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ON THE CONSTRUCTION OF SOLUTIONS OF SPATIAL AXISYMMETRIC STATIONARY WITH PARTIALLY UNKNOWN BOUNDARIES PROBLEMS OF THE THEORY OF JET FLOWS

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Abstract. In the present work we present a general mathematical method of constructing stationary solutions of spatial axisymmetric, with partially unknown boundaries, problems of jet theory, in particular, we consider a liquid flow of finite width round a spatial circular wedge. Unknown functions (velocity potential, flow function) and their arguments on each interval of the boundary satisfy two inhomogeneous boundary conditions. The system of differential equations with respect to the velocity potential and flow function are reduced to the normal form. Unknown functions are represented by a sum of holomorphic and generalized analytic functions. One problem from the theory of spatial jets is solved.

რეზიუმე. ნაშრომში გადმოცემულია ჭავლური თეორიის სივრცითი ღერძსიმეტრიული ნაწილობრივ უცნობსაზღვრიანი სტაციონარული ამოცანების აგების ზოგადი მათემატიკური მეთოდი. კერძოდ, განხილულია წრიული სივრცითი გარსდენის ამოცანა სასრული სისქის სითხის ნაკადით.

საძიებელმა ფუნქციებმა (სიჩქარის პოტენციალმა, დენის ფუნქციამ) და ამ ფუნქციების არგუმენტებმა საზღვრის ყველა უბანზე უნდა დააკმაყოფილოს ორი არაერთგვაროვანი სასაზღვრო პირობა. სიჩქარის პოტენციალის და დენის ფუნქციის მიმართ დიფერენციალურ განტოლებათა სისტემა დაყვანილია ნორმალურ დიფერენციალურ განტოლებებამდე. საძიებელი ფუნქციები წარმოდგენილია ჰოლომორფული და განზოგადებული ანალიზური ფუნქციების ჯამის სახით.

ამოხსნილია ჭავლური თეორიის ერთ-ერთი სივრცითი ამოცანა.

1. Statement of the Problem and Mathematical Methods of Their Solutions

The liquid motion is said to be spatially axisymmetric, if all vectors of velocity lie in half-planes passing through a straight line which is called the

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symmetry axis, and a picture of flow at all such half-planes is the same. The field of velocities of axisymmetric motion is completely circumscribed by a plane field on every such half-plane ([1–21]).

Solution of spatial stationary jet problems with partially unknown boundaries is of great mathematical difficulty. At present we are aware only of the works connected with axisymmetric jet flows. However, even for that simplest particular case one did not succeed in creating a mathematical apparatus convenient to that of the theory of functions of a complex variable. The authors of works engaged in axisymmetric jet flows either restrict themselves to approximate numerical solutions of the problems, or prove theorems of general character ([1–4]).

In this as well as in the earlier works ([12–14]) the author presents a general mathematical method of constructing solutions of spatial axisymmetric stationary, with partially unknown boundaries, problems of jet theory. In the present paper the use will be made of the essentials of the general theory of generalized analytic functions ([1–21]).

Consider a stationary axisymmetric irrotational flow with partially unknown boundaries of a perfect weightless incompressible liquid. Let the x-axis coincide with the symmetry axis of the flow. The velocity potential φ and the flow function ψ are the functions of only cylindrical coordinates x and y, where y is the distance from the x-axis. Owing to the axial symmetry, it suffices to study the flow in an arbitrarily chosen meridional half-plane with the right system of Cartesian coordinates x and y. The domain occupied by the moving liquid on an arbitrarily chosen meridional half-plane we denote by $S(z)$, where $z=x+iy$. On the planes $\omega_0(z)=\varphi_0(x,y)+i\psi_0(x,y)$ and $w=\omega_0'(z)/z'(z)$, to the domain $S(z)$ there correspond, respectively, the domains $S(\omega_0'(z))$ and $S(w)$, where $w=\omega_0'(z)/z'(z)$ ([1–21]).

As is known ([1–4]), the functions $\varphi(x,y)$ and $\psi(x,y)$ satisfy the following equations:

$$\frac{\partial\varphi}{\partial x}=\frac{1}{y}\,\frac{\partial\psi}{\partial y}=v_x, \tag{1.1}$$

$$\frac{\partial\varphi}{\partial y}=-\frac{1}{y}\,\frac{\partial\psi}{\partial x}=v_y, \tag{1.2}$$

where v_x and v_y are the projections of velocity on the x and y-axes. Excluding φ and ψ from (1.1) and (1.2), we obtain

$$\Delta\varphi+\frac{1}{y}\,\frac{\partial\varphi}{\partial y}=0, \tag{1.3}$$

$$\Delta\psi-\frac{1}{y}\,\frac{\partial\psi}{\partial y}=0, \tag{1.4}$$

where

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

is the Laplace operator.

Recall that the hydrodynamical problems may be assumed to be solved if either of the functions $\varphi(x, y)$ or $\psi(x, y)$ is known. Besides equations (1.3) and (1.4), for determination of the above functions we have the following boundary conditions: the normal velocity on the free surface and on the body surface is equal to zero, $\frac{\partial\varphi}{\partial n} = 0$, where n is the normal, directed inside the liquid. On the surfaces under consideration, the flow function ψ is constant, $\psi = \text{const}$. The condition $\psi = \text{const}$ is equivalent to the condition $\frac{\partial\varphi}{\partial n} = 0$ for φ. The constant in the condition $\psi = \text{const}$ may take different values on different boundaries. For example, Fig. 1 of [1, §48] shows one-half of the meridional plane xoy for the problem of a liquid flowing round a circular cone in a tube. Since the flow function is defined within the constant summand, we can put $\psi = 0$ on the symmetry axis x, on the cone and on the free surface. But the difference of values ψ on the flow surfaces is equal to the liquid discharge between these surfaces, divided by 2π, hence on the tube walls $\psi = \pi v_\infty h^2/(2\pi)$, where h is the tube radius and v_∞ is velocity of accumulating flow from the left at infinity. The form of free surfaces is beforehand unknown, but we have a supplementary condition for the velocity modulus v to be constant, which is equivalent to that of pressure constancy. This condition can be written as

$$\frac{1}{y^2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] = \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = v_0^2,$$

where v_0 is equal to v on the free surface ([1]).

