot g_\alpha = g_{\alpha'} \cdot p_{\alpha\alpha'}.$$

Thus, the family (g_α, ψ) is a morphism of $\boldsymbol{X}$ to $\boldsymbol{Y}$ and $(g_\alpha, \psi) \sim (f_\alpha, \varphi)$. □

Theorem 11. *Let $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ be an object of the category* **inj**–$\boldsymbol{\mathcal{T}}$*. Then there exists a direct system $\boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ isomorphic to $\boldsymbol{X}$ and indexed by a directed cofinite ordered set B with cardinality $|B| \le |A|$. Moreover, each term Y_β and bonding morphism $q_{\beta\beta'}$ of $\boldsymbol{Y}$ are term and bonding morphism of $\boldsymbol{X}$, respectively.*

Proof. This theorem is clear in that case when A is a finite set. Assume the cardinality $|A| \ge \aleph_0$ and A is antisymmetric. Consider the set B of all finite subsets β of A having maximal elements $\max \beta$. The elements of B form a directed ordered set provided $\beta \le \beta'$ means $\beta \subseteq \beta'$. It is clear that B is a cofinite set and $|B| = |A|$. Consider a direct system $\boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$, where $Y_\beta = X_{\max\beta}$ for each index $\beta \in B$ and $q_{\beta\beta'} = p_{\max\beta,\max\beta'}$ for each pair $\beta \leq \beta'$.

Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ be the functions given by formulas:

$$\varphi(\alpha) = \{\alpha\}, \quad \alpha \in A,$$
$$\psi(\beta) = \max\beta, \quad \beta \in B.$$

Assume that $f_\alpha = 1_{X_\alpha} : X_\alpha \longrightarrow Y_{\varphi(\alpha)} = X_\alpha$ for each index $\alpha \in A$ and $g_\beta = 1_{X_{\max\beta}} : Y_\beta = X_{\max\beta} \longrightarrow X_{\max\beta}$ for each index $\beta \in B$.

For each pairs $\alpha \leq \alpha'$ and $\beta \leq \beta'$ we have equalities

$$q_{\varphi(\alpha)\varphi(\alpha')} \cdot f_\alpha = 1_{X_\alpha} = p_{\alpha\alpha'} = 1_{X_{\alpha'}} \cdot p_{\alpha\alpha'} = f_{\alpha'} \cdot p_{\alpha\alpha'}$$

and

$$p_{\psi(\beta)\psi(\beta')} \cdot g_\beta = p_{\max\beta,\max\beta'} \cdot 1_{X_{\max\beta}} = p_{\max\beta,\max\beta'}$$
$$= 1_{X_{\max\beta'}} \cdot p_{\max\beta,\max\beta'} = g_{\beta'} \cdot q_{\beta\beta'}.$$

Consequently, we have obtained the morphisms $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $(g_\beta, \psi) : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ of the category $\mathbf{dir} - \boldsymbol{\mathcal{T}}$.

It is clear that $(g_\beta, \psi)\cdot(f_\alpha, \varphi) \sim (1_{X_\alpha}, 1_A)$ and $(f_\alpha, \varphi)\cdot(g_\beta, \psi) \sim (1_{Y_\beta}, 1_B)$. Indeed, for each indexes $\alpha \in A$ and $\beta \in B$ hold equalities

$$g_{\varphi(\alpha)} \cdot f_\alpha = 1_{X_\alpha} \cdot 1_{X_\alpha} = 1_{X_\alpha}$$

and

$$q_{\varphi(\psi(\beta))\beta} \cdot f_{\psi(\beta)} \cdot g_\beta = 1_{X_{\max\beta}} \cdot 1_{X_{\max\beta}} \cdot 1_{X_{\max\beta}} = 1_{X_{\max\beta}} = 1_{Y_\beta}.$$

Thus, $\boldsymbol{g} \cdot \boldsymbol{f} = 1_{\boldsymbol{X}}$ and $\boldsymbol{f} \cdot \boldsymbol{g} = 1_{\boldsymbol{Y}}$. □

Let $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ and $\boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ be objects of $\mathbf{dir}\text{–}\boldsymbol{\mathcal{T}}$. A morphism (f_α, φ) between $\boldsymbol{X}$ and $\boldsymbol{Y}$ is said special morphism if it satisfies the following conditions: $\varphi = 1_A$ and for each pair $\alpha \leq \alpha'$, $q_{\alpha\alpha'} \cdot f_\alpha = f_{\alpha'} \cdot p_{\alpha\alpha'}$.

Theorem 12. *For each morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ of $\mathbf{inj}\text{–}\boldsymbol{\mathcal{T}}$ there exist direct systems $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ indexed by directed cofinite ordered set N such that every term and bonding morphism of $\boldsymbol{X}'$ ($\boldsymbol{Y}'$) is also one in $\boldsymbol{X}$ ($\boldsymbol{Y}$). Moreover, there exist isomorphisms $\boldsymbol{i} : \boldsymbol{X} \longrightarrow \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \longrightarrow \boldsymbol{Y}'$ of $\mathbf{inj}\text{–}\boldsymbol{\mathcal{T}}$ and a special morphism $(f'_\nu, 1_N) : \boldsymbol{X}' \longrightarrow \boldsymbol{Y}'$ of $\mathbf{dir}\text{–}\boldsymbol{\mathcal{T}}$ such that $\boldsymbol{j} \cdot \boldsymbol{f} = \boldsymbol{f}' \cdot \boldsymbol{i}$, $\boldsymbol{f}' = [(f'_\nu, 1_N)]$.*

Proof. By Lemma 10 there exists a representative (f_α, φ) of morphism $\boldsymbol{f}$ such that $\varphi : A \longrightarrow B$ is an increasing function. Then $q_{\varphi(\alpha)\varphi(\alpha')} \cdot f_\alpha = f_{\alpha'} \cdot p_{\alpha\alpha'}$ for each pair $\alpha \leq \alpha'$.

Let $N = \{\nu = (\alpha, \beta) \mid (\alpha, \beta) \in A \times B,\ \varphi(\alpha) \leq \beta\}$. Define a relation $\leq$ on the set N. By definition

$$\nu = (\alpha, \beta) \leq \nu' = (\alpha', \beta') \Leftrightarrow \alpha \leq \alpha' \wedge \beta \leq \beta'.$$

Let $\boldsymbol{X}' = (X'_\nu, p'_{\nu\nu'}, N)$ and $\boldsymbol{Y}' = (Y'_\nu, q'_{\nu\nu'}, N)$ be direct systems on directed cofinite ordered set N, where $X'_\nu = X_\alpha$, $Y'_\nu = Y_\beta$, $p'_{\nu\nu'} = p_{\alpha\alpha'}$, and $q_{\nu\nu'} = q_{\beta\beta'}$.

Let $f'_\nu = q_{\varphi(\alpha)\beta} \cdot f_\alpha$. Show that $(f'_\nu, 1_N)$ is a morphism of $\boldsymbol{X}'$ to $\boldsymbol{Y}'$ in **dir**–$\boldsymbol{\mathcal{T}}$. For each pair $\nu \leq \nu'$ we have

$$q_{\varphi(\alpha)\varphi(\alpha')} \cdot f_\alpha = f_{\alpha'} \cdot p_{\alpha\alpha'}, \quad q_{\varphi(\alpha)\beta} \cdot f_\alpha = f'_\nu, \quad q_{\varphi(\alpha')\beta'} \cdot f_{\alpha'} = f'_{\nu'},$$
$$q_{\beta\beta'} \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\varphi(\alpha')\beta'} \cdot q_{\varphi(\alpha)\varphi(\alpha')} \cdot f_\alpha.$$

Consequently,

$$q'_{\nu\nu'} \cdot f'_\nu = q_{\beta\beta'} \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\varphi(\alpha')\beta'} \cdot q_{\varphi(\alpha)\varphi(\alpha')} \cdot f_\alpha$$
$$= q_{\varphi(\alpha')\beta'} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'} = f'_{\nu'} \cdot p'_{\nu\nu'}.$$

Hence the morphism $(f'_\nu, 1_N)$ is a special morphism.

Let $\boldsymbol{f}' = [(f'_\nu, 1_N)]$. Consider a function $i' : N \longrightarrow A$ and a morphism $i'_\nu : X'_\nu \longrightarrow X_{i'(\nu)}$ given by formulas:

$$i'(\nu) = \alpha, \quad \nu = (\alpha, \beta) \in N,$$
$$i'_\nu = 1_{X_\alpha}, \quad \nu = (\alpha, \beta) \in N.$$

For pair $\nu \leq \nu'$ we have

$$p_{i'(\nu)i'(\nu')} \cdot i'_\nu = p_{\alpha\alpha'} \cdot 1_{X_\alpha} = p_{\alpha\alpha'} = 1_{X_{\alpha'}} \cdot p_{\alpha\alpha'} = i'_{\nu'} \cdot p'_{\nu\nu'}.$$

Thus the family (i'_ν, i') induces a morphism $\boldsymbol{i}' = [(i'_\nu, i')] : \boldsymbol{X}' \longrightarrow \boldsymbol{X}$ of the category **inj**–$\boldsymbol{\mathcal{T}}$.

For each index $\alpha \in A$ the pair $(\alpha, \varphi(\alpha)) \in N$. There exists a function $i : A \longrightarrow N$ defined by $i(\alpha) = (\alpha, \varphi(\alpha))$, $\alpha \in A$. The function i is a increasing function because for each pair $\alpha \leq \alpha'$

$$i(\alpha) = (\alpha, \varphi(\alpha)) \leq (\alpha', \varphi(\alpha')) = i(\alpha').$$

Consider a family (i_α, i), $\alpha \in A$, where i_α is a morphism

$$i_\alpha : X_\alpha \longrightarrow X'_{i(\alpha)} = X'_{(\alpha,\varphi(\alpha))} = X_\alpha$$

given by $i_\alpha = 1_{X_\alpha}$. Then for each pair $\alpha \leq \alpha'$ we have

$$p'_{i(\alpha)i(\alpha')} \cdot i_\alpha = p_{\alpha\alpha'} \cdot 1_{X_\alpha} = p_{\alpha\alpha'} = 1_{X_{\alpha'}} \cdot p_{\alpha\alpha'} = i_{\alpha'} \cdot p_{\alpha\alpha'}.$$

Thus $\boldsymbol{i} = [(i_\alpha, i)] : \boldsymbol{X} \longrightarrow \boldsymbol{X}'$ is a morphism of **inj**–$\boldsymbol{\mathcal{T}}$.

Let $j' : N \longrightarrow B$ and $j'_\nu : Y'_\nu \longrightarrow Y_{j'(\nu)}$ be a function and a morphism defined by formulas

$$j'(\nu) = \beta, \quad \nu = (\alpha, \beta) \in N$$

and

$$j'_\nu = 1_{Y_\beta}, \quad \nu = (\alpha, \beta) \in N,$$

respectively. Observe that

$$q_{j'(\nu)j'(\nu')} \cdot j'_\nu = q_{\beta\beta'} \cdot 1_{Y_\beta} = q_{\beta\beta'} = 1_{Y_{\beta'}} \cdot q_{\beta\beta'} = j'_{\nu'} \cdot q_{\nu\nu'}.$$

Hence, the family (j'_ν, j') induces a morphism $\boldsymbol{j}' = [(j'_\alpha, j')] : \boldsymbol{Y}' \longrightarrow \boldsymbol{Y}$ in **inj**–$\boldsymbol{\mathcal{T}}$.

Let α_0 be an fixed index of the set A. Since B is directed set there exists an index $\zeta(\beta) \in B$ such that $\zeta(\beta) \geq \beta, \varphi(\alpha_0)$. By this way is defined a function $\zeta : B \longrightarrow B$. By Lemma 2 there exist an increasing function $\psi : B \longrightarrow B$ such that $\psi(\beta) \geq \zeta(\beta)$ for each $\beta \in B$, i.e. $\psi(\beta) \geq \beta, \varphi(\alpha_0)$. Now define a function $j : B \longrightarrow N$ and a morphism $j_\beta : Y_\beta \longrightarrow Y'_{j(\beta)}$. By definition,

$$j(\beta) = (\alpha_0, \psi(\beta)), \quad \beta \in B,$$
$$j_\beta = q_{\beta\psi(\beta)} : Y_\beta \longrightarrow Y'_{j(\beta)} = Y'_{(\alpha_0, \psi(\beta))} = Y_{\psi(\beta)}, \ \ \beta \in B.$$

For each pair $\beta \leq \beta'$ we have

$$j(\beta) = (\alpha_0, \psi(\beta)) \leq (\alpha_0, \psi(\beta')) = j(\beta')$$

and

$$q'_{j(\beta)j(\beta')} \cdot j_\beta = q_{\psi(\beta)\psi(\beta')} \cdot q_{\beta\psi(\beta)} = q_{\beta\psi(\beta')} = q_{\beta'\psi(\beta')} \cdot q_{\beta\beta'} = j_{\beta'} \cdot q_{\beta\beta'}$$

which show that the family (j_β, j) induces a morphism $\boldsymbol{j} = [(j_\beta, j)] : \boldsymbol{Y} \longrightarrow \boldsymbol{Y}'$ in **inj**–$\boldsymbol{\mathcal{T}}$.

Now show that

$$\boldsymbol{i} \cdot \boldsymbol{i}' = 1_{\boldsymbol{X}'}, \ \ \boldsymbol{i}' \cdot \boldsymbol{i} = 1_{\boldsymbol{X}}, \ \ \boldsymbol{j} \cdot \boldsymbol{j}' = 1_{\boldsymbol{Y}'}, \ \ \boldsymbol{j}' \cdot \boldsymbol{j} = 1_{\boldsymbol{Y}}.$$

For each index $\nu = (\alpha, \beta) \in N$ we have

$$p'_{i(i'(\nu))\nu} \cdot i_{i'(\nu)} \cdot i'_\nu = 1_{X_\alpha} \cdot 1_{X_\alpha} \cdot 1_{X_\alpha} = 1_{X_\alpha} = 1_{X'_\nu}.$$

Consequently, the composition $(i_\alpha, i) \cdot (i'_\nu, i') = (i_{i'(\nu)} \cdot i'_\nu, i \cdot i') : \boldsymbol{X}' \longrightarrow \boldsymbol{X}'$ is equivalent to the identity morphism $(1_{X'_\nu}, 1_N) : \boldsymbol{X}' \longrightarrow \boldsymbol{X}'$. Hence, $\boldsymbol{i} \cdot \boldsymbol{i}' = 1_{\boldsymbol{X}'}$.

For each index $\alpha \in A$ we clearly have

$$p_{i'(i(\alpha))\alpha} \cdot i'_{i(\alpha)} \cdot i_\alpha = 1_{X_\alpha} \cdot 1_{X_\alpha} \cdot 1_{X_\alpha} = 1_{X_\alpha}.$$

The composition $(i'_\nu, i') \cdot (i_\alpha, i) = (i'_{i(\alpha)} \cdot i_\alpha, i' \cdot i) : \boldsymbol{X} \longrightarrow \boldsymbol{X}$ is equivalent to the identity morphism $(1_{X_\alpha}, 1_A) : \boldsymbol{X} \longrightarrow \boldsymbol{X}$. Hence, $\boldsymbol{i}' \cdot \boldsymbol{i} = 1_{\boldsymbol{X}}$.

Let $\nu = (\alpha, \beta) \in N$. By direction of the set A there exists an index $\alpha' \geq \alpha_0, \psi(\beta)$. Clearly,

$$\nu' = (\alpha', \psi(\beta)) \geq j(j'(\nu)) = (\alpha_0, \psi(\beta)), \ \ \nu = (\alpha, \beta).$$

On the other hand

$$q'_{j(j'(\nu))\nu'} \cdot j_{j'(\nu)} \cdot j'_\nu = 1_{Y_{\psi(\beta)}} \cdot q_{\beta\psi(\beta)} \cdot 1_{Y_\beta} = q_{\beta\psi(\beta)} \cdot 1_{Y_\beta} = q'_{\nu\nu'} \cdot 1_{Y'_\nu}.$$

Consequently, the composition $(j_\beta, j) \cdot (j'_\nu, j') = (j_{j'(\nu)} \cdot j'_\nu, j \cdot j') : \boldsymbol{Y}' \longrightarrow \boldsymbol{Y}'$ and the identity morphism $(1_{Y'_\nu}, 1_N) : \boldsymbol{Y}' \longrightarrow \boldsymbol{Y}'$ are equivalent. Thus $\boldsymbol{j} \cdot \boldsymbol{j}' = 1_{\boldsymbol{Y}'}$.

For each index $\beta \in B$ we have $\psi(\beta) = j'(j(\beta)) \geq \beta$. Since we have

$$j'_{j(\beta)} \cdot j_\beta = 1_{Y_{\psi(\beta)}} \cdot q_{\beta\psi(\beta)} = q_{\beta\psi(\beta)} \cdot 1_{Y_\beta} = q_{\beta j'(j(\beta))} \cdot 1_{Y_\beta}$$

the composition $(j'_\nu, j') \cdot (j_\beta, j) = (j'_{j(\beta)} \cdot j_\beta, j' \cdot j) : \boldsymbol{Y} \longrightarrow \boldsymbol{Y}$ and the identity morphism $(1_{Y_\beta}, 1_B) : \boldsymbol{Y} \longrightarrow \boldsymbol{Y}$ are equivalent. Thus, $\boldsymbol{j}' \cdot \boldsymbol{j} = 1_{\boldsymbol{Y}}$.

Now show that $\boldsymbol{j}' \cdot \boldsymbol{f}' = \boldsymbol{f} \cdot \boldsymbol{i}'$. It is clear that

$$\boldsymbol{j}' \cdot \boldsymbol{f}' = \big[(j'_\nu, j')\big] \cdot \big[(f'_\nu, 1_N)\big] = \big[(j'_\nu, j') \cdot (f'_\nu, 1_N)\big],$$
$$\boldsymbol{f} \cdot \boldsymbol{i}' = \big[(f_\alpha, \varphi)\big] \cdot \big[(i'_\nu, i')\big] = \big[(f_\alpha, \varphi) \cdot (i'_\nu, i')\big].$$

For each index $(\alpha, \beta) \in N$, which by definition satisfies the condition $\varphi(\alpha) \leq \beta$, hold equalities

$$j'_\nu \cdot f'_\nu = 1_{Y_\beta} \cdot q_{\varphi(\alpha)\beta} \cdot f_\alpha, \quad q_{\varphi(i'(\nu))j'(\nu)} \cdot f_{i'(\nu)} \cdot i'_\nu = q_{\varphi(\alpha)\beta} \cdot f_\alpha \cdot 1_{X_\alpha}.$$

Thus,

$$j'_\nu \cdot f'_\nu = q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\varphi(i'(\nu))j'(\nu)} \cdot f_{i'(\nu)} \cdot i'_\nu.$$

Hence it follows that

$$(j'_\nu, j') \cdot (f'_\nu, 1_N) \sim (f_\alpha, \varphi) \cdot (i'_\nu, i),$$

i.e. $\boldsymbol{j}' \cdot \boldsymbol{f} = \boldsymbol{f} \cdot \boldsymbol{i}'$. □

Theorem 13. *A morphism $\boldsymbol{f} : \boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A) \longrightarrow \boldsymbol{Y} = (Y_\alpha, q_{\alpha\alpha'}, A)$ of* **inj**–$\boldsymbol{\mathcal{T}}$ *given by a special morphism $(f_\alpha, 1_A) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is an isomorphism of* **inj**–$\boldsymbol{\mathcal{T}}$ *of and only if each index $\alpha \in A$ admits an index $\alpha' \geq \alpha$ and a morphism $g_\alpha : Y_\alpha \longrightarrow X_{\alpha'}$ of $\boldsymbol{\mathcal{T}}$ such that $g_\alpha \cdot f_\alpha = p_{\alpha\alpha'}$ and $f_{\alpha'} \cdot g_\alpha = q_{\alpha\alpha'}$.*

Proof. First we assume that $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is an isomorphism. Then there exists a morphism $\boldsymbol{h} : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ such that $\boldsymbol{f} \cdot \boldsymbol{h} = 1_{\boldsymbol{Y}}$ and $\boldsymbol{h} \cdot \boldsymbol{f} = 1_{\boldsymbol{X}}$. Let $(h_\alpha, \varphi) : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ be a representative of morphism $\boldsymbol{h}$. It is clear that $(f_\alpha, 1_A) \cdot (h_\alpha, \varphi) \sim (1_{Y_\alpha}, 1_A)$ and $(h_\alpha, \varphi) \cdot (f_\alpha, 1_A) \sim (1_{X_\alpha}, 1_A)$. For each index $\alpha \in A$ there exists an index $\alpha' \geq \alpha, \varphi(\alpha)$ such that

$$q_{\varphi(\alpha)\alpha'} \cdot f_{\varphi(\alpha)} \cdot h_\alpha = q_{\alpha\alpha'}, \quad q_{\varphi(\alpha)\alpha'} \cdot h_\alpha \cdot f_\alpha = p_{\alpha\alpha'}, \quad q_{\varphi(\alpha)\alpha'} \cdot f_{\varphi(\alpha)} = f_{\alpha'} \cdot p_{\varphi(\alpha)\alpha'}.$$

Let $g_\alpha = p_{\varphi(\alpha)\alpha'} \cdot h_\alpha : Y_\alpha \longrightarrow X_{\alpha'}$. Thus we get $g_\alpha \cdot f_\alpha = p_{\alpha\alpha'}$ and $f_{\alpha'} \cdot g_\alpha = g_{\alpha\alpha'}$.

