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AN APPROXIMATE SOLUTION OF EQUATION OF AXISYMMETRIC MAGNETOHYDRODYNAMIC LAMINAR BOUNDARY LAYER UNDER THE ACTION OF A STRONG MAGNETIC FIELD

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Abstract. We consider the motion in an axisymmetric magnetohydrodynamic boundary layer under the action of an exterior strong magnetic field. The problem is solved in an induction-free approximation. The first two approximations of the problem are found, and physical characteristics of the boundary layer are defined.

რეზიუმე. განხილულია ღერძსიმეტრიული მაგნიტოჰიდროდინამიკური ლამინარულ სასაზღვრო ფენაში ბლანტი არაკუმშვადი გამტარი სითხის მოძრაობა ძლიერი გარეგანი მაგნიტური ველის მოქმედებისას. ამოცანა ამოხსნილია არაინდუქციურ მიახლოებაში. ნაპოვნია ამოცანის ორი პირველი მიახლოება და განსაზღვრულია სასაზღვრო ფენის ფიზიკური მახასიათებლები.

We place the axes x and y in the plane of meridional channel cross-section along the generatrix and normal to it. We denote the corresponding components of the velocity vector on the boundary layer by u and v. Since in the case under consideration the velocity vector of the external flow has, besides the axial constituent U_∞, the circumferential constituent W_∞, the velocity in the boundary layer will likewise have a circumferential constituent which we denote by w.

In this case, the system of equations of the magnetohydrodynamic boundary layer in an incompressible isotropic conductive liquid takes the form (1, 2]):

$$
\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial P}{\partial x}-\frac{\mu^2\sigma H_y^2}{\rho}u+\nu\frac{\partial^2 u}{\partial y^2},\\
u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}&=-\frac{\mu\sigma E_x H_y}{\rho}-\frac{\mu^2\sigma K_y^2}{\rho}w+\nu\frac{\partial^2 w}{\partial y^2},
\end{aligned}
\tag{1}
$$

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$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

with the boundary conditions

$$\begin{gathered} u=v=w=0 \quad \text{for} \quad y=0, \\ u\to U_\infty(x), \quad w\to W_\infty(x) \quad \text{as} \quad y\to\infty. \end{gathered} \tag{2}$$

It can be shown that the components of stress of electric and magnetic fields are the functions of the longitudinal component, i.e.,

$$E_x=E_x(x), \quad H_y=H_y(x).$$

We omit the pressure gradient and the component of electric field strength by means of the Euler equations

$$\begin{gathered} U_\infty\frac{dU_\infty}{dx}+\frac{\mu^2\sigma H_y^2}{\rho}U_\infty=-\frac{1}{\rho}\frac{\partial P}{\partial x}, \\ U_\infty\frac{dW_\infty}{dx}+\frac{\mu\sigma}{\rho}E_xH_y+\frac{\mu^2\sigma H_y^2}{\rho}W_\infty=0 \end{gathered} \tag{3}$$

The latter equation of the system (3) shows that the electric and magnetic fields are connected with hydrodynamic characteristics of the external flow ([1, 2]).

Thus using the relation (3), the basic system of equations (1) can be rewritten as

$$\begin{gathered} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_\infty\frac{dU_\infty}{dx}+\frac{\mu^2\sigma H_y^2}{\rho}(U_\infty-u)+\nu\frac{\partial^2 u}{\partial y^2}, \\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=U_\infty\frac{dW_\infty}{dx}+\frac{\mu^2\sigma H_y^2}{\rho}(W_\infty-w)+\nu\frac{\partial^2 w}{\partial y^2}, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \end{gathered} \tag{4}$$

with the boundary conditions (2).

The last equation of the system (4) allows us to determine the velocity

$$v(x,y)=-\int_0^y\frac{\partial u}{\partial x}dy,$$

which we substitute into the first two equations (4). Thus we obtain

$$\begin{gathered} \nu\frac{\partial^2 u}{\partial y^2}+N(U_\infty-u)=u\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\int_0^y\frac{\partial u}{\partial x}dy-U_\infty\frac{dU_\infty}{dx}, \\ \nu\frac{\partial^2 w}{\partial y^2}+N(W_\infty-w)=u\frac{\partial w}{\partial x}-\frac{\partial w}{\partial y}\int_0^y\frac{\partial u}{\partial x}dy-U_\infty\frac{dW_\infty}{dx}, \end{gathered} \tag{5}$$

where we have introduced the notation

$$N = \frac{\sigma\mu^2 H_y^2}{\rho}. \tag{6}$$

We now pass to the following new variables:

$$\xi = x, \quad z = \sqrt{\frac{N}{\nu}}y, \quad u(x,y) = f(\xi,z), \quad w(x,y) = \varphi(\xi,z). \tag{7}$$

Then the system (5) takes the form

$$\begin{aligned}
&\frac{\partial^2 f}{\partial z^2} - f = \frac{1}{N}\left[f\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial z}\int\limits_0^z \frac{\partial f}{\partial \xi}dz - U_\infty\frac{dU_\infty}{d\xi}\right] + U_\infty,\\
&\frac{\partial^2 \varphi}{\partial z^2} - \varphi = \frac{1}{N}\left[f\frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial z}\int\limits_0^z \frac{\partial f}{\partial \xi}dz - U_\infty\frac{dW_\infty}{d\xi}\right] + W_\infty,
\end{aligned} \tag{8}$$

and the boundary conditions (2) take the form

$$\begin{gathered}
f(\xi,z) = 0, \quad \varphi(\xi,z) = 0, \quad \text{for} \quad z = 0,\\
f(\xi,z) = U_\infty(\xi), \quad \varphi(\xi,z) = W_\infty(\xi) \quad \text{as} \quad z \to \infty.
\end{gathered} \tag{9}$$