Unknown functions φ and ψ can be represented as follows: $\varphi = y^{-1/2}\varphi_1$, $\psi = y^{1/2}\psi_1$ then (1.3) and (1.4) can be written in the form

$$\Delta\varphi_1(x, y) + \frac{1}{4}\, y^{-2}\varphi_1(x, y) = 0, \tag{1.5}$$

$$\Delta\psi_1(x, y) - \frac{3}{4}\, y^{-2}\psi_1(x, y) = 0, \tag{1.6}$$

When solving the system (1.3) and (1.4), φ and ψ must satisfy the system (1.1) and (1.2). The system (1.5) and (1.6) is, obviously, not equivalent. For the system (1.5) and (1.6) to remain equivalent, we act as follows. For example, this means that we have first to define φ_1 from (1.5) and then $\varphi = y^{-1/2}\varphi_1$. Having defined φ, we find the function ψ from (1.1) and (1.2). Having found the function ψ_1 from (1.6), we define $\psi = y^{1/2}\psi_1$. Knowing ψ, we can define φ from (1.1) and (1.2).

We map conformally the half-plane $\operatorname{Im}(\zeta) \geq 0$ (or $\operatorname{Im}(\zeta) < 0$) of the auxiliary complex plane $\zeta = \xi + i\eta$ onto the domain $S(z)$ occupied by the moving liquid and onto the domains $S(\omega_0)$ and $S(w)$, where $w = \omega_0'(\zeta)/z'(\zeta)$. We obtain

$$\frac{\partial\varphi}{\partial\xi}=\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\eta}, \tag{1.7}$$

$$\frac{\partial\varphi}{\partial\eta}=-\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\xi}. \tag{1.8}$$

The system (1.7) and (1.8) can be rewritten as

$$\frac{\partial}{\partial\xi}\left(y(\xi,\eta)\frac{\partial\varphi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(y(\xi,\eta)\frac{\partial\varphi}{\partial\eta}\right)=0, \tag{1.9}$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{y(\xi,\eta)}\frac{\partial\psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{y(\xi,\eta)}\frac{\partial\varphi}{\partial\eta}\right)=0, \tag{1.10}$$

or

$$\Delta\varphi(\xi,\eta)+\frac{1}{y(\xi,\eta)}\left[\frac{\partial y}{\partial\xi}\,\frac{\partial\varphi}{\partial\xi}+\frac{\partial y}{\partial\eta}\,\frac{\partial\varphi}{\partial\eta}\right]=0, \tag{1.11}$$

$$\Delta\psi(\xi,\eta)-\frac{1}{y(\xi,\eta)}\left[\frac{\partial y}{\partial\xi}\,\frac{\partial\psi}{\partial\xi}+\frac{\partial y}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}\right]=0. \tag{1.12}$$

Assuming that there exist analytic (holomorphic) functions

$$\begin{aligned}&z=f(\zeta)=x(\xi,\eta)+iy(\xi,\eta), \quad \zeta=\xi+i\eta,\\ &\omega_0(\zeta)=\varphi_0(\xi,\eta)+i\psi_0(\xi,\eta), \quad w(\zeta)=\omega_0'(\zeta)/z(\zeta)\end{aligned} \tag{1.13}$$

we can map conformally the domain $y_n(\zeta)\geq 0$ onto the domains $S(z)$, $S(\omega_0(s))$ and $S(w)$.

It is easily seen from (1.1) and (1.2) that $\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}=0$. Consequently, the lines $\varphi(x,y)=\text{const}$ and $\psi(x,y)=\text{const}$ are orthogonal, but the mapping $f_0(x,y)=\varphi(x,y)+i\psi(x,y)$ is not conformal. It transforms infinitely small squares into infinitely small rectangles. Thus the mappings $f_0(x,y)$ under consideration are quasi-conformal ([1–21]).

Recall that from the very beginning we deal with the surfaces of rotation. This implies that $\varphi(x,y)$ and $\psi(x,y)$ are the functions of y^2, i.e., $\varphi(x,y)=\varphi_0(x,y^2)$, $\psi(x,y)=\psi_0(x,y^2)$. Note that here our notation differs from that introduced at the beginning and at the end of the paper.

The surface F is said to be the surface of rotation, if it is formed by the rotation of a curve around the x-axis. The lines of intersection of the surface and planes passing through the axis of revolution are called meridians, and the lines of intersection with planes, perpendicular to the axis, are called parallels.

We can prove that the functions $\varphi(x,y)$ and $\psi(x,y)$ are the functions of y^2, i.e., the equalities

$$\varphi(x,-y)=\varphi(x,y) \quad \text{and} \quad \psi(x,-y)=\psi(x,y) \tag{1.14}$$

hold. We write the system (1.3) and (1.4) as follows:

$$\frac{\partial^2\varphi}{\partial x^2}+4\alpha\frac{\partial^2\varphi}{\partial\alpha^2}+4\frac{\partial\varphi}{\partial\alpha^2}=0,\quad \alpha=y^2, \tag{1.15}$$

$$\frac{\partial^2\psi}{\partial x^2}+4\alpha\frac{\partial^2\psi}{\partial\alpha^2}=0,\quad \alpha=y^2. \tag{1.16}$$

Compose the equation of the surface of rotation which is formed under the rotation of some curve γ around the z-axis. Assume that we have the coordinate system x, y, z ([15])

$$x=\varphi(u),\quad z=\psi(u), \tag{1.17}$$

lying in the plane xz near the z-axis. The point $(\varphi(u), 0, \psi(u))$ of the curve γ under the rotation of the curve through the angle v turns into the point

$$\big(\varphi(u)\cos v,\ \varphi(u)\sin v,\ \psi(u)\big), \tag{1.18}$$

whence we obtain the equation of the surface of rotation:

$$x=\varphi(u)\cos v,\ \ y=\varphi(u)\sin v,\ \ z=\psi(u). \tag{1.19}$$

The lines v = const are the surface meridians and u = const are the parallels. We can see that meridians and parallels of the surface of rotation form orthogonal net.

We transform the system (1.11) and (1.12) in the form

$$\varphi(\xi,\eta)=y^{-1/2}\varphi_2^*(\xi,\eta),\quad \psi(\xi,\eta)=y^{1/2}\psi_2^*(\xi,\eta), \tag{1.20}$$

where

$$\varphi_2^*(\xi,\eta)=\varphi_{10}(\xi,\eta)+\varphi_2(\xi,\eta),\ \ \psi_2^*(\xi,\eta)=\psi_{10}(\xi,\eta)+\psi_2(\xi,\eta) \tag{1.21}$$

where $\zeta=\xi+i\eta$. We obtain

$$\Delta\varphi_2^*(\xi,\eta)+\frac{1}{4}\left|\frac{f'(\zeta)}{y(\xi,\eta)}\right|^2\varphi_2^*(\xi,\eta)=0, \tag{1.22}$$

$$\Delta\psi_2^*(\xi,\eta)-\frac{3}{4}\left|\frac{f'(\zeta)}{y(\xi,\eta)}\right|^2\psi_2^*(\xi,\eta)=0. \tag{1.23}$$

Consider (1.5) and (1.6). Passing to new independent variables ξ,η, we put $\zeta=\xi+i\eta$, $z=x+iy=f(\zeta)$, where the function $f(\zeta)$ is analytic. Thus we obtain

$$\begin{aligned}\Delta\varphi_1(\xi,\eta)+\frac{1}{4}\left|\frac{f'(\zeta)}{y(\xi,\eta)}\right|^2\varphi_1(\xi,\eta)=0,\\ \Delta\psi_1(\xi,\eta)-\frac{3}{4}\left|\frac{f'(\zeta)}{y(\xi,\eta)}\right|^2\psi_1(\xi,\eta)=0.\end{aligned} \tag{1.23$_1$}$$

We can see from (1.23$_1$) that the systems (1.22), (1.23) and (1.23$_1$) coincide.