Conversely, assume there exist an index $\alpha' = \psi(\alpha) \geq \alpha$ and morphism $g_\alpha : Y_\alpha \longrightarrow X_{\alpha'} = X_{\psi(\alpha)}$ such that the conditions of theorem are satisfied. By this way is defined a function $\psi : A \longrightarrow A$. Now we prove that the family (g_α, ψ) is a morphism of $\boldsymbol{Y}$ to $\boldsymbol{X}$. Note that if $\alpha'' \geq \psi(\alpha)$, then $p_{\psi(\alpha)\psi(\alpha'')} \cdot g_\alpha = g_{\alpha''} \cdot q_{\alpha\alpha''}$ because hold the following equalities

$$f_{\psi(\alpha)} \cdot g_\alpha = q_{\alpha\varphi(\alpha)}, \quad q_{\psi(\alpha)\alpha''} \cdot f_{\psi(\alpha)} = f_{\alpha''} \cdot p_{\psi(\alpha)\alpha''}, \quad g_{\alpha''} \cdot f_{\alpha''} = p_{\alpha''\psi(\alpha'')}.$$

For a pair $\alpha \leq \alpha'$ we choose an index $\alpha'' \geq \psi(\alpha), \psi(\alpha')$. By definition of function ψ we have $\psi(\alpha'') \geq \alpha''$ and $p_{\psi(\alpha')\psi(\alpha'')} \cdot g_{\alpha'} = g_{\alpha''} \cdot q_{\alpha'\alpha''}$.

Consequently,

$$p_{\psi(\alpha)\psi(\alpha'')} \cdot g_{\alpha'} = g_{\alpha''} \cdot q_{\alpha\alpha''} = g_{\alpha''} \cdot q_{\alpha'\alpha''} \cdot q_{\alpha\alpha'} = p_{\psi(\alpha')\psi(\alpha'')} \cdot g_{\alpha'} \cdot q_{\alpha\alpha'}.$$

Thus, $(g_\alpha, \varphi) : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ is a morphism in **dir**–$\boldsymbol{\mathcal{T}}$. Let $\boldsymbol{g} = [(g_\alpha, \varphi)] : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$. It is clear that $\boldsymbol{g} \cdot \boldsymbol{f} = 1_{\boldsymbol{X}}$ and $\boldsymbol{f} \cdot \boldsymbol{g} = 1_{\boldsymbol{Y}}$. □

Let $\mathbf{A}$ be a category satisfying the following conditions:

i) There is a small subcategory $\mathbf{A}'$ of $\mathbf{A}$ such that for each object $\alpha \in \mathbf{A}$ there exist an object $\alpha' \in \mathbf{A}'$ and a morphism $u : \alpha \longrightarrow \alpha'$.

ii) For each two objects α', $\alpha'' \in \mathbf{A}$ there exist an object $\alpha \in \mathbf{A}'$ and morphisms $u' : \alpha' \longrightarrow \alpha$ and $u'' : \alpha'' \longrightarrow \alpha$.

iii) For each two morphisms u', $u'' : \alpha \longrightarrow \alpha'$ there exist an object α'' and morphism $u : \alpha' \longrightarrow \alpha''$ such that $u \cdot u' = u \cdot u''$.

A generalized direct system in $\boldsymbol{\mathcal{T}}$ is defined as covariant functor $\mathbf{X}$: $\mathbf{A} \longrightarrow \boldsymbol{\mathcal{T}}$ of the category $\mathbf{A}$ to the category $\boldsymbol{\mathcal{T}}$. For each object $\alpha \in \mathbf{A}$ we have an object $\boldsymbol{X}(\alpha) = X_\alpha$ and for each morphism $u' : \alpha \longrightarrow \alpha'$ of $\mathbf{A}$ we have bonding morphism $\boldsymbol{X}(u) = p_u : X_\alpha \longrightarrow X_{\alpha'}$. A generalized direct system $\boldsymbol{X}$ we denote by $\boldsymbol{X} = (X_\alpha, p_u, \mathbf{A})$. It is clear that for each bonding morphisms $p_u : X_\alpha \longrightarrow X_{\alpha'}$ and $p_{u'} : X_{\alpha'} \longrightarrow X_{\alpha''}$ the composition $p_{u'} \cdot p_u : X_\alpha \longrightarrow X_{\alpha''}$ is bonding morphism $p_{u' \cdot u} : X_\alpha \longrightarrow X_{\alpha''}$ and for each two bonding morphisms $p_{u_1} : X_\alpha \longrightarrow X_{\alpha'}$, $p_{u_2} : X_\alpha \longrightarrow X_{\alpha'}$ there is a bonding morphism $p_{u'} : X_{\alpha'} \longrightarrow X_{\alpha''}$ such that $p_{u'} \cdot p_{u_1} = p_{u'} \cdot p_{u_2}$.

A morphism of generalized direct system $\boldsymbol{X} = (X_\alpha, p_u, \mathbf{A})$ to a generalized direct system $\boldsymbol{Y} = (Y_\beta, q_v, \mathbf{B})$ is a family $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ consisting of a function $\varphi : \mathbf{A} \longrightarrow \mathbf{B}$ which maps the objects of $\mathbf{A}$ into the objects of $\mathbf{B}$ and morphisms $f_\alpha : X_\alpha \longrightarrow Y_{\varphi(\alpha)}$, $\alpha \in \mathbf{A}$, such that whenever $u : \alpha \longrightarrow \alpha'$ is a morphism in $\mathbf{A}$, then there is an index β in $\mathbf{B}$ and there are morphisms $v : \varphi(\alpha) \longrightarrow \beta$ and $v' : \varphi(\alpha') \longrightarrow \beta$ in $\mathbf{B}$ such that $q_v \cdot f_\alpha = q_{v'} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'}$.

The composition of morphisms generalized direct systems $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $(g_\beta, \psi) : \boldsymbol{Y} \longrightarrow \boldsymbol{Z} = (Z_\gamma, r_w, \mathbf{C})$ are defined in usual manner: $(g_\beta, \psi) \cdot (f_\alpha, \varphi) = (g_{\varphi(\alpha)} \cdot f_\alpha, \psi \cdot \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Z}$. The family $(1_{X_\alpha}, 1_{\mathbf{A}})$ is the identity morphism of $\boldsymbol{X}$ inself. By **Dir**–$\boldsymbol{\mathcal{T}}$ denote the obtained category.

We say that morphisms $(f_\alpha, \varphi), (g_\alpha, \psi) : \boldsymbol{X} = (X_\alpha, p_u, \mathbf{A}) \longrightarrow \boldsymbol{Y} = (Y_\beta, q_v, \mathbf{B})$ are equivalent morphisms and write $(f_\alpha, \varphi) \sim (q_\alpha, \psi)$ if for each $\alpha \in A$ there is an object β in $\mathbf{B}$ and there are morphisms $v : \varphi(\alpha) \longrightarrow \beta$ and $v' : \psi(\alpha) \longrightarrow \beta$ such that $q_v \cdot f_\alpha = q_{v'} \cdot g_\alpha$.

As before we can prove that this relation is an equivalence relation. Composition of equivalence classes is defined by composing representatives. It is easy to see that the generalized systems in $\boldsymbol{\mathcal{T}}$ and the equivalence classes of morphisms of the category **Dir**–$\boldsymbol{\mathcal{T}}$ form a category which we denote by **Inj**–$\boldsymbol{\mathcal{T}}$.

As before we can show that if $\mathbf{A}'$ is a cofinal subcategory of category $\mathbf{A}$ then a subsystem $\boldsymbol{X}' = (X_\alpha, p_u, \mathbf{A}')$ of generalized direct system $\boldsymbol{X} =$

$(X_\alpha, p_u, \mathbf{A})$ is isomorphic to $\boldsymbol{X}$ in the category $\mathbf{Inj}$–$\boldsymbol{\mathcal{T}}$. Thus every generalized direct system is isomorphic to a generalized direct system indexed by a small category with properties ii) and iii).

Proposition 14. *Let* $\mathbf{A}$ *be a category with properties* ii) *and* iii) *and let* $v_i', v_i'' : \alpha_i \longrightarrow \alpha'$, $i = 1, 2, \ldots, n$ *be morphisms in* $\mathbf{A}$. *Then there is a morphism* $u' : \alpha' \longrightarrow \alpha''$ *such that* $u' \cdot v_i' = u' \cdot v_i''$, $i = 1, 2, \ldots, n$.

Proof. Indeed, by the condition iii) there exist morphisms $v_i : \alpha' \longrightarrow \alpha_i'$, $i = 1, 2, \ldots, n$, such that $v_i \cdot v_i' = v_i \cdot v_i''$, $i = 1, 2, \ldots, n$. By the condition ii) also exist morphisms $\widetilde{v}_i : \alpha_i' \longrightarrow \alpha^*$, $i = 1, 2, \ldots, n$. Applying again the condition iii) we conclude that there exists a morphism $v : \alpha^* \longrightarrow \alpha''$ for which

$$v \cdot (\widetilde{v}_1 \cdot v_1) = v \cdot (\widetilde{v}_2 \cdot v_2) = \cdots = v \cdot (\widetilde{v}_n \cdot v_n) = u'.$$

Thus, $u' \cdot v_i' = u' \cdot v_i''$, $i = 1, 2, \ldots, n$. □

A finite diagram β in the category $\mathbf{A}$ we call a diagram β consists of a finite set of objects of $\mathbf{A}$ and a finite set of morphisms of $\mathbf{A}$ between these objects. We say that an object $\alpha_0 \in \beta$ is a maximal object of β if for each $\alpha \in \beta$ there are unique morphisms $u_\alpha : \alpha \longrightarrow \alpha_0$. Note that $u_{\alpha_0} = 1_{\alpha_0}$. The diagram β is called a commutative diagram at a maximal object α_0 if for any morphism $u : \alpha \longrightarrow \alpha'$ of β holds equality $u_{\alpha'} \cdot u = u_\alpha$.

We have the following useful result.

Theorem 15. *Let* $\boldsymbol{X}$ *be a generalized direct system of* $\mathbf{Inj}$–$\boldsymbol{\mathcal{T}}$. *Then there exists isomorphic to* $\boldsymbol{X}$ *in* $\mathbf{Inj}$–$\boldsymbol{\mathcal{T}}$ *a direct system* $\boldsymbol{Y}$ *such that the index set of* $\boldsymbol{Y}$ *is a directed cofinite ordered set, each term and bonding morphism of* $\boldsymbol{Y}$ *are term and bonding morphism of* $\boldsymbol{X}$, *respectively.*

Proof. We can assume that $\boldsymbol{X}$ is indexed by a small category with properties ii) and iii). Consider an order relation $\leq$ on the set B of all finite diagrams β in $\mathbf{A}$ which are commutative at some maximal object $\max \beta$:

$$\beta \leq \beta' \;\Leftrightarrow\; \beta \subseteq \beta', \quad \beta, \beta' \in B.$$

It is clear that the set B with the order $\leq$ is a cofinite ordered set. Now show that B also is a directed set.

Let β_1 and β_2 be diagrams in $\mathbf{B}$ and let α_1 and α_2 be maximal objects of β_1 and β_2, respectively. By condition ii) there is an object $\alpha' \in \mathbf{A}$ and there are morphisms $u_1 : \alpha_1 \longrightarrow \alpha'$ and $u_2 : \alpha_2 \longrightarrow \alpha'$. If $\alpha_i = \alpha'$, then we put $u_i = 1_{\alpha_i}$. If an object $\alpha \in \beta_1 \cap \beta_2$, then we have two morphisms in $\mathbf{A}$

$$v_\alpha' = u_1 \cdot u_{\alpha_1},\; v_\alpha'' = u_2 \cdot u_{\alpha_2} : \alpha \longrightarrow \alpha'.$$

There exist an object α'' and a morphism $u' : \alpha' \longrightarrow \alpha''$ for which $(u' \cdot u_1) \cdot u_{\alpha_1} = (u' \cdot u_2) \cdot u_{\alpha_2}$ for all $\alpha \in \beta_1 \cap \beta_2$. Let $v_1 = u' \cdot u_1 : \alpha_1 \longrightarrow \alpha''$ and $v_2 = u' \cdot u_2 : \alpha_2 \longrightarrow \alpha''$. Consequently we have two morphisms $v_1 : \alpha_1 \longrightarrow \alpha''$ and $v_2 : \alpha_2 \longrightarrow \alpha''$ with $v_1 \cdot u_{\alpha_1} = v_2 \cdot u_{\alpha_2}$ for each $\alpha \in \beta_1 \cap \beta_2$. In the role of object $\beta \geq \beta_1, \beta_2$ we can take the diagram whose objects are all objects of $\beta_1 \cup \beta_2$ and the object α'', and whose morphisms are all the morphisms of $\beta_1 \cup \beta_2$ and the morphisms $1_{\alpha''}$, $v_1 \cdot u_{\alpha_1}$ for $\alpha \in \beta_1$ and $v_2 \cdot u_{\alpha_2}$ for $\alpha \in \beta_2$. It is clear that β is a finite diagram with maximal object α''. Besides, it also is commutative at α''. Indeed, if $u : \alpha \longrightarrow \alpha'$ is a morphism in β_1, then $v_1 \cdot (u_{\alpha'_1} \cdot u) = v_1 \cdot u_{\alpha_1}$, because $u_{\alpha_1} = u_{\alpha'_1} \cdot u$. Analogously, if $u : \alpha \longrightarrow \alpha'$ is a morphism in β_2, then from equality $u_{\alpha_2} = u_{\alpha'_2} \cdot u$ it follows that $v_2 \cdot u_{\alpha_2} = v_2 \cdot (u_{\alpha'_2} \cdot u)$.

Define on the set $(B, \leq)$ a direct system $\boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$. Let $Y_\beta = X_{\max\beta}$. Let $\beta \leq \beta'$. Then $\max\beta \in \beta'$ and, consequently, there exists unique morphism $u : \max\beta \longrightarrow \max\beta'$. Let $p_u : X_{\max\beta} \longrightarrow X_{\max\beta'}$ be the corresponding to u morphism in $\boldsymbol{X}$. Assume that $q_{\beta\beta'} = p_u$. Note that for each triple $\beta \leq \beta' \leq \beta''$

$$p_{\beta'\beta''} \cdot p_{\beta\beta'} = p_{\beta\beta''}.$$

Now define morphisms $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $(g_\beta, \psi) : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$, where

$$\varphi(\alpha) = \{\alpha\}, \quad \alpha \in A;$$
$$\psi(\beta) = \max\beta, \quad \beta \in B;$$
$$f_\alpha = 1_{X_\alpha} : X_\alpha \longrightarrow Y_{\varphi(\alpha)} = X_\alpha, \quad \alpha \in A;$$
$$g_\beta = 1_{X_{\max\beta}} : Y_\beta = X_{\max\beta} \longrightarrow X_{\psi(\beta)} = X_{\max\beta}, \quad \beta \in B.$$

These morphisms satisfy the following conditions

$$(f_\alpha, \varphi) \circ (g_\beta, \psi) \sim (1_{Y_\beta}, 1_B), \quad (g_\beta, \psi) \cdot (f_\alpha, \varphi) \sim (1_{X_\alpha}, 1_{\mathbf{A}}).$$

Hence, $\boldsymbol{f} \cdot \boldsymbol{g} = 1_{\boldsymbol{Y}}$ and $\boldsymbol{g} \cdot \boldsymbol{f} = 1_{\boldsymbol{X}}$. □

3. Abstract Coshape Category

In this section we introduce the foundations of abstract coshape theory.

Let $\boldsymbol{\mathcal{P}}$ be a full subcategory of category $\boldsymbol{\mathcal{T}}$. Now we define a dual version of expansion of object ([24], Ch. I, §2.1).

Let X be an object of the category $\boldsymbol{\mathcal{T}}$. A $\boldsymbol{\mathcal{T}}$– coexpansion of X is a morphism $\boldsymbol{p} : \boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A) \longrightarrow (X)$ in **inj**–$\boldsymbol{\mathcal{T}}$ of direct system $\boldsymbol{X}$ in the category $\boldsymbol{\mathcal{T}}$ to direct system (X) with the condition:

For each direct system $\boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ in the subcategory $\boldsymbol{\mathcal{P}}$ and each morphism $g : \boldsymbol{Y} \longrightarrow (X)$ in **inj**–$\boldsymbol{\mathcal{T}}$ there exists a unique morphism $\boldsymbol{f} : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ in **inj**–$\boldsymbol{\mathcal{T}}$ such that $\boldsymbol{p} \cdot \boldsymbol{f} = \boldsymbol{g}$.

If $\boldsymbol{X}$ and $\boldsymbol{f}$ are object and morphism of **inj**–$\boldsymbol{\mathcal{P}}$ then we say that $\boldsymbol{p}$ is a $\boldsymbol{\mathcal{P}}$–coexpansion of X. In this case we also say that $\boldsymbol{X}$ is coassociated with X.

Note that if $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ and $\boldsymbol{p}' : \boldsymbol{X}' \longrightarrow (X)$ are two $\boldsymbol{\mathcal{P}}$–coexpansions of object $X \in \boldsymbol{\mathcal{T}}$ then there is an isomorphism $\boldsymbol{i} : \boldsymbol{X} \longrightarrow \boldsymbol{X}'$ of the category **inj**–$\boldsymbol{\mathcal{P}}$.

The following theorem gives necessary and sufficient conditions for $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ to be a $\boldsymbol{\mathcal{T}}$–coexpansion ($\boldsymbol{\mathcal{P}}$–coexpansion).

Theorem 16. *Let* $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A) \in$ **inj**–$\boldsymbol{\mathcal{T}}$ (**inj**–$\boldsymbol{\mathcal{P}}$). *A morphism* $\boldsymbol{p} = [(p_\alpha)] : \boldsymbol{X} \longrightarrow (X)$ *is a* $\boldsymbol{\mathcal{T}}$*–coexpansion* ($\boldsymbol{\mathcal{P}}$*–coexpansion*) *if and only if the morphisms* $p_\alpha : X_\alpha \longrightarrow X$, $a \in A$ *satisfy the following conditions:*

CAE1) *For arbitrary morphism* $h : P \longrightarrow X$ *in* $\boldsymbol{\mathcal{T}}$, $P \in \boldsymbol{\mathcal{P}}$, *there exist an index* $a \in A$ *and a morphism in* $\boldsymbol{\mathcal{T}}$ (*in* $\boldsymbol{\mathcal{P}}$) $f : P \longrightarrow X_\alpha$ *for which* $h = p_\alpha \cdot f$.

CAE2) *If for morphisms* $f, f' : P \longrightarrow X_\alpha$ *holds equality* $p_\alpha \cdot f = p_\alpha \cdot f'$, *then there exists an index* $\alpha' \geq \alpha$ *such that* $p_{\alpha\alpha'} \cdot f = p_{\alpha\alpha'} \cdot f'$.