Recall that the Green's function for the upper half-plane has the form

$$G(\xi,\eta;x,y)=\frac{1}{2\pi}\ln\Big\{\big[(\xi-x)^2+(\eta+y)^2\big]\big[(\xi-x)^2+(\eta-y)^2\big]^{-1}\Big\}. \quad (1.24)$$

Here we describe in detail the method of obtaining equations (1.22) and (1.23). In the meridional plane we introduce the coordinates ξ,η putting $x=x(\xi,\eta)$, $y=y(\xi,\eta)$. If ξ and η are the orthogonal coordinates and the Laplace equation has a solution $\varphi(x,y)$, $\varphi(x,y)=y^{-1/2}\varphi_*$, then we can choose the coordinates ξ,η, such that the mapping of the (x,y) plane onto the (ξ,η) plane is conformal. In addition, we put $x+iy=f(\zeta)$, where $f(\zeta)$ is the analytic function, $\varphi(\xi,\eta)=y^{-1/2}\varphi_*(\xi,\eta)$. Our reasoning is analogous for equation (1.4) (or for (1.6)).

The Green's function belongs to a class the fundamental solutions of the Laplace equation. This function is defined as harmonic, symmetric with respect to P and Q, equal to zero on the boundary l, and analytic at all points P of the domain, with the exclusion of the points $P=Q$ at which it has logarithmic singularity, i.e., at the point P of the neighborhood Q the following relation is fulfilled ([16–20]):

$$G(P;Q)=\frac{-1}{2\pi}\ln r(P;Q)+g(P;Q),$$

$g(P;Q)$ is the function, analytic everywhere, $r=\sqrt{(x-\xi)^2+(y-\eta)^2}$ is the distance between the points P and Q, and also, being the function of P, it has everywhere inside of D_i, with the exception of the point Q, the continuous derivatives up to the second order and satisfies both the Laplace equation and the limiting condition on the boundary. Next, being the function of P, it has at the point Q singularity corresponding to the finite charge (or mass) concentrated at the point Q. The Green's function of the Laplace operator for the plane simply-connected domain under the limiting condition $u|_l=0$ is tightly connected with the function of conformal mapping of the above-mentioned domain onto the circle $|w|=1$.

$g(P;Q)$ is the harmonic in the domain D_i function both of the coordinates (x,y) and of the coordinates (ξ,η) ([1–21]).

If d is diameter of the domain D_i, then the inequality

$$0\le G(P;Q)\le \ln(d/r)$$

is valid.

The Green's function for the circle of radius $R=1$ has the form

$$G(P;Q)=\frac{1}{2\pi}\ln\Big(\frac{\rho r_1}{r}\Big),$$

where $\rho=\sqrt{\xi^2+\eta^2}$ is the distance of the point $Q(\xi,\eta)$ from center of the circle, and r_1 is defined as follows:

$$r_1=\sqrt{(x-\xi/\rho^2)^2+(y-\eta/\rho^2)^2}.$$

Consider the inhomogeneous equation

$$\Delta u(x,y) = -\varphi_0(x,y). \tag{1.24$_1$}$$

We seek for a solution of (1.24_1), continuous up to the contour of the domain and satisfying the limiting equation $u|_l = 0$. Such a solution may be only unique ([1–21]).

An unknown solution has the form

$$u(x,y) = \iint\limits_{D_i} G(x,y;\xi,\eta)\,\varphi_0(\xi,\eta)\,d\xi\,d\eta, \tag{1.25}$$

or

$$u(x,y) = \frac{1}{2\pi}\iint\limits_{D_i} \varphi_0(x,y)\ln\frac{1}{r}\,d\xi\,d\eta + \iint\limits_{D_i} g(x,y;\xi,\eta)\,\varphi_0(\xi,\eta)\,d\xi\,d\eta. \tag{1.26}$$

The first summand in the right-hand side of (1.26) has inside of D_i the continuous derivatives up to the second order, and its Laplace operator is equal to $[-\varphi_0(\xi,\eta)]$. It is proved that the second summand in (1.26) can be differentiated with respect to the coordinates (x,y) of the point $P(x,y)$ as much as desired under the integral sign. This implies that this summand is the function, harmonic in D_i, because $g(P;Q)$ is the harmonic function of the point $P(x,y)$. $g(P;Q)$ is the harmonic function of the point Q with limiting values $(-\frac{1}{2\pi}\ln r)$, where $r = \sqrt{(x-\xi)^2+(y-\eta)^2}$. It is assumed that $P(x,y)$ lies inside of D_i. Formula (1.25) provides us with a solution of equation (1.24_1) satisfying the condition $u|_l = 0$.

Recall that there exists a generalized solution of the integral equation (1.25) ([1–20]).

The linear-fractional conformal mapping of the half-plane $\mathrm{Im}(\zeta) > 0$ onto the circle $|w_0| < 1$ has the form

$$w_0 = \frac{1+i\zeta}{i+\zeta}\,, \quad \zeta = \xi + i\eta, \quad w_0 = u + iv, \tag{1.27}$$

It follows from (1.27) that

$$u = \frac{2\xi}{\xi^2+(1+\eta)^2}\,, \quad v = \frac{\xi^2+\eta^2-1}{\xi^2+(1+\eta)^2}\,. \tag{1.28}$$

On the other hand, from (1.27) follows

$$\zeta = \frac{iw_0}{1+iw_0}\,, \quad \xi = \frac{2u}{u^2+(1-v)^2}\,, \quad \eta = \frac{1-v^2-u^2}{u^2+(1-v)^2}\,. \tag{1.29}$$

2. One Essential Remark Regarding the Solutions of Fredholm Equations of Second Kind

Consider the simplest Fredholm integral equation of second kind ([15–20]),

$$u(x) - \lambda \int\limits_a^b K(x,t)\, u(t)dt = f(x), \tag{2.1}$$

where the unknown function $u(x)$ depends on the real variable x which varies in the same interval $[a,b]$ as the integration variable t.