Proof. Necessity. Let $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ be a $\boldsymbol{\mathcal{P}}$–coexpansion of X and let $h : P \longrightarrow X$ be a arbitrary morphism of the category $\boldsymbol{\mathcal{T}}$. We can consider h as a morphism $\boldsymbol{h} = [(h)] : (P) \longrightarrow (X)$ of the category **inj**–$\boldsymbol{\mathcal{T}}$. By assumption there exists a morphism $\boldsymbol{f} : (P) \longrightarrow \boldsymbol{X}$ such that $\boldsymbol{p} \cdot \boldsymbol{f} = \boldsymbol{h}$. It is clear that the morphism $\boldsymbol{f}$ is given by some morphism $f : P \longrightarrow X_\alpha$, $\alpha \in A$. The representatives $p_\alpha \cdot f$ and h of $\boldsymbol{p} \cdot \boldsymbol{f}$ and $\boldsymbol{h}$ are equivalent morphisms. Consequently, $h = p_\alpha \cdot f$.

Now we assume that $f, f' : P \longrightarrow X_\alpha$ are morphisms such that $p_\alpha \cdot f = p_\alpha \cdot f'$. Let $\boldsymbol{f}, \boldsymbol{f}' : (P) \longrightarrow \boldsymbol{X}$ be morphisms induced by morphisms f and f', respectively. It is clear that $\boldsymbol{p} \cdot \boldsymbol{f} = \boldsymbol{p} \cdot \boldsymbol{f}'$. By uniqueness, it follows that $\boldsymbol{f} = \boldsymbol{f}'$. Consequently, f and f' are equivalent morphisms. Hence, there exists an index $\alpha' \geq \alpha$ for which $p_{\alpha\alpha'} \cdot f = p_{\alpha\alpha'} \cdot f'$.

Sufficiency. Assume that $\boldsymbol{X} \in$ **inj**–$\boldsymbol{\mathcal{P}}$ and the morphism $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ satisfies the conditions CAE1) and CAE2). We will show that $\boldsymbol{p}$ is a $\boldsymbol{\mathcal{P}}$–coexpansion. Consider a arbitrary morphism $\boldsymbol{h} = [(h_\beta)] : \boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B) \longrightarrow (X)$. By condition CAE1) for each index $\beta \in B$ there exist an index $\alpha \in A$ and a morphism $f_\beta : Y_\beta \longrightarrow X_\alpha$ such that $p_\alpha \cdot f_\beta = h_\beta$. Let $\beta = \varphi(\alpha)$. This correspondence defines a function $\varphi : B \longrightarrow A$. For each pair $\beta \leq \beta'$ we have

$$p_{\varphi(\beta)} \cdot f_\beta = h_\beta = h_{\beta'} \cdot q_{\beta\beta'} = p_{\varphi(\beta')} \cdot f_{\beta'} \cdot q_{\beta\beta'}.$$

If $\alpha \geq \varphi(\beta), \varphi(\beta')$, then

$$p_\alpha \cdot p_{\varphi(\beta)\alpha} \cdot f_\beta = p_\alpha \cdot p_{\varphi(\beta')\alpha} \cdot f_{\beta'} \cdot q_{\beta\beta'}.$$

By condition CAE2) there exists an index $\alpha' \geq \alpha$ such that

$$p_{\alpha\alpha'} \cdot p_{\varphi(\beta)\alpha} \cdot f_\beta = p_{\alpha\alpha'} \cdot p_{\varphi(\beta')\alpha} \cdot f_{\beta'} \cdot q_{\beta\beta'},$$

i.e. $p_{\varphi(\beta)\alpha'} \cdot f_\beta = p_{\varphi(\beta')\alpha'} \cdot f_{\beta'} \cdot q_{\beta\beta'}$.

Consequently, the family (f_α, φ) is a morphism of the category **dir**–$\boldsymbol{\mathcal{T}}$ and it induces a morphism $\boldsymbol{f} : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ of the category **inj**–$\boldsymbol{\mathcal{P}}$. It is clear that $\boldsymbol{p} \cdot \boldsymbol{f} = \boldsymbol{h}$. Assume that there exists another morphism $\boldsymbol{f}' = [(f'_\beta, \varphi')] : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ with this property. Note that

$$p_{\varphi(\beta)} \cdot f_\beta = h_\beta = p_{\varphi'(\beta)} \cdot f'_\beta.$$

Let $\alpha' \geq \varphi(\beta), \varphi'(\beta)$. It is clear that

$$p_{\alpha'} \cdot p_{\varphi(\beta)\alpha'} \cdot f_\beta = p_{\varphi(\beta)} \cdot f_\beta = p_{\varphi'(\beta)} \cdot f'_\beta = p_{\alpha'} \cdot q_{\varphi'(\beta)\alpha'} \cdot f'_\beta.$$

By condition CAE2) there exists an index $\alpha'' \geq \alpha$ such that

$$p_{\alpha'\alpha''} \cdot p_{\varphi(\beta)\alpha'} \cdot f_\beta = p_{\alpha'\alpha''} \cdot p_{\varphi'(\beta)\alpha'} \cdot f'_\beta.$$

Thus we have $p_{\varphi(\beta)\alpha''} \cdot f_\beta = p_{\varphi'(\beta)\alpha''} \cdot f'_\beta$. Consequently, $(f_\beta, \varphi) \sim (f'_\beta, \varphi')$. Hence, $\boldsymbol{f} = \boldsymbol{f}'$.

In case when $\boldsymbol{p}$ is $\boldsymbol{\mathcal{T}}$–coexpansion the proof is similarly. □

A subcategory $\boldsymbol{\mathcal{P}} \subset \boldsymbol{\mathcal{T}}$ is called a codence subcategory of category $\boldsymbol{\mathcal{T}}$ provided each object $X \in \boldsymbol{\mathcal{T}}$ admits a $\boldsymbol{\mathcal{P}}$–coexpansion.

Let $X^{\boldsymbol{\mathcal{P}}}$ be the category whose objects are all morphisms $f : P \longrightarrow X$, $P \in \boldsymbol{\mathcal{P}}$ and whose morphisms $u : f \longrightarrow f' : P' \longrightarrow X$ are all morphisms $u : P \longrightarrow P'$ in $\boldsymbol{\mathcal{P}}$ such that $f = f' \cdot u$.

Note that the category $X^{\boldsymbol{\mathcal{P}}}$ satisfies the condition i) if and only if for each morphism $f : P \longrightarrow X$, $P \in \boldsymbol{\mathcal{P}}$ in $\boldsymbol{\mathcal{T}}$, there exist a morphism $f' : P' \longrightarrow X$, $P' \in \boldsymbol{\mathcal{P}}$ in $\boldsymbol{\mathcal{T}}$ and a morphism $u : P \longrightarrow P'$ in $\boldsymbol{\mathcal{P}}$ with $f' \cdot u = f$.

Also note that the category $X^{\boldsymbol{\mathcal{P}}}$ satisfy the conditions ii) and iii) if and only if it has the following two properties:

For each two morphisms $f_1 : P_1 \longrightarrow X$, $P_1 \in \boldsymbol{\mathcal{P}}$ and $f_2 : P_2 \longrightarrow X$, $P_2 \in \boldsymbol{\mathcal{P}}$ in $\boldsymbol{\mathcal{T}}$, there is a morphism $f : P \longrightarrow X$, $P \in \boldsymbol{\mathcal{P}}$ in $\boldsymbol{\mathcal{T}}$ and there are morphisms $u_1 : P_1 \longrightarrow P$, $u_2 : P_2 \longrightarrow P$ in $\boldsymbol{\mathcal{P}}$ such that $f \cdot u_1 = f_1$ and $f \cdot u_2 = f_2$.

If $f : P' \longrightarrow X$ and $u_1, u_2 : P \longrightarrow P'$, $P, P' \in \boldsymbol{\mathcal{P}}$, are morphism in $\boldsymbol{\mathcal{T}}$ and in $\boldsymbol{\mathcal{P}}$, respectively, and $f \cdot u_1 = f \cdot u_2$, then there exist a morphism $f' : P'' \longrightarrow X$, $P'' \in \boldsymbol{\mathcal{P}}$ in $\boldsymbol{\mathcal{T}}$ and a morphism $u : P' \longrightarrow P''$ in $\boldsymbol{\mathcal{P}}$ for which $f' \cdot u = f$ and $u \cdot u_1 = u \cdot u_2$.

We have the following theorem.

Theorem 17. *A subcategory $\boldsymbol{\mathcal{P}} \subseteq \boldsymbol{\mathcal{T}}$ is codence subcategory of the category $\boldsymbol{\mathcal{T}}$ if and only if for each object $X \in \boldsymbol{\mathcal{T}}$ the category $X^{\boldsymbol{\mathcal{P}}}$ satisfies the conditions* i), ii) *and* iii).

Proof. Let $\boldsymbol{\mathcal{P}}$ be a codence subcategory of the category $\boldsymbol{\mathcal{T}}$. For each object X of $\boldsymbol{\mathcal{T}}$ there exists a $\boldsymbol{\mathcal{P}}$–coexpansion $\boldsymbol{p} : \boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A) \longrightarrow (X)$, where A is a directed set. For each morphism $f : P \longrightarrow X$, where $P \in \boldsymbol{\mathcal{P}}$, there exist an index $\alpha \in A$ and a morphism $u : P \longrightarrow X_\alpha$ such that $f = p_\alpha \cdot u$. Note that the family $\{X_\alpha,\ \alpha \in A\}$ is a set. Let $\boldsymbol{\mathcal{P}}'$ be the full subcategory of the category $\boldsymbol{\mathcal{P}}$ whose objects class is a set $\{X_\alpha,\ \alpha \in A\}$. It is clear that $\boldsymbol{\mathcal{P}}'$ is a small category. The category $X^{\boldsymbol{\mathcal{P}}}$ satisfies the condition i).

For each morphisms $f_1 : P_1 \longrightarrow X$ and $f_2 : P_2 \longrightarrow X$ there exist morphisms $u'_1 : P_1 \longrightarrow X_{\alpha_1}$ and $u'_2 : P_2 \longrightarrow X_{\alpha_2}$ such that $p_{\alpha_1} \cdot u'_1 = f_1$ and $p_{\alpha_2} \cdot u'_2 = f_2$. Let $\alpha \geq \alpha_1, \alpha_2$. From equalities $p_\alpha \cdot p_{\alpha_1\alpha} = p_{\alpha_1}$ and $p_\alpha \cdot p_{\alpha_2\alpha} = p_{\alpha_2}$ it follows that $p_\alpha(p_{\alpha_1\alpha} \cdot u'_1) = f_1$ and $p_\alpha(p_{\alpha_2\alpha} \cdot u'_2) = f_2$.

Let $p_{\alpha_1\alpha} \cdot u_1' = u_1$ and $p_{\alpha_2\alpha} \cdot u_2' = u_2$. Thus, $p_\alpha \cdot u_1 = f_1$ and $p_\alpha \cdot u_2 = f_2$. Consequently, the category $X^{\boldsymbol{\mathcal{P}}}$ satisfies the condition ii).

Consider morphisms $f : P' \longrightarrow X$ and $u_1, u_2 : P \longrightarrow P'$, $P \in \boldsymbol{\mathcal{P}}$ with $f \cdot u_1 = f \cdot u_2$. Since $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ is a $\boldsymbol{\mathcal{P}}$–coexpansion there exists a morphism $v : P' \longrightarrow X_\alpha$ such that $p_\alpha \cdot v = f$. It is clear that

$$p_\alpha \cdot v \cdot u_i = f \cdot u_1 = f \cdot u_2 = p_\alpha \cdot v \cdot u_2.$$

Besides, there is an index $\alpha' \geq \alpha$ such that

$$p_{\alpha\alpha'} \cdot v \cdot u_1 = p_{\alpha\alpha'} \cdot v \cdot u_2.$$

Let $P'' = X_{\alpha'}$, $f' = p_{\alpha'}$ and $u = p_{\alpha\alpha'} \cdot v$. Thus we get

$$f' \cdot u = p_{\alpha'} \cdot p_{\alpha\alpha'} \cdot v = p_\alpha \cdot v = f,$$
$$u \cdot u_1 = p_{\alpha\alpha'} \cdot v \cdot u_1 = p_{\alpha\alpha'} \cdot v \cdot u_2 = u \cdot u_2.$$

The category $X^{\boldsymbol{\mathcal{P}}}$ satisfies the condition iii).

Conversely, assume that for each object $X \in \boldsymbol{\mathcal{P}}$ the category $X^{\boldsymbol{\mathcal{P}}}$ satisfy the conditions i), ii) and iii) and show that $\boldsymbol{\mathcal{P}}$ is a codense subcategory. Consider a generalized direct system $\boldsymbol{X} = (X_\alpha, p_u, A)$, where $A = X^{\boldsymbol{\mathcal{P}}}$, $X_\alpha = P$ for each object $\alpha = f : P \longrightarrow X$ of $A = X^{\boldsymbol{\mathcal{P}}}$ and $p_u = u : X_\alpha = P \longrightarrow X_{\alpha'} = P'$ for each morphism $u : P \longrightarrow P'$ with $f' \cdot u = f$.

Let $p_\alpha = f : P = X_\alpha \longrightarrow X$ for each morphism $\alpha = f : P \longrightarrow X$. Note that for each pair $\alpha \leq \alpha'$, $p_{\alpha'} \cdot p_u = f' \cdot u = f = p_\alpha$. Thus, we have a morphism $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$. Now show that $\boldsymbol{p}$ satisfies the conditions CAE1) and CAE2).

Let $\boldsymbol{f} : P \longrightarrow X$, $P \in \boldsymbol{\mathcal{P}}$, be an arbitrary morphism. Note that $\alpha = f \in X^{\boldsymbol{\mathcal{P}}}$ and $p_\alpha = f$. Hence, $f = p_\alpha \cdot 1_P$. Thus the condition CAE1) holds.

Let $p_{\alpha'} \cdot u_1 = p_{\alpha'} \cdot u_2$ for each morphisms $u_1, u_2 : P \longrightarrow X_{\alpha'} = P'$, $P' \in \boldsymbol{\mathcal{P}}$, i.e. $f' \cdot u_1 = f' \cdot u_2$. The category $X^{\boldsymbol{\mathcal{P}}}$ satisfies the condition iii). Consequently, there are morphisms $f'' : P'' \longrightarrow X$ and $u : P' \longrightarrow P''$, $P'' \in \boldsymbol{\mathcal{P}}$ such that $f'' \cdot u = f'$, $u \cdot u_1 = u \cdot u_2$. It is clear that $p_u \cdot u_1 = p_u \cdot u_2$, where $u : \alpha' \longrightarrow \alpha'' = f'' \in A = X^{\boldsymbol{\mathcal{P}}}$, $p_u = u$. Thus the condition CAE2) holds. □

Now we define the coshape category for arbitrary category $\boldsymbol{\mathcal{T}}$ and its full codence subcategory $\boldsymbol{\mathcal{P}}$. Let $p : \boldsymbol{X} \longrightarrow (X)$, $p' : \boldsymbol{X'} \longrightarrow (X)$ and $q : \boldsymbol{Y} \longrightarrow Y$, $q' : \boldsymbol{Y'} \longrightarrow (Y)$ be $\boldsymbol{\mathcal{P}}$–coexpansions of X and Y, respectively. Then there are isomorphisms $\boldsymbol{i} : \boldsymbol{X} \longrightarrow \boldsymbol{X'}$ and $j : \boldsymbol{Y} \longrightarrow \boldsymbol{Y'}$. We say that morphisms $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{f'} : \boldsymbol{X'} \longrightarrow \boldsymbol{Y'}$ are equivalent if $\boldsymbol{f'} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{f}$. The equivalence class of $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ we denote by F and call a coshape morphism of X to Y. The composition $G \cdot F : X \longrightarrow Z$ of two coshape morphisms $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ we can define as equivalence class of morphism $\boldsymbol{g} \cdot \boldsymbol{j} \cdot \boldsymbol{f}$, where $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y'} \longrightarrow \boldsymbol{Z}$ are representatives of F and G, respectively. Let I_X be the equivalence class of the identity morphism $1_{\boldsymbol{X}} : \boldsymbol{X} \longrightarrow \boldsymbol{X}$. It is clear that $I_Y \cdot F = F \cdot I_X = F$

and $H \cdot (G \cdot F) = (H \cdot G) \cdot F$ for each coshape morphisms $F : X \longrightarrow Y$, $G : Y \longrightarrow Z$ and $H : Z \longrightarrow W$. We have obtained the abstract coshape category $\mathbf{CSH}_{(\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{P}})}$, whose objects are all objects of category $\boldsymbol{\mathcal{T}}$ and whose morphisms are all coshape morphisms.

For each morphism $f : X \longrightarrow Y$ of the category $\boldsymbol{\mathcal{T}}$ and for any $\boldsymbol{\mathcal{P}}$–coexpansions $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ and $\boldsymbol{q} : \boldsymbol{Y} \longrightarrow (Y)$ there exists a unique morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ in **inj**–$\boldsymbol{\mathcal{P}}$ such that $f \cdot \boldsymbol{p} = \boldsymbol{q} \cdot \boldsymbol{f}$. Indeed, for $\boldsymbol{\mathcal{P}}$–coexpansion $\boldsymbol{q} : \boldsymbol{Y} \longrightarrow (Y)$ and morphism $f \cdot \boldsymbol{p} : \boldsymbol{X} \longrightarrow (Y)$ there exists a unique morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ for which $\boldsymbol{q} \cdot \boldsymbol{f} = f \cdot \boldsymbol{p}$. Let $\boldsymbol{p}' : \boldsymbol{X}' \longrightarrow (X)$ and $\boldsymbol{q}' : \boldsymbol{Y} \longrightarrow (Y)$ be other $\boldsymbol{\mathcal{P}}$–coexpansions of X and Y, respectively. Then also exists a unique morphism $\boldsymbol{f}' : \boldsymbol{Y}' \longrightarrow \boldsymbol{X}'$ such that $\boldsymbol{q}' \cdot \boldsymbol{f}' = f \cdot \boldsymbol{p}'$. Using equalities $\boldsymbol{p} \cdot \boldsymbol{i} = \boldsymbol{p}'$ and $\boldsymbol{q} \cdot \boldsymbol{j} = \boldsymbol{p}'$ we obtain

$$\boldsymbol{q} \cdot (\boldsymbol{j} \cdot \boldsymbol{f}') = \boldsymbol{q}' \cdot \boldsymbol{f} = f \cdot \boldsymbol{p}' = f \cdot (\boldsymbol{p} \cdot \boldsymbol{i}) = (\boldsymbol{q} \cdot \boldsymbol{f}) \cdot \boldsymbol{i} = \boldsymbol{q} \cdot (\boldsymbol{f} \cdot \boldsymbol{i}).$$

By uniqueness, $\boldsymbol{j} \cdot \boldsymbol{f}' = \boldsymbol{f} \cdot \boldsymbol{i}$. Hence, $\boldsymbol{f} \sim \boldsymbol{f}'$. Consequently, each morphism $f : X \longrightarrow Y$ in $\boldsymbol{\mathcal{T}}$ induces a coshape morphism with representative $\boldsymbol{f}$. Let $\mathrm{CS}(f)$ denote the equivalence class of the morphism $\boldsymbol{f}$. If we put $\mathrm{CS}(X) = X$ for each object $X \in \boldsymbol{\mathcal{T}}$ then we obtain a functor CS: $\boldsymbol{\mathcal{T}} \longrightarrow \mathbf{CSH}_{(\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{P}})}$ called the coshape functor. For any $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ morphism in **inj**–$\boldsymbol{\mathcal{P}}$ there exists a unique coshape morphism $F : X \longrightarrow Y$ such that $\boldsymbol{q} \cdot \boldsymbol{f} = F \cdot \boldsymbol{p}$. If the objects X and Y are isomorphic in the coshape category $\mathbf{CSH}_{(\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{P}})}$ then we say that they have some coshape and write $\mathrm{csh}(X) = \mathrm{csh}(Y)$.

Theorem 18. *For each coshape morphism $F : P \longrightarrow X$ of $P \in \boldsymbol{\mathcal{P}}$ to $X \in \boldsymbol{\mathcal{T}}$ there exists a unique morphism $f : P \longrightarrow X$ in $\boldsymbol{\mathcal{T}}$ such that $F = \mathrm{CS}(f)$.*

Proof. The identity morphism $1_P : P \longrightarrow P$, $P \in \boldsymbol{\mathcal{P}}$, induces the $\boldsymbol{\mathcal{P}}$–coexpansion $1_P : (P) \longrightarrow P$. Let $\boldsymbol{f} : (P) \longrightarrow \boldsymbol{X}$ be a representative of coshape morphism $F : P \longrightarrow X$. It is clear that $F = F \cdot 1_P = \boldsymbol{p} \cdot \boldsymbol{f}$ and morphism $\boldsymbol{f}$ is given by a morphism $f_\alpha : P \longrightarrow X_\alpha$, where α is a some fixed index of set A. Let $f = p_\alpha \cdot f_\alpha : P \longrightarrow X$. Thus, $F = \mathrm{CS}(f)$. It is clear that f is a unique morphism. □

From Theorem 18 follows that the category $\boldsymbol{\mathcal{P}}$ and the full subcategory of the category $\mathbf{CSH}_{(\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{P}})}$, restricted to objects of $\boldsymbol{\mathcal{P}}$ are isomorphical.

Theorem 19. *Let $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ be a morphism of the category* **inj**–$\boldsymbol{\mathcal{P}}$ *and let $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ and $\boldsymbol{q} : \boldsymbol{Y} \longrightarrow (Y)$ be $\boldsymbol{\mathcal{P}}$–coexpansions of X and Y, respectively. Then the coshape morphism $F : X \longrightarrow Y$ induced by $\boldsymbol{f}$ is a unique morphism for which $\boldsymbol{q} \cdot \boldsymbol{f} = F \cdot \boldsymbol{p}$ in* **inj**–**CSH**.

Proof. We must show that for each index $\alpha \in A$ an equality $F \cdot p_\alpha = q_{\varphi(\alpha)} \cdot f_\alpha$ holds in the category $\mathbf{CSH}_{(\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{P}})}$. The composition $F \cdot p_\alpha$ as a coshape morphism is induced by a morphism $\boldsymbol{h} = \boldsymbol{f} \cdot \boldsymbol{g} : (X_\alpha) \longrightarrow \boldsymbol{Y}$, where $\boldsymbol{g} : (X_\alpha) \longrightarrow \boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ is given by $1_{X_\alpha} : X_\alpha \longrightarrow X_\alpha$. We can say that $\boldsymbol{h}$ is the morphism of the category **inj**–$\boldsymbol{\mathcal{P}}$ and it is determined by $f_\alpha : X_\alpha \longrightarrow Y_{\varphi(\alpha)}$. The composition $\boldsymbol{q}\cdot\boldsymbol{f}$ is given by $q_{\varphi(\alpha)}\cdot f_\alpha$. Now we recall that $q_{\varphi(\alpha)} \cdot f_\alpha$ as a coshape morphism is defined by $\boldsymbol{h}$. Thus, $\boldsymbol{q} \cdot \boldsymbol{f} = F \cdot \boldsymbol{p}$. The uniqueness of coshape morphism $F : X \longrightarrow Y$ follows from the next proposition. □

Proposition 20. *Let $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ be a $\boldsymbol{\mathcal{P}}$–coexpansion of X and let $F, F' : X \longrightarrow Y$ be coshape morphisms such that $F \cdot p_\alpha = F' \cdot p_\alpha$ for each index $\alpha \in A$. Then $F = F'$.*

Proof. Let $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{f}' : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ be representatives of F and F', respectively. Note that

$$\boldsymbol{q} \cdot \boldsymbol{f} = F \cdot \boldsymbol{p} = F' \cdot \boldsymbol{p} = \boldsymbol{q} \cdot \boldsymbol{f}'.$$

By definition of a coexpansion it follows that $\boldsymbol{f} = \boldsymbol{f}'$. Hence, $F = F'$. □

Proposition 21. *There exists a one-to-one correspodence between the coshape morphisms $F : X \longrightarrow Y$ and the morphisms $\boldsymbol{h} : \boldsymbol{X} \longrightarrow Y$ of* **inj**–$\boldsymbol{\mathcal{T}}$.

Proof. Let $Y \in \boldsymbol{\mathcal{T}}$ and let $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ be a $\boldsymbol{\mathcal{P}}$–coexpansion of X. For each coshape morphism $F : X \longrightarrow Y$ consider the composition $F \cdot \boldsymbol{p} : \boldsymbol{X} \longrightarrow Y$ as a morphism in **inj**–**CSH**. Since $X_\alpha \in \boldsymbol{\mathcal{P}}$, $\alpha \in A$ we can consider $F \cdot p_\alpha : X_\alpha \longrightarrow Y$, $\alpha \in A$ as a some morphism $h_\alpha : X_\alpha \longrightarrow Y$, $\alpha \in A$ of the category $\boldsymbol{\mathcal{T}}$. Note that for each pair $\alpha \leq \alpha'$

$$h_\alpha \cdot p_{\alpha\alpha'} = F \cdot p_{\alpha'} \cdot p_{\alpha\alpha'} = F \cdot p_\alpha = h_\alpha.$$

Consequently, $\boldsymbol{h} = (h_\alpha) : \boldsymbol{X} \longrightarrow Y$ is a morphism in **inj**–$\boldsymbol{\mathcal{T}}$ and $F \cdot \boldsymbol{p} = \boldsymbol{h}$.

Conversely, suppose $\boldsymbol{h} : \boldsymbol{X} \longrightarrow Y$ is a morphism of **inj**–$\boldsymbol{\mathcal{T}}$. For each $\boldsymbol{\mathcal{P}}$–coexpansion $\boldsymbol{q} : \boldsymbol{Y} \longrightarrow (Y)$ there is a morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ such that $\boldsymbol{g} \cdot \boldsymbol{f} = \boldsymbol{h}$. By Theorem 19 there exists a unique coshape morphism $F : X \longrightarrow Y$ for which $\boldsymbol{q} \cdot \boldsymbol{f} = F \cdot \boldsymbol{p}$, i.e. $\boldsymbol{h} = F \cdot \boldsymbol{p}$. By Proposition 20 F is uniquely defined by $\boldsymbol{h}$. □

Theorem 22. *Let $Y \in \boldsymbol{\mathcal{T}}$ and $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ be a $\boldsymbol{\mathcal{T}}$–coexpansion of X. For each morphism $\boldsymbol{h} : \boldsymbol{X} \longrightarrow (Y)$ of* **inj**–$\mathbf{CSH}_{(\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{P}})}$ *there exists a unique coshape morphism $F : X \longrightarrow Y$ such that $\boldsymbol{h} = F \cdot \boldsymbol{p}$.*

Proof. Let $\boldsymbol{p}' : \boldsymbol{X}' \longrightarrow (X)$ be a $\boldsymbol{\mathcal{P}}$–coexpansion of X. There exists a unique morphism $\boldsymbol{g} : \boldsymbol{X}' \longrightarrow \boldsymbol{X}$ with $\boldsymbol{p} \cdot \boldsymbol{g} = \boldsymbol{p}'$. By Theorem 18 the composition $\boldsymbol{h} \cdot \boldsymbol{g} : \boldsymbol{X}' \longrightarrow Y$ can be considered as a morphism of the category **inj**–$\boldsymbol{\mathcal{T}}$. Also observe that there exists a unique coshape morphism $F : X \longrightarrow Y$ for which $\boldsymbol{h} \cdot \boldsymbol{g} = F \cdot \boldsymbol{p}'$.

Now prove an equality $\boldsymbol{h} = F \cdot \boldsymbol{p}$. To achieve this first prove that for each morphism $\boldsymbol{u} : P \longrightarrow \boldsymbol{X}$, $P \in \boldsymbol{\mathcal{P}}$ in **inj**–$\boldsymbol{\mathcal{T}}$ holds an equality $\boldsymbol{h} \cdot \boldsymbol{u} = F \cdot \boldsymbol{p} \cdot \boldsymbol{u}$. Indeed, for each index $\alpha \in A$ and for each morphisms $\boldsymbol{u} : P \longrightarrow \boldsymbol{X}$ we have $h_\alpha \cdot u = F \cdot p_\alpha \cdot u$. By Proposition 20, $h_\alpha = F \cdot p_\alpha$. For the morphism

$\boldsymbol{p} \cdot \boldsymbol{u} : P \longrightarrow X$ there exists a morphism $\boldsymbol{v} : (P) \longrightarrow \boldsymbol{X}'$ with $\boldsymbol{p} \cdot \boldsymbol{u} = \boldsymbol{p}' \cdot \boldsymbol{v}$. Clearly, $\boldsymbol{p} \cdot \boldsymbol{u} = \boldsymbol{p} \cdot \boldsymbol{g} \cdot \boldsymbol{v}$. Since $\boldsymbol{p} : \boldsymbol{X} \longrightarrow \boldsymbol{X}$ is $\boldsymbol{\mathcal{P}}$–coexpansion it follows that $\boldsymbol{u} = \boldsymbol{g} \cdot \boldsymbol{v}$. Consequently,

$$F \cdot \boldsymbol{p} \cdot \boldsymbol{u} = F \cdot \boldsymbol{p} \cdot \boldsymbol{g} \cdot \boldsymbol{v} = F \cdot \boldsymbol{p}' \cdot \boldsymbol{v} = \boldsymbol{h} \cdot \boldsymbol{g} \cdot \boldsymbol{v} = \boldsymbol{h} \cdot \boldsymbol{u}.$$

Let $F' : X \longrightarrow Y$ be an another coshape morphism which satisfies a condition $F \cdot \boldsymbol{p} = \boldsymbol{h} = F' \cdot \boldsymbol{p}$. Using the Proposition 20 and equalities

$$F' \cdot \boldsymbol{p}' = F' \cdot \boldsymbol{p} \cdot \boldsymbol{g} = F \cdot \boldsymbol{p} \cdot \boldsymbol{g} = F' \cdot \boldsymbol{p}'$$

we obtain $F = F'$. □

Let $F : X \longrightarrow Y$ be a coshape morphism of $X \in \boldsymbol{\mathcal{T}}$ to $Y \in \boldsymbol{\mathcal{T}}$ and let $f : P \longrightarrow X$ be a morphism of $P \in \boldsymbol{\mathcal{P}}$ to X. By Theorem 18 the coshape morphism $F \cdot f : P \longrightarrow Y$ is some morphism $g : P \longrightarrow Y$ of the category $\boldsymbol{\mathcal{T}}$. Then there is a function $F_P : \boldsymbol{\mathcal{T}}(P, X) \longrightarrow \boldsymbol{\mathcal{T}}(P, Y)$ such that $F_P(f) = g = F \cdot f$. Let $v : P \longrightarrow P'$, $P' \in \boldsymbol{\mathcal{P}}$ be a morphism and let $f' : P' \longrightarrow X$ be a morphism of the category $\boldsymbol{\mathcal{T}}$ such that

$$f' \cdot v = f. \tag{1}$$

We have $g' \cdot v = F \cdot f' \cdot v = F \cdot f = g$. Consequently (1) implies

$$g' \cdot v = g. \tag{2}$$

Let $F, F' : X \longrightarrow Y$ be two coshape morphisms such that $F_P = F'_P$ for each $P \in \boldsymbol{\mathcal{P}}$. Then $F = F'$. Let $G_P : \boldsymbol{\mathcal{T}}(P, X) \longrightarrow \boldsymbol{\mathcal{T}}(P, Y)$ be a map such that (1) implies (2). Then there is a coshape morphism $F : X \longrightarrow Y$ such that $G_P = F_P$.

The composition $G \cdot F$ of coshape morphisms $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ assigns to each morphism $f : P \longrightarrow X$ the morphism $G \cdot P \cdot f : P \longrightarrow Z$ so that

$$(GF)_P(f) = G_P(F_P(f)). \tag{3}$$

For identity coshape morphism $I_X : X \longrightarrow X$ we have $(I_X)_P(f) = f$. Consequently a coshape morphism $F : X \longrightarrow Y$ is a collection of functions $F_P : \boldsymbol{\mathcal{T}}(P, X) \longrightarrow \boldsymbol{\mathcal{T}}(P, Y)$, $P \in \boldsymbol{\mathcal{P}}$, such that (1) implies (2). The identity coshape morphism $I_X : X \longrightarrow X$ is defined by the identity functions $\boldsymbol{\mathcal{T}}(P, X) \longrightarrow \boldsymbol{\mathcal{T}}(P, X)$, $P \in \boldsymbol{\mathcal{P}}$, and the composition is given by formula (3).

Let $\boldsymbol{\mathcal{T}}(-, X) : \boldsymbol{\mathcal{P}} \longrightarrow$**Set** be the functor with assigns to each object $P \in \boldsymbol{\mathcal{P}}$ the set $\boldsymbol{\mathcal{T}}(P, X)$ and to morphism $v : P' \longrightarrow P$ of $\boldsymbol{\mathcal{P}}$ the function $v_X = \boldsymbol{\mathcal{T}}(v, X) : \boldsymbol{\mathcal{T}}(P, X) \longrightarrow \boldsymbol{\mathcal{T}}(P', X)$ given by formula:

$$v_X(f) = f' = f \cdot v, \quad f \in \boldsymbol{\mathcal{T}}(P, X).$$

For each coshape morphism $F : X \longrightarrow Y$ we have defined functions $F_P : \boldsymbol{\mathcal{T}}(P, X) \longrightarrow \boldsymbol{\mathcal{T}}(P, Y)$, $P \in \boldsymbol{\mathcal{P}}$, such that (1) implies (2) and $F_{P'} \cdot v_X = v_Y \cdot F_P$. Consequently F_P, $P \in \boldsymbol{\mathcal{P}}$, is a natural transformation of functor $\boldsymbol{\mathcal{T}}(-, X)$ to functor $\boldsymbol{\mathcal{T}}(-, Y)$. Thus, we have the following

Theorem 23. *Let* $\mathbf{M}$ *be category whose objects are the objects of category* $\boldsymbol{\mathcal{T}}$ *and whose morphisms* $X \longrightarrow Y$ *are the natural transformations* $\boldsymbol{\mathcal{T}}(-, X) \longrightarrow \boldsymbol{\mathcal{T}}(-, Y)$. *The functor* $\Lambda : \mathbf{CSH}_{(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{P}})} \longrightarrow \mathbf{M}$ *which assigns to object* $X \in \mathbf{CSH}_{(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{P}})}$ *the same object* X *and to coshape morphism* $F : X \longrightarrow Y$ *the natural transformation* (F_P), $P \in \boldsymbol{\mathcal{P}}$ *is an isomorphism.*

4. The Topological Coshape Category

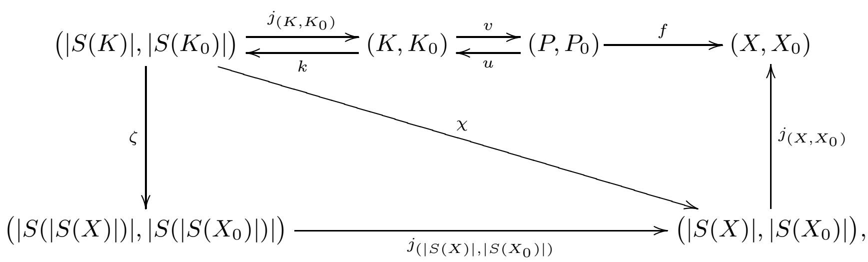
Now we construct the coshape category $\mathbf{CSH}^2 = \mathbf{CSH}_{(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{P}})}$, where $\boldsymbol{\mathcal{T}} = \mathbf{HTop}^2$ and $\boldsymbol{\mathcal{P}} = \mathbf{HCW}^2_{\boldsymbol{f}}$. To achieve this aim we establish the following main theorem.

Theorem 24. *The homotopy category* $\mathbf{HCW}^2_{\boldsymbol{f}}$ *is a codense subcategory of the homotopy category* $\mathbf{HTop}^2$.

The proof of this theorem is based on the Theorems 10, 11 and 12 and on the following two lemmas.

Lemma 25. *Let* $f : (P, P_0) \longrightarrow (X, X_0)$ *be a map of pair* $(P, P_0) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ *to pair* $(X, X_0) \in \mathbf{Top}^2$. *Then it factors through a pair of finite* CW*–simplicial complexes of a small subcategory of the category* $\mathbf{HCW}^2_{\boldsymbol{f}}$.

Proof. By condition of lemma there is a pair (K, K_0) of a finite CW–complex K and its subcomplex K_0 and maps $u : (P, P_0) \longrightarrow (K, K_0)$ and $v : (K, K_0) \longrightarrow (P, P_0)$ such that $v \cdot u \simeq 1_{(P,P_0)}$ and $u \cdot v \simeq 1_{(K,K_0)}$. Consider the following diagram



where $j_{(K,K_0)}, k, \zeta$ and χ are maps such that

$$j_{(K,K_0)} \cdot k = 1_{(K,K_0)}, \quad j_{(X,X_0)} \cdot \chi = f \cdot v \cdot j_{(K,K_0)}, \quad j_{(|S(X)|,|S(X_0)|)} \cdot \zeta = \chi.$$

The existence of these maps follows from Proposition 1. Let $h = \zeta \cdot k \cdot u : (P, P_0) \longrightarrow \big(|S(|S(X)|)|, |S(|S(X_0)|)|\big)$ and $j = j_{(X,X_0)} \cdot j_{(|S(X)|,|S(X_0)|)} : \big(|S(|S(X)|)|, |S(|S(X_0)|)|\big) \longrightarrow (X, X_0)$. Note that

$$\begin{aligned} j \cdot h &= j_{(X,X_0)} \cdot j_{(|S(X)|,|S(X_0)|)} \cdot \zeta \cdot k \cdot u = j_{(X,X_0)} \cdot \chi \cdot k \cdot u \\ &= f \cdot v \cdot j_{(K,K_0)} \cdot k \cdot u = f \cdot v \cdot 1_{(K,K_0)} \cdot u = f \cdot v \cdot u \simeq f \cdot 1_{(P,P_0)} = f. \end{aligned}$$

Thus, $f \simeq j \cdot h$. It is clear that the pair $\big(|S(|S(X)|)|, |S(|S(X_0)|)|\big)$ is pair of CW–simplicial complexes (see [22], Lemma 4.10 of Ch. III, Sec. 4 and Corollary 3.6 of Ch.IV, Sec. 3).

Let $\mathscr{P}' = \{(X_\alpha, X_{0\alpha}) \,|\, \alpha \in A\}$ be the set of all pairs of finite CW–simplicial subcomplexes of pair $\big(|S(|S(X)|)|, |S(|S(X_0)|)|\big)$.

We have the following inclusion

$$\begin{aligned} h\big((P,P_0)\big) &= (\zeta \cdot k \cdot u)\big((P,P_0)\big) \subset \zeta k\big(u(P,P_0)\big) \\ &\subset \zeta k\big((K,K_0)\big) = \big(\zeta k(K), \zeta k(K_0)\big). \end{aligned}$$

The compact pair $\big(\zeta k(K), \zeta k(K_0)\big)$, and hence the pair $h((P,P_0))$, is contained in some pair $(X_\alpha, X_{0\alpha}) \in \mathscr{P}'$.

Let $j_\alpha = j_{|(X_\alpha, X_{0\alpha})} : (X_\alpha, X_{0\alpha}) \longrightarrow (X, X_0)$ and let $h_\alpha = h^{|(X_\alpha, X_{0\alpha})}$: $(P,P_0) \longrightarrow (X_\alpha, X_{0\alpha})$. Clearly, $f \simeq j_\alpha \cdot h_\alpha$. This is the dezired factorization. □

Lemma 26. *Let $(X, X_0) \in \mathbf{HTop}^2$, $(P,P_0), (P', P'_0) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ and let $f' : (P', P'_0) \longrightarrow (X, X_0)$, $h_1, h_2 : (P,P_0) \longrightarrow (P', P'_0)$ be maps such that $f' \cdot h_1 \simeq f' \cdot h_2$. Then there exist a pair $(P'', P''_0) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ and maps $f'' : (P'', P''_0) \longrightarrow (X, X_0)$ and $h : (P', P'_0) \longrightarrow (P'', P''_0)$ such that $f'' \cdot h = f'$ and $h \cdot h_1 \simeq h \cdot h_2$.*

Proof. Let $H : (P,P_0) \times I \longrightarrow (X, X_0)$ be a homotopy between $f' \cdot h_1$ and $f' \cdot h_2$. Let $f'_0 = f'_{|P'_0} : P'_0 \longrightarrow X_0$, $h_{01} = h_{1|P_0} : P_0 \longrightarrow P'_0$, $h_{02} = h_{2|P_0}$: $P_0 \longrightarrow P'_0$. Note that $H_{|P_0 \times I} = H_0 : f'_0 \cdot h_{01} \simeq f'_0 \cdot h'_{02}$. Consider the pair $(S, S_0) = (P \times I \cup \mathrm{Cyl}(g), P_0 \times I \cup \mathrm{Cyl}(g_0))$, where $\mathrm{Cyl}(g)$ and $\mathrm{Cyl}(g_0)$ is the mapping cylinders of maps $g = h_1 \oplus h_2 : P^1 \oplus P^2 \longrightarrow P'$, $P^1 = P$, $P^2 = P$ and $g_0 = h_{01} \oplus h_{02} : P^1_0 \oplus P^2_0 \longrightarrow P'_0$, $P^1_0 = P_0$, $P^2_0 = P_0$, respectively.