As is known, this requirement is applied, without exception, to all classes of integral equations we consider in the present work. The interval $[a,b]$ may be finite or infinite. The functions $K(x,t)$ and $f(x)$ are assumed to be given and defined almost everywhere respectively in the square $a \le x \le b$, $a \le t \le b$ and in the interval $a \le x \le b$. The function $K(x,t)$ is called the kernel of the integral equation. It is assumed that the kernel $K(x,t)$ of the Fredholm equation satisfies the inequality

$$\int\limits_a^b \int\limits_a^b |K(x,t)|^2 dx\, dt < \infty, \tag{2.2}$$

and the free term of the Fredholm equation satisfies the inequality

$$\int\limits_a^b |f(x)|^2 dx < \infty. \tag{2.3}$$

We have to consider the Fredholm equation of a more general type.

Let Ω be a measurable set in a space of an arbitrary number of variables, x and t be the points of the set, and μ be nonnegative measure defined on Ω. The equation

$$u(x) - \lambda \int\limits_\Omega K(x,t)\, u(t) d\mu(x) = f(x), \tag{2.4}$$

is likewise called the Fredholm equation whose kernel $K(x,t)$ and free term $f(x)$ satisfy, respectively, the inequalities

$$\int\limits_\Omega \int\limits_\Omega |K(x,t)|^2 d\mu(x)\, d\mu(t) < 0, \quad \int\limits_\Omega |f(x)|^2 d\mu(x) < 0. \tag{2.5}$$

The kernel $K(x,t)$ satisfying (2.5) is called the Fredholm kernel. The unknown function $u(x)$ is quadratically summable in (a,b), and hence it belomgs to the Fredholm inequality $L_2(a,b)$. A solution of equation (2.4) belongs to the space $L_2(\mu,\Omega)$ of functions which are summable quadratically in Ω in measure μ. Inequalities (2.3) and (2.5) imply that the free term of the equation belongs to the same space. The parameter λ may take real and complex values.

We denote the volume element by dx and the integral (2.5) by B_k^2 ([18]):

$$\int\limits_\Omega \int\limits_\Omega |K(x,t)|^2 dx\, dt = B_k^2. \tag{2.6}$$

As is known, the Fredholm equation has either a finite, or a countable set of characteristic numbers; if those numbers is a countable set, then they tend to infinity. But there exist kernels having no at all characteristic numbers, for example, Volterra's numbers. A complete characteristic of such kernels is given in the following Lalesco's theorem. Let $K(x,t)$ be the Fredholm kernel and $K_n(x,t)$ be its iterated kernel. For the kernel $K(x,t)$ to have no characteristic nymbers, it is necessary and sufficient that

$$A_n = \int\limits_\Omega k_n(x,x)dx = 0, \quad n = 3, 4, 5, \dots\ . \tag{2.7}$$

Note that A_n are called the traces of the kernel $K(x,t)$. Lalesco proved his theorem for the case of bounded kernels, and a general proof of the theorem has been given by S. N. Krachkovscii ([18]).

The determinant and the Fredholm minors are represented as a quotient of arbitrary two integral functions of λ. In addition, the resolvent poles, the characteristic numbers of the kernel $K(x,t)$, do not depend on x and t. Thus the resolvent has the form ([18])

$$\Gamma(x,t;\lambda) = D(x,t;\lambda)/D(\lambda), \tag{2.8}$$

where $D(x,t;\lambda)$ and $D(\lambda)$ are the integral functions of λ. If we manage in constructing these functions, then the resolvent becomes known, and a solution of the integral equation can be constructed by the well-known formula.

For the numerator and denominator of the fraction (2.8) we give representations in the form of the following series which are called the Fredholm series ([1–21]):

$$D(x,t) = \sum_{n=1}^\infty \frac{(-1)^n}{n!}\, B_n(x,t)\lambda^n, \quad D(\lambda) = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\, C_n\lambda^n, \tag{2.9}$$

where the formulas

$$\begin{gathered} C_0 = 1, \quad B_0(x,t) = K(x,t), \quad C_n = \int\limits_\Omega B_{n-1}(x,x)dx, \quad n > 0, \\ B_n(x,t) = C_n K(x,t) - n\int\limits_\Omega K(x,\tau)\, B_{n-1}(\tau,t)d\tau, \end{gathered} \tag{2.10}$$

allow one to calculate recursively the coefficients $B_n(x,t)$ and C_n.

Below, we will need the well-known ([16–18]) formula

$$D'(\lambda)/D(\lambda) = -\sum_{n=1}^{\infty} A_n \lambda^{n-1}, \tag{2.11}$$

where

$$A_n = \int_{\Omega} K_n(x,x)dx, \quad n = 1,2,3,\dots, \tag{2.12}$$

are the above-mentioned traces of the kernel $K(x,t)$.

If the kernel is non-continuous and has the second kind discontinuities, then the integrals (2.12) defining the coefficients $C_1, C_2, C_3, \dots$, from formula (2.10) make no sense. When the kernel $K(x,t)$ contains as a multiplier the Green's function $G(P,Q)$ which vanishes on the boundary, then this function has at the point $P = Q$ a logarithmic singularity. Consequently, the kernel $K(x,t)$ has logarithmic singularity, and the integral

$$\int_{\Omega} K(x,x)dx,$$

defining the coefficient C_1 is meaningless. This difficulty can be handled by putting, for example, $C_1 = 0$ ([18]).

The iterated kernel $K_2(s,t)$ has the form

$$K_2(s,t) = \int_a^b K(s,t_1)\, K(t_1,t)dt. \tag{2.13}$$

The function $K_2(s,t)$ makes sense for any positions of s and t in $[a,b]$, since in the most unfavorable case, where s and t coincide, we will have ([16–21]) the following estimate for the integrand function:

$$\big|K(s,t_1)\, K(t_1,s)\big| \le \frac{M_1}{|s-t_1|^{\varepsilon_1}}\,, \ \ \varepsilon_1 > 0. \tag{2.14}$$

It is proved that $K_2(s,t)$ is the continuous function in the square $a \le x \le b$, $a \le t_1 \le b$. Analogously, we estimate the functions

$$K_n(s,t) = \int_a^b K(s,t_1)\, K_{n-1}(t_1,t)\, dt_1, \quad n = 1,2,3,\dots, \tag{2.15}$$

$$\big|K(s,t_1)\, K_{n-1}(t_1,t)\big| < \frac{M_{n-1}}{|s-t_1|^{\varepsilon_{n-1}}}\,, \ \ \varepsilon_{n-1} > 0. \tag{2.16}$$

the integral $K_n(s,t)$, $n = 1,2,3,\dots$, makes sense for any positions of s and t in $[a,b]$, and the estimates of the integrand function have the form (2.16).