Consider a relation on S:

$$\begin{aligned} &(p,1) \sim [(p,0)], \ \ (p,1) \in P \times I, \ \ [(p,0)] \in \mathrm{Cyl}(g), \ \ p \in P^1; \\ &(p,0) \sim [(p,0)], \ \ (p,0) \in P \times I, \ \ [(p,0)] \in \mathrm{Cyl}(g), \ \ p \in P^2; \\ &(p,1) \sim [(p,0)], \ \ (p,1) \in P_0 \times I, \ \ [(p,0)] \in \mathrm{Cyl}(g_0), \ \ p \in P^1_0; \\ &(p,0) \sim [(p,0)], \ \ (p,0) \in P_0 \times I, \ \ [(p,0)] \in \mathrm{Cyl}(g_0), \ \ p \in P^2_0. \end{aligned}$$

Let $P'' = S/\sim$ and $P''_0 = S_0/\sim$ and let $q : S \longrightarrow P''$ be the quotient map. It is clear that q maps the pair (S, S_0) onto the pair (P'', P''_0). Now define maps $h : P' \longrightarrow P''$ and $f'' : P'' \longrightarrow X$. By definition,

$$h(p') = [p'], \quad p' \in P';$$

$$f''(z) = \begin{cases} H(p,t), & z = q([(p,t)]), \ \ p \in P, \ \ 0 \le t \le 1, \\ f'h_1(p), & z = q([(p,t)]), \ \ p \in P^1, \ \ 0 \le t \le 1, \\ f'h_2(p), & z = q([(p,t)]), \ \ p \in P^2, \ \ 0 \le t \le 1, \\ f'(p'), & z = q([(p',t)]), \ \ p' \in P'. \end{cases}$$

It is clear that $h(P'_0) \subseteq P''_0$ and $f''(P''_0) \subseteq X_0$, i.e. h and f'' are maps of pairs. The pair (P'', P''_0) and maps $f'' : (P'', P''_0) \longrightarrow (X, X_0)$ and $h : (P', P'_0) \longrightarrow (P'', P''_0)$ satisfy the conditions of lemma. □

Let $\mathbf{HTop}^2_*$ be the pointed homotopy category of pointed pairs and let $\mathbf{HPol}^2_{\boldsymbol{f}*}$ be the pointed homotopy category of pairs which have the homotopy type of pointed pair of finite CW–complexes. Similarly we can prove pointed versions of Lemma 25 and Lemma 26. Consequently we have the following theorem.

Theorem 27. *The pointed homotopy category* $\mathbf{HCW}^2_{\boldsymbol{f}*}$ *is the codence subcategory of the pointed homotopy category* $\mathbf{HTop}^2_*$.

The pointed coshape category $\mathbf{CSH}^2_*$ of pairs of spaces is the abstract coshape category $\mathbf{CSH}_{(\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{P}})}$, where $\boldsymbol{\mathcal{T}} = \mathbf{HTop}^2_*$ and $\boldsymbol{\mathcal{P}} = \mathbf{HCW}^2_{\boldsymbol{f}*}$.

By $\mathrm{csh}(X, X_0)$ ($\mathrm{csh}(X, X_0, *)$) we denote the coshape (pointed coshape) of pair (X, X_0) (pointed pair $(X, X_0, *)$).

Remark 1. Applying Lemma 25, Theorem 4 and arguments used in the proof of Theorem 17 we can conclude that for each pair $(X, X_0) \in \mathbf{HTop}^2((X, X_0, *) \in \mathbf{HTop}^2_*)$ there exists a coassociated with (X, X_0) $((X, X_0, *))$ direct system consisting of pairs (pointed pairs) of CW–simplicial complexes.

CHAPTER II
COSHAPE INVARIANTS

5. On Extensions of Functors

The purpose of this section is to construct of coshape invariant and continuous extensions of covariant (contravariant) functors from the category $\mathbf{HCW}^2_{\boldsymbol{f}}$ ($\mathbf{HCW}^2_{\boldsymbol{f}*}$) to the category $\mathbf{HTop}^2$ ($\mathbf{HTop}^2_*$).

Let $\mathrm{T}: \mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ be a covariant (contravariant) functor of the category $\mathbf{HCW}^2_{\boldsymbol{f}}$ to the category groups $\mathbf{Gr}$. Let $(\boldsymbol{X}, \boldsymbol{X}_0) = ((X_\alpha, X_{0\alpha}), p_{\alpha\alpha'}, A)$ be a direct system in $\mathbf{HCW}^2_{\boldsymbol{f}}$. The covariant (contravariant) functor T forms direct (inverse) system $\mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0) = ((\mathrm{T}(X_\alpha, X_{0\alpha}), \mathrm{T}(p_{\alpha\alpha'}), A)$ in the category $\mathbf{Gr}$. Let $(f_\alpha, \varphi) : (\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow (\boldsymbol{Y}, \boldsymbol{Y}_0) = ((Y_\beta, Y_{0\beta}), q_{\beta\beta'}, B)$ be a morphism of the category $\mathbf{dir}$–$\mathbf{HCW}^2_{\boldsymbol{f}}$. Then we have the morphism $(\mathrm{T}(f_\alpha), \varphi) : \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0)$ $((\mathrm{T}(f_\alpha), \varphi) : \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0))$ of the category $\mathbf{dir}$–$\mathbf{Gr}$ ($\mathbf{inv}$–$\mathbf{Gr}$). It is clear that if $(f_\alpha, \varphi) \sim (f'_\alpha, \varphi')$ then $(\mathrm{T}(f_\alpha), \varphi) \sim (\mathrm{T}(f'_\alpha), \varphi')$ in the category $\mathbf{dir}$–$\mathbf{Gr}$ ($\mathbf{inv}$–$\mathbf{Gr}$). Consequently, a morphism $\boldsymbol{f} = [(f_\alpha, \varphi)] : (\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow (\boldsymbol{Y}, \boldsymbol{Y}_0)$ of the category $\mathbf{inj}$–$\mathbf{HCW}^2_{\boldsymbol{f}}$ induces the morphism $\mathrm{T}(\boldsymbol{f}) = [(\mathrm{T}(f_\alpha), \varphi)] : \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0)$ $(\mathrm{T}(\boldsymbol{f}) = [(\mathrm{T}(f_\alpha), \varphi)] : \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0))$ of the category

inj–Gr (**pro–Gr**). Thus, we have defined covariant (contravariant) functor, which for simplicity we again denote by

$$\mathrm{T}(-,-) : \mathbf{inj\text{–}HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{inj\text{–}Gr}$$
$$\big(\mathrm{T}(-,-) : \mathbf{inj\text{–}HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{pro\text{–}Gr}\big).$$

Let $(X, X_0) \in \mathbf{HTop}^2$ and let $\boldsymbol{p} = [(p_\alpha)] : (\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow (X, X_0)$ be a $\mathbf{HCW}^2_{\boldsymbol{f}}$–coexpansion of (X, X_0). Note that for each another $\mathbf{HCW}^2_{\boldsymbol{f}}$–coexpansion $\boldsymbol{p'} = [(p'_{\alpha'})] : (\boldsymbol{X}, \boldsymbol{X}_0)' \longrightarrow (X, X_0)$ isomorphism $\boldsymbol{i} : (\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow (\boldsymbol{X}, \boldsymbol{X}_0)'$ induces isomorphism $\mathrm{T}(\boldsymbol{i}) : \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0)'$ ($\mathrm{T}(\boldsymbol{i}) : \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0)' \longrightarrow \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0)$). The equivalence class of $\mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0)$ denote by inj–$T(X, X_0)$ (pro–$T(X, X_0)$).

Let $F : (X, X_0) \longrightarrow (Y, Y_0)$ be a coshape morphism and let $\boldsymbol{f} : (\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow (\boldsymbol{Y}, \boldsymbol{Y}_0)$ be its some representative. For another representative $\boldsymbol{f'} : (\boldsymbol{X}, \boldsymbol{X}_0)' \longrightarrow (\boldsymbol{Y}, \boldsymbol{Y}_0)'$ we have $\boldsymbol{f'} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{f}$. Consequently,

$$\mathrm{T}(\boldsymbol{f'}) \cdot \mathrm{T}(\boldsymbol{i}) = \mathrm{T}(\boldsymbol{j}) \cdot \mathrm{T}(\boldsymbol{f}) \;\; \big(\mathrm{T}(\boldsymbol{f}) \cdot \mathrm{T}(\boldsymbol{j}) = \mathrm{T}(\boldsymbol{i}) \cdot \mathrm{T}(\boldsymbol{f'})\big).$$

The morphisms $\mathrm{T}(\boldsymbol{f}) : \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0)$ and $T(\boldsymbol{f'}) : \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0)' \longrightarrow \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0)'$ (($\mathrm{T}(\boldsymbol{f}) : \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0)$) and $\mathrm{T}(\boldsymbol{f'}) : \mathrm{T}(\boldsymbol{Y}, \boldsymbol{Y}_0)' \longrightarrow \mathrm{T}(\boldsymbol{X}, \boldsymbol{X}_0)'$) are coincide. Thus, the coshape morphism $F : (X, X_0) \longrightarrow (Y, Y_0)$ induces a morphism

$$\text{inj–}T(F) : \text{inj–}T(X, X_0) \longrightarrow \text{inj–}T(Y, Y_0)$$
$$\big(\text{pro–}T(F) : \text{pro–}T(Y, Y_0) \longrightarrow \text{pro–}T(X, X_0)\big).$$

Thus, we have defined covariant (contravariant) functor

$$\text{inj–T}(-,-) : \mathbf{CSH}^2 \longrightarrow \mathbf{inj} - \mathbf{Gr}$$
$$\big(\text{pro–T}(-,-) : \mathbf{CSH}^2 \longrightarrow \mathbf{pro} - \mathbf{Gr}\big).$$

By definition,

$$(\text{inj–T})((X, X_0)) = \text{inj} - T(X, X_0), \quad (X, X_0) \in \mathbf{CSH}^2,$$
$$(\text{pro–T})((X, X_0)) = \text{pro} - T(X, X_0), \quad (X, X_0) \in \mathbf{CSH}^2,$$
$$(\text{inj–T})(F) = \text{inj} - T(F), \quad F \in \mathbf{CSH}^2,$$
$$(\text{pro–T})(F) = \text{pro} - T(F), \quad F \in \mathbf{CSH}^2.$$

Analogously we can define the covariant (contravariant) functor

$$\text{inj–T}(-,-) : \mathbf{CSH}^2_* \longrightarrow \mathbf{inj} - \mathbf{Gr}$$
$$\big(\text{pro–T}(-,-) : \mathbf{CSH}^2_* \longrightarrow \mathbf{pro} - \mathbf{Gr}\big).$$

The objects of the category **inj–Gr** are called inj–groups ([5], [30]) and the objects of the category **pro–Gr** are called pro–groups [24].

We have obtained the following propositions.

Proposition 28. *Let* $(X, X_0), (Y, Y_0) \in \mathbf{HTop}^2$ *and* $\operatorname{csh}(X, X_0) = \operatorname{csh}(Y, Y_0)$. *Then* inj$-T(X, X_0) =$ inj$-T(Y, Y_0)$ *and* pro$-T(X, X_0) =$ pro$-T(Y, Y_0)$.

Proposition 29. *Let* $(X, X_0, *), (Y, Y_0, *) \in \mathbf{HTop}^2_*$ *and* $\operatorname{csh}(X, X_0, *) = \operatorname{csh}(Y, Y_0, *)$. *Then* inj$-T(X, X_0, *) =$ inj$-T(Y, Y_0, *)$ *and* pro$-T(X, X_0, *)$ $=$ pro$-T(Y, Y_0, *)$.

For each pair (X, X_0) and coshape morphism $F : (X, X_0) \longrightarrow (Y, Y_0)$ define spectral groups

$$\hat{T}(X, X_0) = \varinjlim \ \text{inj}{-}T(X, X_0)$$

$$\big(\overset{\vee}{T}(X, X_0) = \varprojlim \ \text{pro}{-}T(X, X_0)\big)$$

and homomorphisms

$$\hat{F} = \varinjlim \ \text{inj}{-}T(F) : \hat{T}(X, X_0) \longrightarrow \hat{T}(Y, Y_0)$$

$$\big(\overset{\vee}{F} = \varprojlim \ \text{pro}{-}T(F) : \overset{\vee}{T}(Y, Y_0) \longrightarrow \overset{\vee}{T}(X, X_0)\big).$$

Thus, the covariant (contravariant) functor $\mathrm{T} : \mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ induces the covariant (contravariant) functor $\overset{\wedge}{\mathrm{T}} : \mathbf{CSH}^2 \longrightarrow \mathbf{Gr}$ ($\overset{\vee}{\mathrm{T}} : \mathbf{CSH}^2 \longrightarrow \mathbf{Gr}$). By definition,

$$\overset{\wedge}{\mathrm{T}}((X, X_0)) = \hat{T}(X, X_0), \quad (X, X_0) \in \mathbf{CSH}^2$$

$$\big(\overset{\vee}{\mathrm{T}}((X, X_0)) = \overset{\vee}{T}(X, X_0), \quad (X, X_0) \in \mathbf{CSH}^2\big),$$

$$\overset{\wedge}{\mathrm{T}}(F) = \hat{F}, \quad F \in \mathbf{CSH}^2$$

$$\big(\overset{\vee}{\mathrm{T}}(F) = \overset{\vee}{F}, \quad F \in \mathbf{CSH}^2\big).$$

Analogously, a covariant (contravariant) functor T: $\mathbf{HCW}^2_{\boldsymbol{f}}* \longrightarrow \mathbf{Gr}$ induces a covariant (contravariant) functor $\overset{\wedge}{\mathrm{T}} : \mathbf{CSH}^2_* \longrightarrow \mathbf{Gr}$ ($\overset{\vee}{\mathrm{T}} : \mathbf{CSH}^2_* \longrightarrow \mathbf{Gr}$).

The composition $\overset{\wedge}{\mathrm{T}}\cdot\mathrm{CS}(\overset{\vee}{\mathrm{T}}\cdot\mathrm{CS})$ of constructed functor $\overset{\wedge}{\mathrm{T}}\,(\overset{\vee}{\mathrm{T}})$ with coshape functor CS is coshape invariant extension of functor T. For simplicity it we again denote by $\overset{\wedge}{\mathrm{T}}\,(\overset{\vee}{\mathrm{T}})$. Hence, we have the following propositions.

Proposition 30. *If* $(X, X_0), (Y, Y_0) \in \mathbf{HTop}^2$ *and* $\operatorname{csh}(X, X_0) = \operatorname{csh}(Y, Y_0)$, *then* $\hat{T}(X, X_0) = \hat{T}(Y, Y_0)$ *and* $\overset{\vee}{T}(X, X_0) = \overset{\vee}{T}(Y, Y_0)$.

Proposition 31. *If* $(X, X_0, *), (Y, Y_0, *) \in \mathbf{HTop}^2_*$ *and* $\operatorname{csh}(X, X_0, *) = \operatorname{csh}(Y, Y_0, *)$, *then* $\hat{T}(X, X_0, *) = \hat{T}(Y, Y_0, *)$ *and* $\overset{\vee}{T}(X, X_0, *) = \overset{\vee}{T}(Y, Y_0, *)$.

Let $T:\mathbf{HTop}^2\longrightarrow\mathbf{Gr}$ be a covariant (contravariant) functor with the property that any $\mathbf{HCW}^2_{\boldsymbol{f}}$–coexpansion $\boldsymbol{p}:(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow(X,X_0)$ of pair $(X,X_0)\in\mathbf{HTop}^2$ induces a direct (an inverse) limit

$$T(\boldsymbol{p}):T(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow T(X,X_0)\quad\big(T(\boldsymbol{p}):T(X,X_0)\longrightarrow T(\boldsymbol{X},\boldsymbol{X}_0)\big).$$

Theorem 32. *Let* $\boldsymbol{p}=[(p_\alpha)]:(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow(X,X_0)$ *be a* $\mathbf{HTop}^2$*–coexpansion of pair* $(X,X_0)\in\mathbf{HTop}^2$ *and let* $\hat{T}(\boldsymbol{p}):\hat{T}(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow\hat{T}(X,X_0)$ $(\overset{\vee}{T}(\boldsymbol{p}):\overset{\vee}{T}(X,X_0)\longrightarrow\overset{\vee}{T}(\boldsymbol{X},\boldsymbol{X}_0))$ *be the induced morphism of* **inj–Gr** (**pro–Gr**)*. Then the homomorphism*

$$\hat{p}=\varinjlim\hat{T}(\boldsymbol{p}):\varinjlim\hat{T}(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow\hat{T}(X,X_0)$$
$$\big(\overset{\vee}{p}=\varprojlim\overset{\vee}{T}(\boldsymbol{p}):\overset{\vee}{T}(X,X_0)\longrightarrow\varprojlim\overset{\vee}{T}(\boldsymbol{X},\boldsymbol{X}_0)\big)$$

induced by $\hat{T}(\boldsymbol{p})$ $(\overset{\vee}{T}(\boldsymbol{p}))$ *is an isomorphism.*

Proof. For simplisity we denote the object $\hat{T}(X,X_0)$ by $T(X,X_0)$ for each object $(X,X_0)\in\boldsymbol{\mathcal{T}}$, the homomorphism $\hat{T}(f)$ by $\hat{f}$ for each morphism $f:(X,X_0)\longrightarrow(Y,Y_0)$ in $\boldsymbol{\mathcal{T}}$ and the direct system $\hat{T}(\boldsymbol{X},\boldsymbol{X}_0)=(T(X_\alpha,X_{0\alpha}),\hat{p}_{\alpha\alpha'},A)$ in **Gr** by $T(\boldsymbol{X})$ for each direct system $(\boldsymbol{X},\boldsymbol{X}_0)$ in $\boldsymbol{\mathcal{T}}$. Analogously, we denote by $\overset{\wedge}{\boldsymbol{p}}=(\hat{p}_\alpha)$ the morphism $T(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow T(X,X_0)$ given by homomorphisms $\hat{p}_\alpha:T(X_\alpha,X_{0\alpha})\longrightarrow T(X,X_0)$, $\alpha\in A$. Finally, by $\hat{p}$ we denote the homomorphism $\varinjlim H(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow H(X,X_0)$ for which $\hat{p}\cdot\pi_\alpha=\hat{p}_\alpha$, $\alpha\in A$, where $\pi_\alpha:T(X_\alpha,X_{0\alpha})\longrightarrow\varinjlim(\boldsymbol{X},\boldsymbol{X}_0)$ is the injection homomorphism. Besides, also note that for each pair $\alpha\le\alpha'$ holds equality $\pi_{\alpha'}\cdot\hat{p}_{\alpha\alpha'}=\pi_\alpha$.

Let $\boldsymbol{q}:(\boldsymbol{Y},\boldsymbol{Y}_0)=((Y_\beta,Y_{0\beta}),q_{\beta\beta'},B)\longrightarrow(X,X_0)$ be a $\boldsymbol{\mathcal{P}}$–coexpansion of (X,X_0). It is clear that $\overset{\wedge}{\boldsymbol{q}}=(\hat{q}_\beta):T(\boldsymbol{Y},\boldsymbol{Y}_0)\longrightarrow T(X,X_0)$ is a direct limit and there exists a morphism $\boldsymbol{f}:(\boldsymbol{Y},\boldsymbol{Y}_0)\longrightarrow(\boldsymbol{X},\boldsymbol{X}_0)$ of the category **inj–$\boldsymbol{\mathcal{T}}$** such that $\boldsymbol{p}\cdot\boldsymbol{f}=\boldsymbol{q}$. Let (f_β,φ) be some representative of $\boldsymbol{f}$. The homomorphisms $\hat{f}_\beta:T(Y_\beta,Y_{0\beta})\longrightarrow T(X_{\varphi(\beta)},X_{0\varphi(\beta)})$, $\beta\in B$ induce a morphism of inj–groups $\overset{\wedge}{\boldsymbol{f}}=(\hat{f}_\beta,\varphi):T(\boldsymbol{Y},\boldsymbol{Y}_0)\longrightarrow T(\boldsymbol{X},\boldsymbol{X}_0)$. Note that $\overset{\wedge}{\boldsymbol{f}}=\overset{\wedge}{\boldsymbol{p}}\cdot\overset{\wedge}{\boldsymbol{q}}$ and $\overset{\wedge}{\boldsymbol{f}}$ induces a homomorphism of groups $\hat{f}:T(X,X_0)\longrightarrow\varinjlim T(\boldsymbol{X},\boldsymbol{X}_0)$ for which $\boldsymbol{\pi}\cdot\overset{\wedge}{\boldsymbol{f}}=\hat{f}\cdot\overset{\wedge}{\boldsymbol{q}}$, where $\boldsymbol{\pi}:T(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow\varinjlim T(\boldsymbol{X},\boldsymbol{X}_0)$ is a morphism induced by (π_α). For each index $\beta\in B$ we have $\pi_{\varphi(\beta)}\cdot\hat{f}_\beta=$

$\hat{f} \cdot \hat{q}_\beta$. Besides,

$$\hat{p} \cdot \hat{f} \cdot \hat{q}_\beta = \hat{p} \cdot \pi_{\varphi(\beta)} \cdot \hat{f}_\beta = \hat{p}_{\varphi(\beta)} \cdot \hat{f}_\beta = \hat{q}_{_\beta}, \quad \beta \in B.$$

Thus $\hat{p} \cdot \hat{f} \cdot \hat{\boldsymbol{q}} = \hat{\boldsymbol{q}}$. Note that $\hat{\boldsymbol{q}} : T(\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow T(X, X_0)$ is a direct limit of $T(\boldsymbol{Y}, \boldsymbol{Y}_0)$. Consequently, $\hat{p} \cdot \hat{f} = 1_{H(X,X_0)}$.

Now we prove $\hat{p} \cdot \hat{f} = 1_{\varinjlim H(\boldsymbol{X},\boldsymbol{X}_0)}$. Let $\boldsymbol{r} = (r_\gamma) : (\boldsymbol{Z}, \boldsymbol{Z}_0) = ((Z_\gamma, Z_{0\gamma}), r_{\gamma\gamma'}, C) \longrightarrow (X_\alpha, X_{0\alpha})$ be a $\mathbf{HCW}^2_{\boldsymbol{f}}$–coexpansion of $(X_\alpha, X_{0\alpha})$. Since $\boldsymbol{q} : (\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow (X, X_0)$ is a $\mathbf{HCW}^2_{\boldsymbol{f}}$–coexpansion of (X, X_0) and $(Z_\gamma, Z_{0\gamma}) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ there is an index $\beta \in B$ and a morphism $g : (Z_\gamma, Z_{0\gamma}) \longrightarrow (Y_\beta, Y_{0\beta})$ for which $p_\alpha \cdot r_\gamma = q_{_\beta} \cdot g$. Note that $q_{_\beta} = p_{\varphi(\beta)} \cdot f_\beta$, $\beta \in B$ and there exists an index $\alpha' \geq \alpha, \varphi(\beta)$such that

$$p_{\alpha'} \cdot p_{\alpha\alpha'} \cdot r_\gamma = p_{\alpha'} \cdot p_{\varphi(\beta)\alpha'} \cdot f_\beta \cdot g.$$

By the condition CAE2) there also exists an index $\alpha'' \geq \alpha'$ such that

$$p_{\alpha'\alpha''} \cdot p_{\alpha\alpha'} \cdot r_\gamma = p_{\alpha'\alpha''} \cdot p_{\varphi(\beta)\alpha'} \cdot f_\beta \cdot g,$$

i.e. $p_{\alpha\alpha''} \cdot r_\gamma = p_{\varphi(\beta)\alpha''} \cdot f_\beta \cdot g$. Besides,

$$\hat{f} \cdot \hat{p}_\alpha \cdot \hat{r}_\gamma = \hat{f} \cdot \hat{p}_{\alpha''} \cdot \hat{p}_{\alpha\alpha''} \cdot \hat{r}_\gamma = \hat{f} \cdot \hat{p}_{\varphi(\beta)} \cdot \hat{f}_\beta \cdot \hat{g}_{_\beta} = \pi_{\varphi(\beta)} \cdot \hat{f}_\beta \cdot \hat{g}$$
$$= \pi_{\alpha''} \cdot \hat{p}_{\varphi(\beta)\alpha''} \cdot \hat{f}_\beta \cdot \hat{g} = \pi_{\alpha''} \cdot \hat{p}_{\alpha\alpha''} \cdot \hat{r}_\gamma = \pi_\alpha \cdot \hat{r}_\gamma.$$

Since $\hat{\boldsymbol{r}} = (\hat{r}_\gamma) : T(\boldsymbol{Z}, \boldsymbol{Z}_0) \longrightarrow T(X_\alpha, X_{0\alpha})$ is direct limit, $\hat{f} \cdot \hat{p}_\alpha = \pi_\alpha$, $\alpha \in A$. Hence. $\hat{f} \cdot \hat{p} \cdot \pi_\alpha = \pi_\alpha$, $\alpha \in A$, i.e. $\hat{f} \cdot \hat{p} = 1_{\varinjlim T(\boldsymbol{X},\boldsymbol{X}_0)}$.

Analogously we can prove that $\check{T}(X, X_0)$ and $\varprojlim \check{T}(X, X_0)$ are isomorphic objects of the category **Gr**. □

Similar arguments prove a pointed version of Theorem 32.

Theorem 33. *Let* $\boldsymbol{p} = [(p_\alpha)] : (\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow (X, X_0, *)$ *be a* $\mathbf{HTop}^2_*$*–coexpansion of pair* $(X, X_0, *)$ $\in \mathbf{HTop}^2_*$ *and let* $\hat{T}(\boldsymbol{p}) : \hat{T}(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow \hat{T}(X, X_0, *)$ $(\check{T}(\boldsymbol{p}) : \check{T}(X, X_0, *) \longrightarrow T(\boldsymbol{X}, \boldsymbol{X}_0, *))$ *be the induced morphism of* **inj–Gr** (**pro–Gr**). *Then the homomorphism*

$$\hat{p} = \varinjlim \hat{T}(\boldsymbol{p}) : \varinjlim \hat{T}(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow \hat{T}(X, X_0, *)$$
$$\Big(\check{p} = \varprojlim \check{T}(\boldsymbol{p}) : \check{T}(X, X_0, *) \longrightarrow \varprojlim \check{T}(\boldsymbol{X}, \boldsymbol{X}_0, *)\Big)$$

induced by $\hat{T}(\boldsymbol{p})$ $(\check{T}(\boldsymbol{p}))$ *is an isomorphism.*

Let $\mathrm{L} : \mathbf{CW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ be a covariant (contravariant) functor satisfying the homotopy axiom, i.e. if $f \simeq g$, $f, g : (X, X_0) \longrightarrow (Y, Y_0)$, then $\mathrm{L}(f) = \mathrm{L}(g)$. Let T: $\mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ be covariant (contravariant) functor defined by formulas:

$$\mathrm{T}(X, X_0) = \mathrm{L}(X, X_0), \quad (X, X_0) \in \mathbf{HCW}^2_{\boldsymbol{f}},$$
$$\mathrm{T}([f]) = \mathrm{L}(f), \quad \big([f] : (X, X_0) \longrightarrow (Y, Y_0)\big) \in \mathbf{HCW}^2_{\boldsymbol{f}}.$$

Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{Top}^2 & \xrightarrow{\ \mathrm{H}\ } & \mathbf{HTop}^2 & \xrightarrow{\ \mathrm{CS}\ } & \mathbf{CSH}^2 \\
\big\uparrow & & \big\uparrow & \nearrow_{\mathrm{CS}_{|\mathbf{HCW}^2_{\boldsymbol{f}}}} & \big\downarrow \overset{\wedge}{\mathrm{T}}(\overset{\vee}{T}) \\
\mathbf{CW}^2_{\boldsymbol{f}} & \xrightarrow[\mathrm{H}_{|\mathbf{CW}^2_{\boldsymbol{f}}}]{} & \mathbf{HCW}^2_{\boldsymbol{f}} & \xrightarrow[\mathrm{T}]{} & \mathbf{Gr}\,,
\end{array}$$

where H: $\mathbf{Top}^2 \longrightarrow \mathbf{HTop}^2$ is the homotopy functor.

The covariant (contravariant) functor

$$\overset{\wedge}{\mathrm{L}} = \overset{\wedge}{\mathrm{T}} \cdot \mathrm{CS} \cdot \mathrm{H} : \mathbf{Top}^2 \longrightarrow \mathbf{Gr}$$
$$\big(\overset{\vee}{\mathrm{L}} = \overset{\vee}{\mathrm{T}} \cdot \mathrm{CS} \cdot \mathrm{H} : \mathbf{Top}^2 \longrightarrow \mathbf{Gr}\big)$$

satisfies the homotopy axiom and is an extension of covariant (contravariant) functor $\mathrm{L} : \mathbf{CW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$. Note that $\overset{\wedge}{\mathrm{L}}$ $(\overset{\vee}{\mathrm{L}})$ is coshape invariant functor.

Finally, also note that such type extension exists for covariant (contravariant) functor $\mathrm{L} : \mathbf{CW}^2_{\boldsymbol{f}*} \longrightarrow \mathbf{Gr}$ which satisfies the relative homotopy axiom, i.e. if $f \simeq g$ rel$\{*\}$, $f, g : (X, X_0, *) \longrightarrow (Y, Y_0, *)$, then $\mathrm{L}(f) = \mathrm{L}(g)$.

Example 1. Let $\mathrm{T}(-,-) = \mathrm{H}_k(-,-;G) : \mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Ab}$ and $\mathrm{T}(-,-) = \mathrm{H}^k(-,-;G) : \mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Ab}$ be the singular homology and singular cohomology functors with coefficients in abelian group G, respectively. By Theorem 32 the spectral singular homology group $\hat{H}_k(X, X_0; G)$ and the spectral singular cohomology group $\overset{\vee}{H}{}^k(X, X_0; G)$, defined in [8] (see also [21]), induce continuous functors $\overset{\wedge}{\mathrm{H}}_k(-,-;G) : \mathbf{CSH}^2 \longrightarrow \mathbf{Ab}$ and $\overset{\vee}{\mathrm{H}}{}^k(-,-;G) : \mathbf{CSH}^2 \longrightarrow \mathbf{Ab}$, i.e. if $\boldsymbol{p} : (\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow (X, X_0)$ is an $\mathbf{HTop}^2$–coexpansion of (X, X_0) then the morphisms $\hat{H}_k(\boldsymbol{p}) : \hat{H}_k(\boldsymbol{X}, \boldsymbol{X}_0; G) \longrightarrow \hat{H}_k(X, X_0; G)$ and $\overset{\vee}{H}{}^k(\boldsymbol{p}) : \overset{\vee}{H}{}^k(X, X_0; G) \longrightarrow \overset{\vee}{H}{}^k(\boldsymbol{X}, \boldsymbol{X}_0; G)$ induce isomorphisms $\hat{p} = \varinjlim \hat{H}_k(\boldsymbol{p}) : \varinjlim \hat{H}_k(\boldsymbol{X}, \boldsymbol{X}_0; G) \longrightarrow \hat{H}_k(X, X_0; G)$ and $\overset{\vee}{p} = \varprojlim \overset{\vee}{H}{}^k(\boldsymbol{p}) : \overset{\vee}{H}{}^k(X, X_0; G) \longrightarrow \varprojlim (\boldsymbol{X}, \boldsymbol{X}_0; G)$ of groups, respectively.

Example 2. In analogy of the homology inj–groups are defined the homotopy inj–groups inj–$\pi_k(X, X_0, *)$ of pointed pairs $(X, X_0, *) \in \mathbf{HTop}^2_*$. If $\boldsymbol{p} : (\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow (X, X_0, *)$ is an $\mathbf{HCW}^2_{\boldsymbol{f}*}$–coexpansion, then inj–$\pi_k(X, X_0)$ is the class of inj–group $\pi_k(\boldsymbol{X}, \boldsymbol{X}_0, *) = (\pi_k(X_\alpha, X_{0\alpha}, *), \pi_k(p_{\alpha\alpha'}), A)$. We have covariant functors inj–$\pi_k(-,-)$ on $\mathbf{CSH}^2_*$ with values in **inj–Set**$_*$ for $k = 1$, in **inj–Gr** for $k = 2$, and in **inj–Ab** for $k \geq 3$. By Theorem 33, the spectral homotopy groups $\overset{\wedge}{\pi}_k(X, X_0, *)$, defined in [21], induce continuous functors $\overset{\wedge}{\pi}_1(-,-) : \mathbf{CSH}^2_* \longrightarrow \mathbf{Set}_*$, $\overset{\wedge}{\pi}_2(-,-) : \mathbf{CSH}^2_* \longrightarrow \mathbf{Gr}$ and $\overset{\wedge}{\pi}_k(-,-) : \mathbf{CSH}^2_* \longrightarrow \mathbf{Ab}$, $k \geq 3$.

6. Exact Sequences of inj–groups

This section is dedicated to the study of exact sequences of inj–homology, pro–cohomology and inj–homotopy groups.

A morphism $\boldsymbol{f} : \boldsymbol{G} \longrightarrow \boldsymbol{H}$ of the category **inj–Gr** is called a monomorphism if equality $\boldsymbol{f} \cdot \boldsymbol{g} = \boldsymbol{f} \cdot \boldsymbol{g}'$ implies $\boldsymbol{g} = \boldsymbol{g}'$ for each morphisms $\boldsymbol{g}, \boldsymbol{g}' : \boldsymbol{G}' \longrightarrow \boldsymbol{G}$.

A morphism $\boldsymbol{f} : \boldsymbol{G} \longrightarrow \boldsymbol{H}$ of the category **inj–Gr** is called an epimorphism if equality $\boldsymbol{g} \cdot \boldsymbol{f} = \boldsymbol{g}' \cdot \boldsymbol{f}$ implies $\boldsymbol{g} = \boldsymbol{g}'$ for each morphisms $\boldsymbol{g}, \boldsymbol{g}' : \boldsymbol{H} \longrightarrow \boldsymbol{H}'$.

A zero object $\boldsymbol{O}$ in the category **inj–Gr** is an object of **inj–Gr** which is initial and terminal [31], i.e. for each object $\boldsymbol{G}$ of **inj–Gr** there are unique morphisms $\boldsymbol{O} \longrightarrow \boldsymbol{G}$ and $\boldsymbol{G} \longrightarrow \boldsymbol{O}$. The category **inj–Gr** has a zero object.

A morphism $\boldsymbol{G} \longrightarrow \boldsymbol{H}$ is called a zero -morphism of inj–groups provided its factors through a zero object. We denote the zero-morphism by 0. Note that $\boldsymbol{G} = (G_\alpha, p_{al\alpha'}, A)$ is zero-object of the category **inj–Gr** [32] if and only if for each index $\alpha \in A$ there exists an index $\alpha' \geq \alpha$ such that $p_{\alpha\alpha'} = 0$.

A kernel of morphism $\boldsymbol{f} : \boldsymbol{G} \longrightarrow \boldsymbol{H}$ of the category **inj–Gr** is defined as a morphism $\boldsymbol{i} : \boldsymbol{N} \longrightarrow \boldsymbol{G}$ which has the following properties:

i) $\boldsymbol{f} \cdot \boldsymbol{i} = 0$;
ii) For each morphism $\boldsymbol{g} : \boldsymbol{Q} \longrightarrow \boldsymbol{G}$ with the condition $\boldsymbol{f} \cdot \boldsymbol{g} = 0$ there exists a unique morphism $\boldsymbol{h} : \boldsymbol{Q} \longrightarrow \boldsymbol{N}$ such that $\boldsymbol{i} \cdot \boldsymbol{h} = \boldsymbol{g}$.

Theorem 34. *Let $\boldsymbol{G} = (G_\alpha, p_{\alpha\alpha'}, A)$ and $\boldsymbol{H} = (H_\alpha, q_{\alpha\alpha'}, A)$ be* inj*–groups and let $\boldsymbol{f} : \boldsymbol{G} \longrightarrow \boldsymbol{H}$ be a morphism given by a special morphism of direct systems $(f_\alpha) : \boldsymbol{G} \longrightarrow \boldsymbol{H}$. If $i_\alpha : N_\alpha \longrightarrow G_\alpha$, $\alpha \in A$ are the kernels of f_α and $n_{\alpha\alpha'} = p_{\alpha\alpha'|_{N_\alpha}} : N_\alpha \longrightarrow N_{\alpha'}$, $\alpha \leq \alpha'$, then $\boldsymbol{N} = (N_\alpha, n_{\alpha\alpha'}, A) \in$* **inj − Gr**, *$(i_\alpha) : \boldsymbol{N} \longrightarrow \boldsymbol{G}$ is a special morphism of direct systems and the morphism $\boldsymbol{i} = [(i_\alpha)] : \boldsymbol{N} \longrightarrow \boldsymbol{G}$ of* inj*–groups in the kernel of morphism $\boldsymbol{f}$.*

Proof. Let $\boldsymbol{g} : \boldsymbol{K} = (K_\beta, q_{\beta\beta'}, B) \longrightarrow \boldsymbol{G}$ be a morphism of direct systems such that $\boldsymbol{f} \cdot \boldsymbol{g} = 0$. Let $\boldsymbol{g} = [(g_\beta, \psi)]$. By the condition of theorem for each index $\beta \in B$ there exists an index $\psi'(\beta) \geq \psi(\beta)$, such that $q_{\psi(\beta)\psi(\beta')} \cdot f_{\psi(\beta)} \cdot$

$g_\beta = 0$. Note that

$$f_{\psi'(\beta)} \cdot p_{\psi(\beta)\psi'(\beta)} \cdot g_\beta = q_{\psi(\beta)\psi'(\beta)} f_{\psi(\beta)} \cdot g_\beta .$$

Hence $f_{\psi'(\beta)} \cdot p_{\psi(\beta)\psi'(\beta)} \cdot g_\beta = 0$. It is clear that $(g_\beta, \psi) \sim (p_{\psi(\beta)\psi'(\beta)} \cdot g_\beta, \psi')$. Consequently, we can assume that $f_{\psi(\beta)} \cdot g_\beta = 0$. Note that there exists a unique factorization

$$i_{\psi(\beta)} \cdot h_\beta = g_\beta, \quad h_\beta : K_\beta \longrightarrow N_{\psi(\beta)}, \quad \beta \in B.$$

The family (h_β, ψ) is morphism of $\boldsymbol{K}$ to $\boldsymbol{N}$. It induces a morphism $\boldsymbol{h} = [(h_\beta, \psi)]$ of **dir–Gr** for which $\boldsymbol{i} \cdot \boldsymbol{h} = \boldsymbol{g}$. □

A sequence $\boldsymbol{G}' \xrightarrow{\boldsymbol{f}'} \boldsymbol{G} \xrightarrow{\boldsymbol{f}} \boldsymbol{G}''$ of the category **inj–Gr** is called exact at $\boldsymbol{G}$ if it satisfies the following conditions:

i) $\boldsymbol{f} \cdot \boldsymbol{f}' = 0$;

ii) In unique factorization $\boldsymbol{f}' = \boldsymbol{i} \cdot \boldsymbol{h}$, $\boldsymbol{h} : \boldsymbol{G}' \longrightarrow \boldsymbol{N}$, where $\boldsymbol{i} : \boldsymbol{N} \longrightarrow \boldsymbol{G}$ is the kernel of $\boldsymbol{f}$, the morphism $\boldsymbol{h}$ is an epimorphism.