Consequently, we put ([16–21])

$$A_n = \int_a^b K_n(s,s)ds = 0, \quad K_n(s,s] = 0, \quad n = 1,2,3,\dots \,. \tag{2.17}$$

Then $C_n = 0$, $n = 1,2,3,\dots$. Taking into account (2.17), it follows from (2.11) that

$$D'(\lambda) = 0, \tag{2.18}$$

and from (2.18) follows

$$D(\lambda) = 1. \tag{2.19}$$

3. Solution of the System (1.22) and (1.23)

Recall that the system (1.22) and (1.23) is incompatible. Taking into account (1.21), we rewrite the system (1.22) and (1,23) as follows:

$$\varphi_2(\xi,\eta) + \frac{1}{4} \iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2 \varphi_2(x_1,y_1)dx_1\,dy_1 = \\ = \varphi_{10}^*(\xi,\eta), \tag{3.1}$$

$$\psi_2(\xi,\eta) - \frac{3}{4} \iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2 \psi_2(x_1,y_1)dx_1\,dy_1 = \\ = \psi_{10}^*(\xi,\eta), \tag{3.2}$$

where

$$\varphi_{10}^*(\xi,\eta) = -\Big\{\varphi_{10}(\xi,\eta) + \\ + \frac{1}{4} \iint\limits_{\mathrm{Im}(\xi)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2 \varphi_{10}(x_1,y_1)dx_1\,dy_1\Big\}, \tag{3.3}$$

$$\psi_{10}^*(\xi,\eta) = -\Big\{\psi_{10}(\xi,\eta) - \\ - \frac{3}{4} \iint\limits_{\mathrm{Im}(\xi)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2 \psi_{10}(x_1,y_1)dx_1\,dy_1\Big\}. \tag{3.4}$$

It will be assumed that $\varphi_{10}(\xi,\eta)$ and $\psi_{10}(\xi,\eta)$ are the known, self-conjugate harmonic functions satisfying all the boundary conditions. As is seen from (3.1) and (3.2), we have obtained the second kind Fredholm equation. In Section 2 we have proved that the integral equations (3.1) and (3.2) have no characteristic numbers. Consequently, using the method of successive approximations, we can construct a solution of (3.1) and (3.2). But we have to remember that the system (3.1) and (3.2) is incompatible.

The hydrodynamical problem is assumed to be solved, if either of the two functions $\varphi(x,y)$ or $\psi(x,y)$ is known. The functions $\varphi(x,y)$ and $\psi(x,y)$ must satisfy the system (1.1) and (1.2) or the system (1.7 and (1.8). So far, we wrote out these systems due to the fact that these equations are equivalent. This means that having defined $\varphi(x,y)$, we can, by virtue of $\varphi(x,y)$, find ψ, or vice versa, if we are able to define $\psi(x,y)$, then it will not be difficult to find $\varphi(x,y)$. Thus we have first to define $\varphi(x,y)$ and then $\psi(x,y)$.

We seek for solutions of the integral equations (3.1) and (3.2) by using the method of successive approximations in the form of the following series:

$$\varphi_2(\xi,\eta)=\sum_{n=0}^{\infty}\lambda^n\varphi_{2(n)}(\xi,\eta), \tag{3.5}$$

$$\psi_2(\xi,\eta)=\sum_{n=0}^{\infty}\mu^n\psi_{2(n)}(\xi,\eta). \tag{3.6}$$

Substituting the series (3.5) and (3.6), respectively, into the integral equations and then equating the coefficients with equal degrees of parameters λ and μ, we obtain

$$\varphi_{2(0)}(\xi,\eta)=\varphi^*_{10}(\xi,\eta), \tag{3.7}$$

. . . .

$$\varphi_{2(n)}(\xi,\eta)=$$
$$=\iint\limits_{\mathrm{Im}(\zeta)>0}G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2\varphi_{2(n-1)}(x_1,y_1)dx_1\,dy_1, \tag{3.8}$$

. . . .

$$\psi_{20}(\xi,\eta)=\psi^*_{10}(\xi,\eta), \tag{3.9}$$

. . . .

$$\psi_{2(n)}(\xi,\eta)=$$
$$=\iint\limits_{\mathrm{Im}(\zeta)\geq 0}G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2\psi_{2(n-1)}(x_1,y_1)dx_1\,dy_1, \tag{3.10}$$

. . . .

$n=1,2,3,\dots$

Since the integral second kind Fredholm equations have no characteristic numbers, the series (3.5) and (3.6) converge. Moreover, parameters of the integral equations (3.1) and (3.2) are, respectively, equal to $\lambda=1/4$, $\mu=-(3/4)$, hence the series (3.5) and (3.6) are rapidly convergent.

On the basis of (3.7)–(3.10), we can construct general formulas allowing one to express every approximations through the free terms.

Recall that the boundary conditions along the symmetry axis x, when their parts coincide with the boundary $l(x)$, have in terms of the coordinates (x, y) the form

$$\begin{aligned} &f \to 0, \quad \alpha = y^2, \quad \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial y} = \frac{\partial\varphi}{\partial\alpha}\cdot 2y \to 0; \\ &\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\alpha}\frac{\partial\alpha}{\partial y} = \frac{\partial\psi}{\partial\alpha}\cdot 2y \to 0. \end{aligned} \tag{3.11}$$

In terms of the coordinates (ξ, η), the boundary conditions along the axis x have the form

$$\begin{aligned} &y(\xi,\eta) = 0, \quad \frac{\partial y}{\partial\xi} = 0, \quad \frac{\partial y}{\partial\eta} = 0; \\ &\Big(\frac{\partial\varphi}{\partial\xi}\Big)_{y\to 0} = \Big[\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial\xi}\Big]_{y\to 0} \to 0; \quad \Big(\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\xi}\Big)_{y\to 0}; \end{aligned} \tag{3.12}$$

$$\Big[\frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial\xi}\Big]_{y\to 0} \to 0, \quad \Big(\frac{\partial\varphi}{\partial y}\Big)_{y\to 0} = \Big(\frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial y}\Big)_{y\to 0} \to 0, \tag{3.13}$$

$$\Big(\frac{\partial\varphi}{\partial\eta}\Big)_{y\to 0} = \Big(\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial\eta}\Big]_{y\to 0} = \Big(\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\eta}\Big)_{y\to 0}, \quad \frac{\partial y}{\partial\eta} \to 0. \tag{3.14}$$

Suppose that the symmetry axis x or its parts do not coincide with the boundary of the domain $s(z)$

$$\begin{aligned} &y = \text{const} \neq 0, \quad \alpha = y^2, \\ &\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial\alpha}\cdot 2y = \frac{\partial\varphi}{\partial\alpha}\cdot 2\cdot\text{const}; \quad y = \text{const}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\Big(\frac{\partial\psi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial y}\Big)_{\rho\to\text{const}}\cdot\Big(\frac{\partial\varphi}{\partial\alpha}\Big)\cdot 2\,y = \text{const}, \\ &(\psi)_{\rho=\text{const}} = \text{const}, \quad \Big(\frac{\partial\psi}{\partial\xi}\Big)_{\rho\to\text{const}} \to \Big(\frac{\partial\psi}{\partial\eta}\Big)_{y\to 0} \to 0, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \Big(\frac{\partial\varphi}{\partial\eta}\Big)_{y=\text{const}} &= \Big(\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\varphi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial\eta}\Big)_{y\to\text{const}} = \\ &= \Big(\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\eta}\Big)_{\rho=\text{const}}, \quad \frac{\partial\varphi}{\partial\eta} \to 0. \end{aligned} \tag{3.17}$$

Let

$$w(z) = f(z) = u(x,y) + iv(x,y), \qquad z = x + iy \tag{3.18}$$

be a function of a complex variable $z = x + it$, and u and v be real functions of the variables x and y. The necessary and sufficient conditions for the functions to be analytic are the so-called Cauchy-Riemann conditions $u'_x = v'_y$ and $u'_y = -v'_x$ which imply that $\Delta u = 0$ and $\Delta v = 0$, where Δ is the Laplace operator.

Consider the transformations

$$x = x(u,v), \quad y = y(u,v), \tag{3.19}$$

$$u = u(x,y), \quad v = v(x,y), \tag{3.20}$$

where $u(x,y)$ and $v(x,y)$ are the conjugate harmonic functions. Thus the transformations (3.19) and (3.20) are equivalent to the transformation (3.18). By the Cauchy-Riemann conditions, for the functions u and v, the following relations ([1-21)

$$u_x^2 + u_y^2 = u_x^2 + v_x^2 = v_y^2 + v_x^2 = |f'(z)|^2, \;\; u_x v_x + u_y v_y = 0, \tag{3.21}$$

are fulfilled.

Let us find out how the Laplace operator varies for the above transformation. We obtain

$$U = U\big[x(u,v), y(u,v)\big] = \widetilde{U}(u,v), \tag{3.22}$$

$$u''_{xx} + u''_{yy} = \big(\widetilde{U}_{uu} + \widetilde{U}_{vv}\big)\,\big|f'(z)\big|^2. \tag{3.23}$$

As a result of the transformation $w = f(z) = u + iv$, the harmonic in the domain G function $u(x,y)$ transforms into the function $\widetilde{u} = u(x,v)$, harmonic in the domain G', if and only if $|f'(z)|^2 \neq 0$ [6].

On the basis of the formula

$$\Delta U = f(x,y) \tag{3.24}$$

we can conclude that the function $u(x,y)$ is defined by the formula

$$u(x) = -\frac{1}{2\pi} \int\limits_D G(x,y;\xi,\eta)\, f(\xi,\eta)\, d\xi\, d\eta, \tag{3.25}$$

where $G(x,y;\xi,\eta)$ is the Green's function of the Dirichlet problem for harmonic functions in the domain D, and the function $f(x,y)$ is bounded, having continuous first derivatives. This function in the domain D is a regular solution of the Poisson equation ([1–21])

$$\Delta U = f(x,y), \quad (x,y) \in D. \tag{3.26}$$

It is proved that the function $u(x,y)$ satisfies the boundary condition

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = 0, \quad (x,y) \in D, \quad (x_0,y_0) \in S.$$

Taking into account (1.21) and using (1.22) and (1.23), we obtain

$$\begin{aligned}\varphi_{10}(\xi,\eta) + \varphi_2(\xi,\eta) =& \frac{1}{4} \iint\limits_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2 \times \\ &\times \big[\varphi_{10}(x_1,y_1) + \varphi_2(x_1,y_1)\big]\, dx_1\, dy_1, \end{aligned} \tag{3.27}$$

$$\psi_{10}(\xi,\eta)+\psi_2(\xi,\eta) = -\frac{3}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2\times$$
$$\times\left[\psi_{10}(x_1,y_1)+\psi_2(x_1,y_1)\right]dx_1\,dy_1. \tag{3.28}$$

Equalities (3.27) and (3.28) can be rewritten in the form

$$\varphi_2(\xi,\eta) - \frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2\varphi_2(x_1,y_1)dx_1\,dy_1 =$$
$$=\varphi_{10}^*(\xi,\eta), \tag{3.29}$$

$$\psi_2(\xi,\eta) + \frac{3}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2\psi_2(x_1,y_1)dx_1\,dy_1 =$$
$$=\varphi_{10}^*(\xi,\eta), \tag{3.30}$$

where

$$\varphi_{10}^*(\xi,\eta) = -\Bigg[\varphi_{10}(\xi,\eta)-$$
$$-\frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2\varphi_{10}(x_1,y_1)dx_1\,dy_1, \tag{3.31}$$

$$\psi_{10}^*(\xi,\eta) = -\Bigg[\varphi_{10}(\xi,\eta)+$$
$$+\frac{3}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2\psi_{10}(x_1,y_1)dx_1\,dy_1, \tag{3.32}$$

$$\left|\frac{f'(\xi+i\eta)}{y(\xi,\eta)}\right|^2 = \left(\frac{1}{y}\frac{\partial y}{\partial\xi}\right)^2+\left(\frac{1}{y}\frac{\partial y}{\partial\eta}\right)^2 \tag{3.33}$$

Suppose that we have solved the plane problem, i.e., constructed analytic functions allowing us to map conformally the half-plane $\mathrm{Im}(\zeta)>0$ of the plane $\zeta=\xi+i\eta$ onto the circular polygon. For the general discussion, we assume that there is a circular polygon with m vertices. To find such an analytic function, we have to solve the differrential Schwarz equation of third order whose solution reduces to that of the differential Fuchs class equation of second order. The Schwarz equation and, respectively, the corresponding Fuchs class equation contains $2(m-3)$ real unknown parameters. After integration of Schwarz equation, there appears six parameters of integration. We compose a system of higher $2(m-3)$ trascendent equations and a system of six equations for finding parameters of integration. The boundary conditions for the problem of filtration involve additionally unknown parameters.

Further, using the solutions of plane problems, we construct $\varphi(\xi,\eta)$, $\psi(\xi,\eta)$ for the systems (1.7) and (1.8) of differential equations of spatial axisymmetric problems allowing us to construct the functions which map quasi-conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ onto the domain of a complex potential and onto the domains of complex velocity, i.e., onto $S((x)_0)$, $S(\omega_0'(\zeta)/\sigma'(\zeta))$.