Theorem 35. *Let $\boldsymbol{G}^i = (G^i_\alpha, p^i_{\alpha\alpha'}, A)$, $i = 1, 2, 3$, be* inj*–groups and let $\boldsymbol{f}^1 = [(f^1_\alpha)] : \boldsymbol{G}^1 \longrightarrow \boldsymbol{G}^2$ and $\boldsymbol{f}^2 = [(f^2_\alpha)] : \boldsymbol{G}^2 \longrightarrow \boldsymbol{G}^3$ be morphisms of* inj*–groups given by special morphisms. If the sequence of groups*

$$G^1_\alpha \xrightarrow{f^1_\alpha} G^2_\alpha \xrightarrow{f^2_\alpha} G^3_\alpha$$

is exact for each index $\alpha \in A$, then the sequence of inj*–groups*

$$\boldsymbol{G}^1 \xrightarrow{\boldsymbol{f}^1} \boldsymbol{G}^2 \xrightarrow{\boldsymbol{f}^2} \boldsymbol{G}^3$$

is exact.

Proof. Note that $\boldsymbol{f}^2 \cdot \boldsymbol{f}^1 = 0$, because $f^2_\alpha \cdot f^1_\alpha = 0$ for each index $\alpha \in A$. Assume that the unite elements of groups are denoted by $*$. Let $N_\alpha = (f^2_\alpha)^{-1}(*)$, $* \in G^3_\alpha$, $n_{\alpha\alpha'} = p_{\alpha\alpha'|N_\alpha} : N_\alpha \longrightarrow N_{\alpha'}$ and $i_\alpha : N_\alpha \longrightarrow G_\alpha$, $\alpha \in A$ be the inclusion homomorphisms. By Theorem 34 the morphism

$$\boldsymbol{i} = [(\boldsymbol{i}_\alpha, 1_A)] : \boldsymbol{N} = (N_\alpha, n_{\alpha\alpha'}, A) \longrightarrow \boldsymbol{G}$$

is the kernel of morphism $\boldsymbol{f}^2$. For each index $\alpha \in A$ there exists unique morphism $h_\alpha : G^1_\alpha \longrightarrow N_\alpha$ such that $f^1_\alpha = i_\alpha \cdot h_\alpha$. It is clear that morphism $\boldsymbol{h} = [(h_\alpha)] : \boldsymbol{G}^1 \longrightarrow \boldsymbol{N}$ satisfies condition $\boldsymbol{f}^1 = \boldsymbol{i} \cdot \boldsymbol{h}$. For each index $\alpha \in A$ the morphism $h_\alpha : G^1_\alpha \longrightarrow N_\alpha$ is surjective homomorphism. From Corollary 2^* of ([32], Sec. 1) follows that $\boldsymbol{h}$ is an epimorphism. □

Reterning now to the category **LES(Gr)** of long exact sequences in **Gr** we state the fact which we need in next. Let $\{t_\alpha\}_{\alpha \in A}$ ∈**inj-LES(Gr)** be a inj-object consisting of the following exact sequences

$$t_\alpha : \cdots \longrightarrow G^{i+1}_\alpha \longrightarrow G^i_\alpha \xrightarrow{h^i_\alpha} G^{i-1}_\alpha \longrightarrow \cdots, \quad i \in Z, \quad \alpha \in A.$$

Consider the sequence

$$\delta\big(\{t_\alpha\}_{\alpha\in A}\big) : \cdots \longrightarrow \{G_\alpha^{i+1}\} \longrightarrow \{G_\alpha^i\}_{\alpha\in A} \xrightarrow{\boldsymbol{h}^i} \{G_\alpha^{i-1}\}_{\alpha\in A} \longrightarrow \cdots,$$

where $\boldsymbol{h}^i = (\{h_\alpha^i\}_{\alpha\in A}, 1_A)$. By Theorem 35 this sequence is exact. We summarize this result as follows.

Corollary 36. *There is functor* $\delta : \mathbf{inj}-\mathbf{LES}(\mathbf{Gr}) \longrightarrow \mathbf{LES}(\mathbf{inj}-\mathbf{Gr})$.

By Theorem 10 of ([24], Ch. II,§2,3) there exists the functor [14]

$$\gamma : \mathbf{pro}-\mathbf{LES}(\mathbf{Gr}) \longrightarrow \mathbf{LES}(\mathbf{pro}-\mathbf{Gr}),$$

which to each family $\{s_\alpha\}_{\alpha\in A} \in \mathbf{pro}-\mathbf{LES}(\mathbf{Gr})$ of exact sequences

$$s_\alpha : \cdots \longrightarrow G_\alpha^{i+1} \longrightarrow G_\alpha^i \xrightarrow{h_\alpha^i} G_\alpha^{i-1} \longrightarrow \cdots, \quad i\in Z, \ \ \alpha\in A$$

assigns the exact sequence

$$\gamma(\{s_\alpha\}_{\alpha\in A}) : \cdots \longrightarrow \{G_\alpha^{i+1}\}_{\alpha\in A} \longrightarrow \{G_\alpha\}_{\alpha\in A} \xrightarrow{\boldsymbol{h}^i} \{G_\alpha^{i-1}\}_{\alpha\in A} \longrightarrow \cdots,$$

where $\boldsymbol{h}^i = (\{h_\alpha^i\}_{\alpha\in A}, 1_A)$.

Consequently, there is a functor $\gamma : \mathbf{pro}-\mathbf{LES}(\mathbf{Gr}) \longrightarrow \mathbf{LES}(\mathbf{Gr})$.

By Theorem 24 there is an $\mathbf{HCW}^2_{\boldsymbol{f}}$–coexpansion $\boldsymbol{p} = [(p_\alpha)] : (\boldsymbol{X}, \boldsymbol{X}_0) = ((X_\alpha, X_{0\alpha}), p_{\alpha\alpha'}, A) \longrightarrow (X, X_0)$ of pair $(X, X_0) \in \mathbf{HTop}^2$. It is easy to see that the restrictions $\boldsymbol{p} : \boldsymbol{X} \longrightarrow X$ and $\boldsymbol{p}_0 = \boldsymbol{p}|_{\boldsymbol{X}_0} : \boldsymbol{X}_0 \longrightarrow X_0$ are $\mathbf{HCW}^2_{\boldsymbol{f}}$–coexpansions of X and X_0, respectively.

For each index $\alpha\in A$ consider the boundary and coboundary homomorphisms $\partial_k : H_k(X_\alpha, X_{0\alpha}; G) \longrightarrow H_{k-1}(X_{0\alpha}; G)$ and $\delta^k : H^{k-1}(X_{0\alpha}; G) \longrightarrow H^k(X_\alpha, X_{0\alpha}; G)$ [17]. For each pair $\alpha \le \alpha'$ the following diagrams commute

$$\begin{array}{ccc} H_k(X_\alpha, X_{0\alpha}; G) & \xrightarrow{p_{\alpha\alpha'*}} & H_k(X_{\alpha'}, X_{0\alpha'}; G) \\ \Big\downarrow \partial_k^\alpha & & \Big\downarrow \partial_k^{\alpha'} \\ H_{k-1}(X_{0\alpha}; G) & \xrightarrow{(p_{\alpha\alpha'|X_{0\alpha}})_*} & H_{k-1}(X_{0\alpha'}; G), \end{array}$$

$$\begin{array}{ccc} H^k(X_\alpha, X_{0\alpha}; G) & \xleftarrow{p^*_{\alpha\alpha'}} & H^k(X_{\alpha'}, X_{0\alpha'}; G) \\ \Big\uparrow \delta_\alpha^k & & \Big\uparrow \delta_{\alpha'}^k \\ H^{k-1}(X_{0\alpha}; G) & \xleftarrow{(p_{\alpha\alpha'|X_{0\alpha'}})^*} & H^{k-1}(X_{0\alpha'}; G). \end{array}$$

Consequently, we have the following boundary and coboundary morphisms

$$\boldsymbol{\partial}_k : \mathrm{inj}-H_k(X, X_0; G) \longrightarrow \mathrm{inj}-H_{k-1}(X_0; G),$$

$$\boldsymbol{\delta}^k : \mathrm{pro}-H^{k-1}(X_0; G) \longrightarrow \mathrm{pro}-H^k(X, X_0; G).$$

Consider the following sequences of inj–groups and pro–groups of pair $(X, X_0) \in \mathbf{HTop}^2$:

$$\cdots \longrightarrow \mathrm{inj} - H_k(X_0; G) \xrightarrow{\boldsymbol{i}_k} \mathrm{inj} - H_k(X; G) \xrightarrow{\boldsymbol{j}_k}$$

$$\longrightarrow \mathrm{inj} - H_k(X, X_0; G) \xrightarrow{\boldsymbol{\partial}_k} \mathrm{inj} - H_{k-1}(X_0; G) \longrightarrow \cdots \tag{4}$$

$$\cdots \longrightarrow \mathrm{pro} - H^{k-1}(X_0; G) \xrightarrow{\boldsymbol{\delta}^k} \mathrm{pro} - H^k(X, X_0; G) \xrightarrow{\boldsymbol{j}^k}$$

$$\longrightarrow \mathrm{pro} - H^k(X; G) \xrightarrow{\boldsymbol{i}^k} \mathrm{pro} - H^k(X_0; G) \longrightarrow \cdots \tag{5}$$

The morphisms $\boldsymbol{i}_k, \boldsymbol{j}_k$ and $\boldsymbol{i}^k, \boldsymbol{j}^k$ are defined by the special morphisms $(i_\alpha, 1_A) : \boldsymbol{X}_0 \longrightarrow \boldsymbol{X}$ and $(j_\alpha, 1_A) : \boldsymbol{X} \longrightarrow (\boldsymbol{X}, \boldsymbol{X}_0)$ which also induce the following special morphisms

$$(i_{\alpha k}, 1_A) : H_k(\boldsymbol{X}_0; G) \longrightarrow H_k(\boldsymbol{X}; G),$$
$$(j_{\alpha k}, 1_A) : H_k(\boldsymbol{X}; G) \longrightarrow H_k(\boldsymbol{X}, \boldsymbol{X}_0; G)$$

and

$$(i^k_\alpha, 1_A) : H^k(\boldsymbol{X}; G) \longrightarrow H^k(\boldsymbol{X}_0; G),$$
$$(j^k_\alpha, 1_A) : H^k(\boldsymbol{X}, \boldsymbol{X}_0; G) \longrightarrow H^k(\boldsymbol{X}; G).$$

Note that for each index $\alpha \in A$ the following sequences are exact

$$\cdots \longrightarrow H_k(X_{0\alpha}; G) \xrightarrow{i_{\alpha k}} H_k(X_\alpha; G) \xrightarrow{j_{\alpha k}}$$

$$\longrightarrow H_k(X_\alpha, X_{0\alpha}; G) \xrightarrow{\partial^\alpha_k} H_{k-1}(X_{0\alpha}; G) \longrightarrow \cdots \tag{6}$$

$$\cdots \longrightarrow H^{k-1}(X_{0\alpha}; G) \xrightarrow{\delta^k_\alpha} H^k(X_\alpha, X_{0\alpha}; G) \xrightarrow{j^k_\alpha}$$

$$\longrightarrow H^k(X_\alpha; G) \xrightarrow{i^k_\alpha} H^k(X_{0\alpha}; G) \longrightarrow \cdots \tag{7}$$

Using Theorem 35 of Section 1 and Theorem 10 of ([24], Ch. II, §2.3) for sequences (6) and (7) we can conclude that the sequence of inj-groups (4) and the sequence of pro-groups (5) are exact sequences.

Also note that there exists a boundary morphism $\boldsymbol{\partial}_k : \mathrm{inj} - \pi_k(X, X_0, *) \longrightarrow \mathrm{inj} - \pi_{k-1}(X_0, *)$ and sequence of inj-homotopy groups of pointed pair $(X, X_0, *)$

$$\cdots \longrightarrow \mathrm{inj} - \pi_k(X_0, *) \xrightarrow{\boldsymbol{i}_k} \mathrm{inj} - \pi_k(X, *) \xrightarrow{\boldsymbol{j}_k}$$

$$\longrightarrow \mathrm{inj} - \pi_k(X, X_0, *) \xrightarrow{\boldsymbol{\partial}_k} \mathrm{inj} - \pi_{k-1}(X_0, *) \longrightarrow \cdots,$$

which by Theorem 35 is a exact sequence (cf. [30]).

7. Exact Sequences of Maps

In this section we give a concept of the coshape of continuous maps. Just as a topological space X can be approximated by a direct system of topological spaces having the homotopy type of finite CW–complexes, a continuous map of topological spaces $f : X \longrightarrow Y$ also can be approximated by a direct system of continuous maps of topological spaces having the homotopy type of finite CW–complexes.

Let $\boldsymbol{\mathcal{T}}$ be any category and let the symbol $\mathbf{Mor}_{\boldsymbol{\mathcal{T}}}$ denote the category whose objects are all the morphisms $f : X \longrightarrow Y$ of the category $\boldsymbol{\mathcal{T}}$ and whose morphisms are all the pairs $\varphi = (\varphi^1, \varphi^2) : f \longrightarrow f' : X' \longrightarrow Y'$ of morphisms $\varphi^1 : X \longrightarrow X'$ and $\varphi^2 : Y \longrightarrow Y'$ for which holds the equality $f' \cdot \varphi^1 = \varphi^2 \cdot f$. If $\boldsymbol{\mathcal{T}}$ is a category with homotopies, then we can associate to the category $\mathbf{Mor}_{\boldsymbol{\mathcal{T}}}$ the homotopy category $\mathbf{HMor}_{\boldsymbol{\mathcal{T}}}$, whose objects are all the objects of category $\mathbf{Mor}_{\boldsymbol{\mathcal{T}}}$ and whose morphisms are all the homotopy classes of morphisms in $\mathbf{Mor}_{\boldsymbol{\mathcal{T}}}$. In particular, we say that two morphisms $(\varphi^1, \varphi^2), (\psi^1, \psi^2) : f \longrightarrow f'$ are homotopic if there exists a morphism $(F^1, F^2) : f \times 1_I \longrightarrow f'$ consisting of the homotopies $F^1 : \varphi^1 \simeq \psi^1$ and $F^2 : \varphi^2 \simeq \psi^2$ ([13], [31]). Let $[(\varphi^1, \varphi^2)]$ be the equivalence class of (φ^1, φ^2) under this relation. There exists the functor

$$\mathrm{Mor}_{\mathrm{H}} : \mathbf{Mor}_{\boldsymbol{\mathcal{T}}} \longrightarrow \mathbf{HMor}_{\boldsymbol{\mathcal{T}}}$$

such that $\mathrm{Mor}_{\mathrm{H}}(f) = f$ for each object $f \in \mathbf{Mor}_{\boldsymbol{\mathcal{T}}}$ and $\mathrm{Mor}_{\mathrm{H}}((\varphi^1, \varphi^2)) = [(\varphi^1, \varphi^2)]$ for each morphism $(\varphi^1, \varphi^2) : f \longrightarrow f'$ on the category $\mathbf{Mor}_{\boldsymbol{\mathcal{T}}}$.

Let $\mathbf{Ssc}$ be the category of semisimplicial complexes and semisimlicial maps. Let $f : X \longrightarrow Y$ be a continuous map and let $S(f) : S(X) \longrightarrow S(Y)$ be a semisimplicial map of semisimplicial singular complexes $S(X)$ and $S(Y)$, induced by f. Consider the families $\{X_\alpha\}_{\alpha \in A}$ and $\{Y_\beta\}_{\beta \in B}$ of all finite subcomplexes of $S(X)$ and $S(Y)$, respectively. By $i_{\alpha\alpha'} : X_\alpha \longrightarrow X_{\alpha'}$ and $j_{\beta\beta'} : Y_\beta \longrightarrow Y_{\beta'}$ we denote the inclusion semisimplicial maps. The set of all pairs (α, β) for which $S(f)(X_\alpha) \subseteq Y_\beta$ is a directed set $(M, \leq)$ with the following order relation:

$$(\alpha, \beta) \leq (\alpha', \beta') \Leftrightarrow \alpha \leq \alpha', \;\; \beta \leq \beta'.$$

Let $f_{(\alpha,\beta)} = S(f)_{|X_\alpha} : X_\alpha \longrightarrow Y_\beta$. The pair $\pi_{(\alpha,\beta)(\alpha',\beta')} = (i_{\alpha\alpha'}, j_{\beta\beta'})$, $(\alpha, \beta) \leq (\alpha', \beta')$, is a morphism of $f_{(\alpha,\beta)}$to $f_{(\alpha',\beta')}$, because

$$j_{\beta\beta'} \cdot f_{(\alpha,\beta)} = j_{\beta\beta'} \cdot S(f)_{|X_\alpha} = S(f)_{|X_{\alpha'}} \cdot i_{\alpha\alpha'} = f_{(\alpha',\beta')} \cdot i_{\alpha\alpha'}.$$

It is clear that the family $\Omega(f) = \{f_{(\alpha,\beta)}, \pi_{(\alpha,\beta)(\alpha',\beta')}, M\}$ is direct system of the category $\mathbf{Mor}_{\mathbf{Ssc}}$. A morphism $\varphi = (\varphi^1, \varphi^2) : f \longrightarrow f'$ of the category $\mathbf{Mor}_{\mathbf{Top}}$ induces a morphism

$$\Omega(\varphi) : \Omega(f) = \{f_{(\alpha,\beta)}, \pi_{(\alpha,\beta)(\alpha',\beta')}, M\} \longrightarrow \Omega(f') = \{f'_{(\gamma,\delta)}, \pi'_{(\gamma,\delta)(\gamma',\delta')}, M'\}$$

in the obvious way. Assume that $S(\varphi^1)(X_\alpha) = X'_\gamma$ and $S(\varphi^2)(Y_\beta) = Y'_\delta$. Let $\theta : M \longrightarrow M'$ be a map given by $\theta(\alpha,\beta) = (\gamma,\delta)$. A pair $\varphi_{(\alpha,\beta)} = (\varphi^1_{(\alpha,\beta)}, \varphi^2_{(\alpha,\beta)})$, where $\varphi^1_{(\alpha,\beta)} = S(\varphi^1)_{|X_\alpha} : X_\alpha \longrightarrow X'_\gamma$ and $\varphi^2_{(\alpha,\beta)} = S(\varphi^2)_{|Y_\beta} : Y_\beta \longrightarrow Y'_\delta$, is a morphism of $f_{(\alpha,\beta)}$ to $f'_{(\gamma,\delta)}$. It is easy to see that the family $(\varphi_{\alpha,\beta)}, \theta)$ is desired morphism $\Omega(\varphi)$. For simplicity we put $\mu = (\alpha,\beta)$, $X_\mu = X_\alpha$, $Y_\mu = Y_\beta$ for each $(\alpha,\beta) \in M$ and $p_{\mu\mu'} = i_{\alpha\alpha'}$, $q_{\mu\mu'} = j_{\beta\beta'}$ for each $\mu = (\alpha,\beta) \leq \mu' = (\alpha',\beta')$. Consequently, we have obtained the functor

$$\Omega : \mathbf{Mor_{Top}} \longrightarrow \mathbf{dir} - \mathbf{Mor_{Ssc}},$$

which to each object $f : X \longrightarrow Y$ and morphism $\varphi = (\varphi^1, \varphi^2) : f \longrightarrow f'$ of the category $\mathbf{Mor_{Top}}$ assigns a direct system $\Omega(f) = \{f_\mu, \pi_{\mu\mu'}, M\}$ and morphism $\Omega(\varphi) = (\varphi_\mu, \theta)$, respectively. Also note that the direct systems $\boldsymbol{X} = (X_\mu, p_{\mu\mu'}, M)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ are coassociated with $X$and Y , respectively.