For three analytic functions we introduce the notation $z(\zeta) = x(\xi,\eta) + iy(\xi,\eta)$, $\omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta)$, $w(\zeta) = \omega_0(\zeta)/z'(\zeta)$, $\Delta x(\xi,\eta) = 0$, $\Delta y((\xi,\eta) = 0$, $\Delta\varphi_0((\xi,\eta) = 0$, which map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ onto the domain $S(z)$; the liquis motion onto the domains of a complex potential and onto the domains of complex velocity $S(\omega_0)$. The method of finding the function $S(\omega'(\zeta)/\sigma'(\zeta)$ is descrived in [1-21].

4. Flow Round the Barriers by the Bounded Liquid Flows

Example 4.1. Fig. 1 of [1, §48] shows one half of the meridional plane $x \circ y$ for the problem of flow round the circular cone in a tube ([1]).

Example 4.2. In Fig. 2 of [1, §27] we can see a scheme of the problem on flow round a circular cone by liquid of finite depth ([1]).

Since the stream function is defined up to the constant summand, we can put $\psi = 0$ on the symmetry axis x, on the cone and on the free surface. But the difference ψ on the stream surface is equal to the liquid discharge between these surfaces, divided by 2π, hence on the tube walls $\psi = \pi v_\infty h^2/(2\pi)$, where h is radius of the tube, and v_∞ is velocity of accumulating flow from the left at infinity ([1, §48]).

A form of free surfaces is unknown beforehand, but on these surfaces we have the supplementary condition for the constancy of the velocity modulus v which is equivalent to the constancy of pressure. This condition can be written as follows ([1-21]):

$$\frac{1}{y^2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] = \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial x}\right)^2 = v_0^2,$$

where v_0 is equal to v on the free surface.

In Fig. 2 ([1, §27]) we see symmetrical jet flow round a circular cone by the liquid of finite depth, of angle opening $2\pi\mu$. The liquid, flowing out of a cone is bounded by the cone surface.

Here the use will be made of the solution given in [1, §27[. It will be assumed that the above-mentioned solution of the plane problem satisfies all the boundary conditions and this solution is a zero solution of the integral Fredholm equation of second kind. Successive approximation will be obtained by formulas (3.7)–(3.10).

The plane problem under consideration is solved by the mapping onto the unit semi-circle of the plane of parametric variable t (Fig. 3) [1, §6, Figs. 2.1, 2.2] of the domains of variation $dw/(v_0 dz)$ and w for the lower half of the flow shown in Fig. 3 of [1, §36, Figs. 2.1, 2.2].

Here we write the following formulas:

$$\frac{dw}{v_0 dz} = t^{\mu}, \tag{4.1}$$

$$w = \frac{q}{\pi}\ln(t-h) + \frac{q}{\pi}\ln\Big(\frac{1}{h}-t\Big) - \frac{q}{\pi}\ln(t-e^{i\beta}) - \frac{q}{\pi}\ln(t-e^{-i\beta}), \tag{4.2}$$

where $2q$ is the liquid discharge between the walls HA_1 and HA.

In formula (4.1), v_0 denotes velocity on the free surface, h and β are the parameters (Fig. 2).

From (4.1) and (4.2) we find that

$$z(t) = \frac{q}{\pi v_0}\int\limits_0^t \Big(\frac{1}{t-h} + \frac{1}{t-h^{-1}} - \frac{1}{t-e^{i\beta}} - \frac{1}{t-e^{-i\beta}}\Big)\frac{dt}{t^{\mu}}\,. \tag{4.3}$$

If we denote the distance between the walls (diameter of the tube) by $2L$ and the velocity at infinity between the walls v_H by v_∞, then

$$\frac{v_H}{v_0} = \Big(\frac{dw}{v_0 dz}\Big)_H = h^{\mu} \tag{4.4}$$

and

$$L = \frac{q}{v_0 h^{\mu}} \tag{4.5}$$

can be written as

$$2L = \frac{2q}{v_0 h^{\mu}} \quad \text{and} \quad \frac{v_\infty}{v_0} = h^{\mu}. \tag{4.6}$$

The ratios l/L (or a/L) and b/L (Fig. 2 [1, §27]), as well as the angle with size $2\pi\mu$ of the wedge are the geometrical elements defining completely a family of geometrically similar flows. Unknown parameters h and β can be found from the system of equations

$$\frac{b}{L} = \frac{h^{\mu}}{\pi}\int\limits_0^1 \Big[\frac{1}{\xi+h} + \frac{1}{\xi+h^{-1}} - \frac{2(\xi+\cos\beta)}{\xi^2+2\xi+1}\Big]\frac{d\xi}{\xi^{\mu}}\,, \tag{4.7}$$

$$\frac{b}{L} = \frac{l}{L}\cos(\pi\mu) - \frac{h^{\mu}}{\pi}\int\limits_0^1 \frac{dt}{t^{\mu}}\Big[\frac{1}{t-h} + \frac{1}{t-h^{-1}} - \frac{2(t-\cos\beta)}{t^2-2t\cos\beta+1}\Big], \tag{4.8}$$

which define dimensions of the system “Channel-Wedge”; (instead of (4.7) we can use the equality

$$\frac{a}{L} = 1 - \frac{l}{L}\sin(\pi\mu). \tag{4.9}$$

Note that a number of cavitations $Q_1 = v_0^2/v_\infty^2 - 1$ cannot be arbitrary, because according to the second equality (3.5),

$$Q_1 = h^{-2H} - 1. \tag{4.10}$$

This implies that in a general case, for $0 < h < 1$, a number of cavitations is $Q_1 > 0$. The case $Q = 0$ responds partially to the symmetric flow of boundless liquid round the wedge. We will mainly be interested in the wedge resistance. To calculate the wedge resistance, we obtain a formula by using the theorem on the variation of momentum projected on the x-axis to the liquid mass H which is initially bounded by parallel walls, by wedge, free surfaces and by plane sections drawn at infinity from the left and from the right, perpendicular to the flow lines.