The geometric realization functor $\mathrm{R} : \mathbf{Ssc} \longrightarrow \mathbf{CW}$ [22] induces the functor

$$\mathrm{Mor_R} : \mathbf{Mor_{Ssc}} \longrightarrow \mathbf{Mor_{CW}},$$

which to each semisimplicial map f assigns its geometric realization $|f|$. We have obtained the following functors:

$$\mathrm{dir} - \mathrm{Mor_R} : \mathbf{dir} - \mathbf{Mor_{Ssc}} \longrightarrow \mathbf{dir} - \mathbf{Mor_{CW}},$$

$$\mathrm{dir} - \mathrm{Mor_H} : \mathbf{dir} - \mathbf{Mor_{CW}} \longrightarrow \mathbf{dir} - \mathbf{Mor_{CW}}.$$

Let E: $\mathbf{dir} - \mathbf{HMor_{CW}} \longrightarrow \mathbf{inj}\text{–}\mathbf{HMor_{CW}}$ be the quotient functor from the category $\mathbf{dir} - \mathbf{HMor_{CW}}$ to the quotient category $\mathbf{inj}\text{–}\mathbf{HMor_{CW}}$. Assume that

$$\mathrm{T} = \mathrm{E} \cdot (\mathrm{dir}\text{–}\mathrm{Mor_H}) \cdot (\mathrm{dir}\text{–}\ \mathrm{Mor_R}) \cdot \Omega : \mathbf{Mor_{Top}} \longrightarrow \mathbf{inj}\text{–}\mathbf{HMor_{CW}}.$$

Similarly we can define the functor

$$\mathrm{T}_* : \mathbf{Mor_{Top*}} \longrightarrow \mathbf{inj}\text{–}\mathbf{HMor_{CW*}}.$$

The association to each cellular map $f : (X,*) \longrightarrow (Y,*)$ of pointed CW-spaces the long exact sequence

$$\cdots \longrightarrow \pi_i(X,*) \longrightarrow \pi_j(Y,*) \longrightarrow \pi_i(f) \longrightarrow \cdots,$$

where $\pi_i(f) = \pi_i(\mathrm{Cyl}(f), X, *)$, induces the functor

$$\pi : \mathbf{HMor_{CW*}} \longrightarrow \mathbf{inj}\text{–}\mathbf{LES}(\mathbf{Gr}).$$

Let $T(f) = (|f_\mu|, |\varphi_{\mu\mu'}|, M)$. The association of this sequence to each term $|f_\mu| : |X_\mu| \longrightarrow |Y_\mu|$ of $T(f)$ and applying of inj to π yield a functor

$$\mathrm{inj}\text{–}\pi : \mathbf{inj} - \mathbf{HMor_{CW*}} \longrightarrow \mathbf{inj}\text{–}\mathbf{LES}(\mathbf{Gr}).$$

The composition of functors T_*, inj–π and δ gives the functor

$$\delta \cdot (\mathrm{inj} - \pi) \cdot \mathrm{T}_* : \mathbf{Mor_{Top*}} \longrightarrow \mathbf{LES}(\mathbf{Gr}).$$

By the inj–$\pi_i(f)$ denote inj–group $\{\pi_i(|f_\mu|)\}_{\mu\in M}$. The resulting long exact sequence of inj–groups looks as follows.

Theorem 37. *Let $f:(X,*)\longrightarrow(Y,*)$ be a continuous map of pointed spaces. Then there is a long exact sequence*

$$\cdots\longrightarrow \mathrm{inj}-\pi_i(X,*)\longrightarrow \mathrm{inj}-\pi_i(Y,*)\longrightarrow \mathrm{inj}-\pi_i(f)\longrightarrow\cdots$$

Analogously we can prove the homological version of this theorem.

Theorem 38. *For any continuous map $f:X\longrightarrow Y$ there is a long exact sequence*

$$\cdots\longrightarrow \mathrm{inj}-H_i(X;G)\longrightarrow \mathrm{inj}-H_i(Y;G)\longrightarrow \mathrm{inj}-H_i(f;G)\longrightarrow\cdots,$$

where inj-$H_i(f;G)=\{H_i(\mathrm{Cyl}(|f_\mu|),X_\mu;G)\}_{\mu\in M}$ *is* inj*–group consisting of singular homology groups of pair* $(\mathrm{Cyl}(|f_\mu|),X_\mu)$*, $\mu\in M$ with coefficients in the abelian group G.*

Note that for each term $|f_\mu|:X_\mu\longrightarrow Y_\mu$, $\mu\in M$ of direct system $T(f)$ the sequence

$$\cdots\longrightarrow H^i(\mathrm{Cyl}(|f_\mu|),X_\mu;G)\longrightarrow H^i(Y_\mu;G)\longrightarrow H^i(X_\mu;G)\longrightarrow\cdots$$

is exact. Thus we have the functor

$$\mathrm{H}:\mathbf{HMor_{CW}}\longrightarrow\mathbf{LES}(\mathbf{Gr}),$$

which induces the functor

$$\mathrm{inj}-\mathrm{H}:\mathbf{inj}-\mathbf{HMor_{CW}}\longrightarrow\mathbf{pro}-\mathbf{LES}(\mathbf{Gr}).$$

The composition of functors γ, inj–H and T yields the functor

$$\gamma\cdot\mathrm{inj}-\mathrm{H}\cdot\mathrm{T}:\mathbf{Mor_{Top}}\longrightarrow\mathbf{LES}(\mathbf{pro}-\mathbf{Gr}).$$

Thus we have the following

Theorem 39. *For any continuous map $f:X\longrightarrow Y$ there is an exact sequence*

$$\cdots\longrightarrow \mathrm{pro}-H^i(f;G)\longrightarrow \mathrm{pro}-H^i(Y;G)\longrightarrow \mathrm{pro}-H^i(X;G)\longrightarrow\cdots,$$

where pro–$H^i(f;G)=\{H^i(\mathrm{Cyl}(|f_\mu|),X_\mu;G)\}_{\mu\in M}$ *is pro–group consisting of singular cohomology groups of pairs* $(\mathrm{Cyl}(|f_\mu|),X_\mu)$*, $\mu\in M$ with coefficients in the abelian group G.*

8. The Relative Hurewicz Theorem in Coshape Theory

In this section we establish the analogue of the relative Hurewicz theorem [31] in the categories $\mathbf{inj}-\mathbf{HCW}^2_{\boldsymbol{f}}$ and $\mathbf{CSH}^2_*$.

Let $h : (I^k, \partial I^k, s_0) \longrightarrow (X, X_0, *)$ be a representative of element $\alpha \in \pi_k(X, X_0, *)$, $I = [0,1]$. Let a_k be the canonical generator of group $H_k(I^k, \partial I^k; \mathbb{Z}) \approx \mathbb{Z}$ and let $h_* = H_k(h) : H_k(I^k, \partial I^k; \mathbb{Z}) \longrightarrow H_k(X, X_0; \mathbb{Z})$ be the homomorphism induced by the map h. With each pointed pair $(X, X_0, *)$ is associated the relative Hurewicz homomorphism $\varphi = \varphi_{(X,X_0,*)} : \pi_k(X, X_0, *) \longrightarrow H_k(X, X_0; \mathbb{Z})$, $k \geq 1$ which is given by the formula $\varphi(a_k) = h_*(a_k)$. The family $\{\varphi_{(X,X_0,*)} | (X, X_0, *) \in \mathbf{Top}^2_*\}$ is natural transformation of the homotopic functor $\pi_k(-,-,*)$ to the singular homology functor $H_k(-,-;\mathbb{Z})$.

The well-known classical Hurewicz theorem in relative case asserts [31]: Let $(X, X_0, *)$ be a $(n-1)$–connected pointed pair of topological spaces, $n \geq 2$, and let X be path connected. Then $H_k(X, X_0; \mathbb{Z}) = 0$ for $0 \leq k \leq n-1$. If, in addition, $(X_0, *)$ is 1-connected, then the Hurewicz homomorphisms $\pi_n(X, X_0, *) \longrightarrow H_n(X, X_0; \mathbb{Z})$ and $\pi_{n+1}(X, X_0, *) \longrightarrow H_{n+1}(X, X_0; \mathbb{Z})$ are an isomorphism and epimorphism, respectively.

Let $(\boldsymbol{X}, \boldsymbol{X}_0, *) = ((X_\alpha, X_{0\alpha}, *), p_{\alpha\alpha'}, A) \in \mathbf{inj}-\mathbf{HTop}^2_*$. Define the Hurewicz morphism $\boldsymbol{\varphi} : \pi_k(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow H_k(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z})$, $k \geq 1$, as the special morphism (φ_α) of direct systems, where $\varphi_\alpha = \varphi_{(X_\alpha, X_{0\alpha}, *)} : \pi_k(X_\alpha, X_{0\alpha}, *) \longrightarrow H_k(X_\alpha, X_{0\alpha}; \mathbb{Z})$ is the relative Hurewicz homomorphism. For a pair $(X, X_0, *) \in \mathbf{HTop}^2_*$ the morphism $\boldsymbol{\varphi} : \pi_k(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow H_k(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z})$, where $(\boldsymbol{X}, \boldsymbol{X}_0, *) \in \mathbf{inj}-\mathbf{HCW}^2_{\boldsymbol{f}*}$ is coassociated to $(X, X_0, *)$, defines the relative Hurewicz morphism inj–$\pi_k(X, X_0, *) \longrightarrow$ inj $- H_k(X, X_0; \mathbb{Z})$, which for simplicity we again denote by $\boldsymbol{\varphi}$: inj $- \pi_k(X, X_0, *) \longrightarrow$ inj $- H_k(X, X_0; \mathbb{Z})$.

Definition 1. An object $(\boldsymbol{X}, \boldsymbol{X}_0, *) \in \mathbf{inj}-\mathbf{HTop}^2_*$ is n–coconnected if $\pi_k(\boldsymbol{X}, \boldsymbol{X}_0, *) = 0$, $0 \leq k \leq n$.

Definition 2. A pointed pair of spaces $(X, X_0, *)$ is said to be n–coshape coconnected if its $\mathbf{HCW}^2_{\boldsymbol{f}*}$–coexpansions are n–coconnected, i.e. inj–$\pi_k(X, X_0, *) = 0$ for $0 \leq k \leq n$.

Now we prove two lemmas which we need in next.

Lemma 40. *Let $p_i : (X_i, A_i, *) \longrightarrow (X_{i+1}, A_{i+1}, *)$, $n \geq 2$, $i = 0, 1, \dots, n-1$ be maps of pointed pairs of finite* CW*–simplicial complexes such that X_0 is connected and $p_{i\#} : \pi_i(X_i, A_i, *) \longrightarrow \pi_i(X_{i+1}, A_{i+1}, *)$ is equal* 0 *for $i = 1, 2, \dots, n-1$. Then the map*

$$p_{n-1} \cdot p_{n-2} \cdot \ \cdots \ \cdot p_1 \cdot p_0 : (X_0, A_0, *) \longrightarrow (X_n, A_n, *)$$

factors through a finite $(n-1)$–connected CW*–simplicial pair $(Y, B, *)$ with connected Y.*

Proof. It is easy to see that there is a finite triangulation (K, L) of (X_0, A_0) such that L is a complete subcomplex of complex K (see [24], Appendix 1, §1.3). Consider i-th skeleton K^i of K. Let $(Y_i, B_i, *)$, $i = 0, 1, \dots, n-1$, be the pair consisting of the finite CW–simplicial complexes $B_i = (A_0 \times I) \cup (|K^i| \times I)$ and $Y_i = B_i \cup (X_0 \times I)$. Note that

$$(X_0 \times \{0\}, A_0 \times \{0\}, *) \subseteq (Y_0, B_0, *) \subseteq \cdots \subseteq (Y_{n-1}, B_{n-1}, *) = (Y, B, *).$$

As in ([24], see the proof of Lemma 3, Ch. II,§4.2) we can prove that $(Y, B, *)$ is $(n-1)$–connected CW–simplicial pair and there exist the maps $f : (X_0, A_0, *) \longrightarrow (Y, B, *)$ and $g : (Y, B, *) \longrightarrow (X_n, A_n, *)$ such that $p_{n-1} \cdot p_{n-2} \cdot \cdots \cdot p_1 \cdot p_0 = g \cdot f$. □

Lemma 41. *If a direct system* $(\boldsymbol{X}, \boldsymbol{X}_0, *) \in \mathbf{inj} - \mathbf{HCW}^2_{\boldsymbol{f}*}$ *is* $(n-1)$*–coconnected,* $n \geq 2$*, and* $\pi_0(\boldsymbol{X}, *) = 0$*, then for each index* $\alpha \in A$ *there exists an index* $\alpha' \geq \alpha$ *such that* $p_{\alpha\alpha'}$ *factors in* $\mathbf{HCW}^2_{\boldsymbol{f}*}$ *through an* $(n-1)$*–connected* CW*–simplicial pair* $(Y, B, *)$ *with connected* Y*. Moreover, if* $(\boldsymbol{X}_0, *)$ *is* 1*–coconnected, then* $(B, *)$ *also is* 1*–connected.*

Proof. We can assyme that all terms $(X_\alpha, X_{0\alpha}, *)$ of $(\boldsymbol{X}, \boldsymbol{X}_0, *)$ are CW–simplicial pairs and all X_α are connected. Since $\pi_j(\boldsymbol{X}, \boldsymbol{X}_0, *) = 0$, $0 \leq j \leq n-1$ and $\pi_0(\boldsymbol{X}, *) = 0$, by Proposition 3 of [32], for each index $\alpha \in A$ one can find indexes $\alpha = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n = \alpha'$ such that $\pi_{n-i}(p_{\alpha_{i-1}\alpha_i}) = 0$, $i = 1, \dots, n$. Using Lemma 40 we obtain the first assertion of the lemma.

Now establish the second assertion of the lemma. By the first assertion of the lemma for an given index $\alpha \in A$ there is an index $\alpha_1 \geq \alpha$ and a factorization $p_{\alpha\alpha_1} = g \cdot f$ through an $(n-1)$ connected pair $(Y, B, *)$ of CW–simplicial complexes with connected Y. There also exists an index $\alpha' \geq \alpha_1$ such that $p_{\alpha_1\alpha'}$ induces zero homomorphism $\pi_k(X_{0\alpha_1}, *) \longrightarrow \pi_k(X_{0\alpha'}, *)$ for $k = 0, 1$. Let $C = B \cup v|K^1|$ be the union of B and the cone $v|K^1|$ over 1-skeleton K^1 of B, and let $Z = Y \cup C$. It is clear that the composition $p_{\alpha_1\alpha'} \cdot g$ induces zero homomorphisms $\pi_k(B, *) \longrightarrow \pi_k(X_{0\alpha'}, *)$, $k = 0, 1$. Consequently, there exists an H–extension $h : (Z, C, *) \longrightarrow (X_{\alpha'}, X_{0\alpha'}, *)$ of map $p_{\alpha_1\alpha'} \cdot g$. Thus $p_{\alpha_1\alpha'} = h \cdot (i \cdot f)$, where $i : (Y, B, *) \longrightarrow (Z, C, *)$ is the homotopy class of the inclusion map. Note that $(C, *)$ is 1–connected and Z is connected. Proceeding now as in the proof of Lemma 4 of ([24], Ch. II,§4.2), we prove that $(Z, C, *)$ is $(n-1)$–connected. □

Now we state the relative Hurewicz isomorphism theorem in the category $\mathbf{inj}$–$\mathbf{HCW}^2_{\boldsymbol{f}*}$.

Theorem 42. *Let* $(\boldsymbol{X}, \boldsymbol{X}_0, *) \in \mathbf{inj} - \mathbf{HCW}^2_{\boldsymbol{f}*}$ *be* $(n-1)$*–coconnected,* $n \geq 2$*, and let* $\pi_0(\boldsymbol{X}, *) = 0$*. Then*

i) $H_k(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z}) = 0$, $0 \leq k \leq n-1$.
If, in addition, $(\boldsymbol{X}_0, *)$ *is* 1*–coconnected, then*

ii) $\boldsymbol{\varphi}_n : \pi_n(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow H_n(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z})$ *is an isomorphism of* inj*–groups, and*

iii) $\boldsymbol{\varphi}_{n+1} : \pi_{n+1}(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow H_{n+1}(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z})$ *is an epimorphism of* inj*–groups.*

Proof. Let $(\boldsymbol{X}, \boldsymbol{X}_0, *)$ be $(n-1)$-coconnected, $n \geq 2$, and let $\pi_0(\boldsymbol{X}, *) = 0$. By Lemma 41 for an fixed index $\alpha \in A$ there is an index $\alpha' \geq \alpha$ and a factorization $p_{\alpha\alpha'} = g \cdot f$ in $\mathbf{HCW}^2_{\boldsymbol{f}*}$ through a CW–simplicial pair $(Y, B, *)$ which is $(n-1)$–connected with connected Y. For each k we have the following commutative diagram

$$\begin{array}{ccccc} \pi_k(X_\alpha, X_{0\alpha}, *) & \xrightarrow{f_\#} & \pi_k(Y, B, *) & \xrightarrow{g_\#} & \pi_k(X_{\alpha'}, X_{0\alpha'}, *) \\ \downarrow{\scriptstyle \varphi_\alpha} & & \downarrow{\scriptstyle \varphi_{(Y,B,*)}} & & \downarrow{\scriptstyle \varphi_{\alpha'}} \\ H_k(X_\alpha, X_{0\alpha}; \mathbb{Z}) & \xrightarrow[f_*]{} & H_k(Y, B; \mathbb{Z}) & \xrightarrow[g_*]{} & H_k(X_{\alpha'}, X_{0\alpha'}; \mathbb{Z}), \end{array}$$

where $f_\# = \pi_k(f)$, $g_\# = \pi_k(g)$, $f_* = H_k(f)$, $g_* = H_k(g)$ and $g_\# \cdot f_\# = \pi_k(p_{\alpha\alpha'})$, $g_* \cdot f_* = H_k(p_{\alpha\alpha'})$. For each $0 \leq k \leq n-1$ we have $H_k(Y, B; \mathbb{Z}) = 0$. Consequently, $H_k(p_{\alpha\alpha'}) = 0$. From Proposition 3 of [32] follows the assertion i) $H_k(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z}) = 0$.

If in addition, $(\boldsymbol{X}_0, *)$ is 1–coconnected, then we can assume that $(B, *)$ 1–connected. Note that for $k = n$ the homomorphism $\boldsymbol{\varphi}_{(Y,B,*)} : \pi_k(Y, B, *) \rightarrow H_k(Y, B; \mathbb{Z})$ is an isomorphism and the homomorphism $h = g_\# \cdot (\varphi_{(Y,B,*)})^{-1} \cdot f_* : H_n(X_\alpha, X_{0\alpha}; \mathbb{Z}) \longrightarrow \pi_n(X_{\alpha'}, X_{0\alpha'}, *)$ satisfies the conditions $h \cdot \varphi_\alpha = p_{\alpha\alpha'\#}$ and $\varphi_{\alpha'} \cdot h = p_{\alpha\alpha'*}$. As an immediate consequence of Theorem 13, we have that $\boldsymbol{\varphi}_n : \pi_n(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow H_n(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z})$ is an isomorphism.

Let $k = n+1$. In this case the morphism $\varphi_{(Y,B,*)} : \pi_{n+1}(Y, B, *) \longrightarrow H_{n+1}(Y, B; \mathbb{Z})$ is an isomorphism. From the above diagram with $k = n+1$ and equality $p_{\alpha\alpha'*} = g_* \cdot f_*$ follows that

$$\mathrm{Im}(p_{\alpha\alpha'*}) \subseteq \mathrm{Im}(\varphi_{\alpha'}).$$

Using Proposition 2* of [32] we conclude that $\boldsymbol{\varphi}_{n+1} : \pi_{n+1}(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow H_{n+1}(\boldsymbol{X}, \boldsymbol{X}_0; \mathbb{Z})$ is an epimorphism. □

The Theorem 42 yields the relative Hurewicz theorem in coshape theory.

Theorem 43. *Let $(X, X_0, *)$ be a pointed pair of topological spaces and let X be a connected space. If $(X, X_0, *)$ is $(n-1)$–coshape coconnected, $n \geq 2$, then*

i) $\mathrm{inj} - H_k(X, X_0; \mathbb{Z}) = 0, \quad 0 \leq k \leq n-1.$
If, in addition, $(X_0, *)$ is 1–coshape coconnected, then

ii) *the Hurewicz morphism $\boldsymbol{\varphi}_n : \mathrm{inj}{-}\pi_n(X, X_0, *) \longrightarrow \mathrm{inj}{-}H_n(X, X_0; \mathbb{Z})$ is an isomorphism of* inj*–groups, and*

iii) *the Hurewicz morphism* $\boldsymbol{\varphi}_{n+1}$:inj–$\pi_{n+1}(X, X_0, *) \longrightarrow$ inj – $H_{n+1}(X, X_0; \mathbb{Z})$ *is an epimorphism of* inj–*groups.*

Finally, we give the following

Question. *Is there a coshape analog of the shape dimension theory of topological spaces*?

A plan to investigate this question in a future paper.

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Author's address:

Department of Mathematics of
Shota Rustaveli Batumi State University
35, Ninoshvili st., Batumi 6010
Georgia