If we denote pressure and velocity at infinity from the left (the point H) by p_∞ and v_∞, respectively, pressure and velocity on the free surface by p_0 and v_0, distribution of liquid pressure along the wedge sides by $p(y)$, then the resultant of forces acting on the liquid mass M will be equal to $(p_\infty - p_0)2L - X$, where

$$X = \int\limits_{y(B)}^{y(B_1)} (p - p_0)dy \tag{4.11}$$

is the wedge resistance. Denoting by Θ_0 the angle which forms with the axis x the flow velocity of jets $A_1E_1B_1$ (Fig. 2, [1, §27, 6.1]) at infinity, then the increment of momentum of the liquid mass M per unit of time will be equal to

$$2q\rho_0 \cos\Theta_0 - 2q\rho v_\infty. \tag{4.12}$$

Thus the theorem on the variation of momentum for M provides us with

$$(p_\infty - p_0)2L - X = 2q\rho_0 \cos\Theta_0 - 2q\rho v_\infty. \tag{4.13}$$

whence using Bernoulli's integral and replacing $2q$ by $2Lv_\infty$, we obtain

$$X = \frac{\rho}{2}(v_0^2 - v_\infty^2)2L + \rho 2L(v_\infty^2 - v_\infty v_0 \cos\Theta_0), \tag{4.14}$$

or

$$X = 2Lv_\infty^2\Big(\frac{v_0^2}{v_\infty^2} - 2\frac{v_0}{v_\infty}\cos\Theta_0 + 1\Big). \tag{4.15}$$

Thus we find the coefficient of the wedge resistance:

$$C_x = \frac{X}{\rho v_\infty^2 l\sin(\pi\mu)} = \frac{L}{l\sin(\pi\mu)}\Big(\frac{v_0^2}{v_\infty^2} - 2\frac{v_0}{v_\infty}\cos\Theta_0 + 1\Big). \tag{4.16}$$

To find the angle Θ_0, it suffices to put in (3.1) $t = e^{i\beta}$ (the point E), and since at the point E we have $dw/(v_0dz) = e^{i\Theta_0}$, therefore

$$\Theta_0 = \beta\,\mu. \tag{4.17}$$

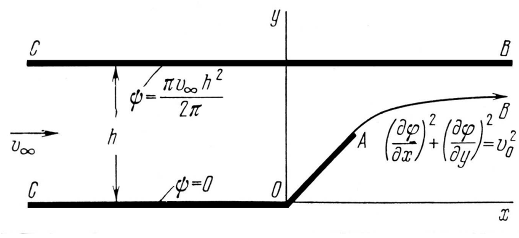

Fig. 1

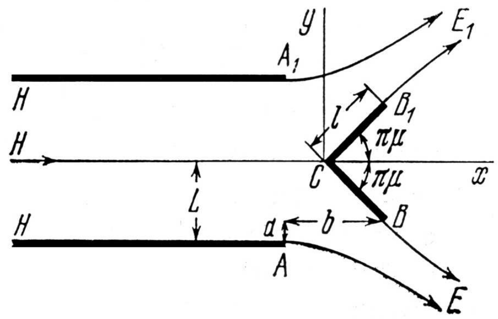

Fig. 2

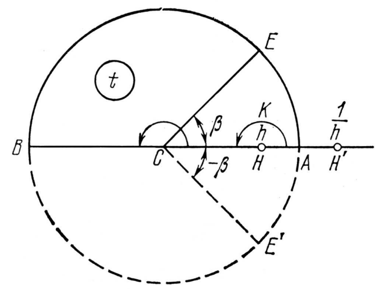

Fig. 3

References

1. M. I. Gurevich, The theory of jets of perfect liquid. (Russian) *Nauka, Glav. Redak. Fiz. Mat. Lit. Moscow*, 1979.
2. M. A. Lavrent'ev and B. V. Shabat, Problems of hydrodynamics and their mathematical models. (Russian) *Izdat. Nauka, Moscow*, 1973.
3. G. Birkhoff, E. Sarantonello, Jets, traces and cavities. Engl. transl. *Izd. Mir, Moscow*, 1964.
4. N. E. Kochin, I. A. Kibel, and N. V. Rose, Theoretical hydromechanics. Part I. (Russian) *Gos. izd. Tekhniko-teoretich. Lit. Moscow*, 1955.

5. P. Ja. Polubarinova-Kochina, Theory of motion of ground water. (Russian) Second edition, revised and enlarged. *Izdat. Nauka, Moscow*, 1977.
6. A. N. Tikhonov and Samarskii, Equations of mathematical physiks. (Russian) *Glavizdat. Gos. Izdat. Tekhn. teoret. Lit. Moscow*, 1953.
7. G. Bateman and A. Erdelyi, Higher transcendent functions, Elliptic and modular functions, Lame and Mathieu functions. (Russian) *Izdat. Nauka, Moscow*, 1967.
8. J. Happel, G. Brenner, Hydrodynamics for low Reynolds numbers. *Mir, Moscow*, 1976.
9. M. A. Lavrent'ev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Gosudarstv. Izdat. Fiz. Mat. Lit. Moscow*, 1958.
10. A. R. Tsitskishvili, Solution of the Schwarz differential equations. *Mem. Differential Equations Math. Phys.* **11**(1997), 129–156.
11. A. R. Tsitskishvili, Construction of analytic functions that conformally map a half plane onto circular polygons. (Russian) *Differentsialnye Uravneniya* **21**(1985), No. 4, 646–656, 734.
12. A. Tsitskishvili, Solution of spatial axially symmetric problems of the theory of filtration with partially unknown boundaries. *Mem. Differential Equations Math. Phys.* **39**(2006), 105–140.
13. A. Tsitskishvili, Exact solution of spatial axisymmetric problems of the filtration theory with partially unknown boundaries. *Mem. Differential Equations Math. Phys.* **42**(2007), 93–125.
14. A. Tsitskishvili, A general method of constructing the solutions of spatial axisymmetric stationary problems of the jet and filtration theories with partially unknown boundaries. *Mem. Differential Equations Math. Phys.* **43**(2008), 119–142.
15. A. V. Pogorelov, Differential geometry. (Russian) *Nauka. Glav. Redakcia Fiz. mat. Lit. Moscow*, 1974.
16. V. I. Smirnov, A course of higher mathematics. (Russian) *Izdat. Tekh. Theor. Lit., Moscow, Leningrad*, 1952.
17. V. I. Smirnov, A course of higher mathematics. (Russian) *Izdat. Tehn. Teor. Lit., Moscow-Leningrad*, 1951.
18. P. P. Zabreiko, A. I. Koshelev, M. A. Krasnoselskii, S. G. Mikhlin, A. S. Rakovshik, and V. Ja. Stetsenko, Integral equations. (Russian) *Nauka. Glav. Redakcia Fiz. Mat. Lit. Moscow*, 1968.
19. Ja. Ber, D. Zaslavskii, and S. Irmey, Physico-mathematical essentials of water filtration. (Russian) *Mir. Moscow*, 1971.
20. I. N. Vekua, Generalized analytic functions. (Russian) *Gos. Izdat. Fiz. Mat. Lit. Moscow*, 1959.
21. A. V. Bitsadze, Equations of mathematical physics. (Russian) *Nauka, Moscow*, 1982.

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