As is assumed, the applied magnetic field H_y is strong, the parameter N will be large, and the value $\frac{1}{N} = \varepsilon$ will be small. Therefore the solution (8) can be represented as follows:

$$\begin{aligned}
f(\xi,z) &= f_0(\xi,z) + \varepsilon\int\limits_0^\infty \left[f\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta}\int\limits_0^\eta \frac{\partial f}{\partial \xi}d\eta - U_\infty\frac{dU_\infty}{d\xi}\right]G(z,\eta)d\eta,\\
\varphi(\xi,z) &= \varphi_0(\xi,z) + \varepsilon\int\limits_0^\infty \left[f\frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \eta}\int\limits_0^\eta \frac{\partial f}{\partial \xi}d\eta - U_\infty\frac{dW_\infty}{d\xi}\right]G(z,\eta)d\eta,
\end{aligned} \tag{10}$$

where $f_0(\xi,z)$ and $\varphi_0(\xi,z)$ are the solutions of the problems

$$\begin{aligned}
&\frac{\partial^2 f_0}{\partial z^2} - f_0 = -U_\infty, \quad f_0(\xi,0) = 0, \quad f_0(\xi,\infty) = U_\infty,\\
&\frac{\partial^2 \varphi_0}{\partial z^2} - \varphi_0 = -W_\infty, \quad \varphi_0(\xi,0) = 0, \quad \varphi_0(\xi,\infty) = W_\infty,
\end{aligned} \tag{11}$$

and $G(z,\eta)$ is the Green's function of the problem

$$\frac{\partial^2 G}{\partial z^2} - G = 0, \quad G(0,\eta) = 0, \quad G(\infty,\eta) = 0.$$

It has the form

$$G(z,\eta) = \begin{aligned}
G_1(z,\eta) &= \frac{1}{2}\left[e^{-(\eta+z)} - e^{-(\eta-z)}\right] \quad 0 < z < \eta,\\
G_2(z,\eta) &= -\frac{1}{2}\left[e^{-(z-\eta)} - e^{-(z+\eta)}\right] \quad \eta < z < \infty,
\end{aligned} \tag{12}$$

A solution of the system (10) is sought in the form of a series in degrees ε ([3, 6]):

$$f(\xi,z)=\sum_{k=0}^{\infty}\varepsilon^k f_k(\xi,z),\quad \varphi(\xi,z)=\sum_{k=0}^{\infty}\varepsilon^k \varphi_k(\xi,z) \tag{13}$$

Substituting the series (13) into the system (10) and equating the coefficients with the same degrees ε, we obtain the following recurrence relations:

$$f_0(\xi,z)=U_0(1-e^{-z}), \tag{14}$$

$$f_1(\xi,z)=\int\limits_0^{\infty}\left[f_0\frac{\partial f_0}{\partial\xi}-\frac{\partial f_0}{\partial\eta}\int\limits_0^{\eta}\frac{\partial f_0}{\partial\xi}d\eta-U_\infty\frac{dU_\infty}{d\xi}\right]G(z,\eta)d\eta, \tag{15}$$

. . . .

$$f_{k+1}(\xi,z)=\int\limits_0^{\infty}\sum_{m=0}^{k}\left[f_m\frac{\partial f_{k-m}}{\partial\xi}-\frac{\partial f_m}{\partial\eta}\int\limits_0^{\eta}\frac{\partial f_{k-m}}{\partial\xi}d\eta\right]G(z,\eta)d\eta, \tag{16}$$

. . . .

$$\varphi_0(\xi,z)=W_\infty(1-e^{-z}), \tag{17}$$

$$\varphi_1(\xi,z)=\int\limits_0^{\infty}\left[f_0\frac{\partial\varphi_0}{\partial\xi}-\frac{\partial\varphi_0}{\partial\eta}\int\limits_0^{\eta}\frac{\partial f_0}{\partial\xi}d\eta-U_\infty\frac{dW_\infty}{d\xi}\right]Gd\eta, \tag{18}$$

. . . .

$$\varphi_{k+1}(\xi,z)=\int\limits_0^{\infty}\left[f_m\frac{\partial\varphi_{k-m}}{\partial\xi}-\frac{\partial\varphi_m}{\partial\eta}\int\limits_0^{\eta}\frac{\partial f_{k-m}}{\partial\xi}d\eta\right]Gd\eta. \tag{19}$$

The above relations in the first two approximations result in

$$f_0(\xi,z)=U_\infty(1-e^{-z}),\quad \varphi_0(\xi,z)=W_\infty(1-e^{-z}), \tag{20}$$

$$f_1(\xi,z)=-\frac{1}{2}U_\infty\frac{dU\infty}{d\xi}ze^{-z}, \tag{21}$$

$$\begin{aligned}\varphi_1(\xi,z)=&U_\infty\frac{dW_\infty}{d\xi}\Big(-\frac{1}{6}e^{-z}+\frac{1}{6}e^{-2z}\Big)+\\&+W_\infty\frac{dU_\infty}{d\xi}\Big(-\frac{1}{2}ze^{-z}+\frac{1}{6}e^{-2z}-\frac{1}{6}e^{-z}\Big).\end{aligned} \tag{22}$$

Finally, for the velocity components we have the following expressions:

$$\begin{aligned}U(x,y)\approx&f_0(\xi,z)+\frac{1}{N}f_1(\xi,z)=U_\infty(\xi)(1-e^{-z})-\\&-\frac{1}{2}\frac{1}{N}U_\infty\frac{dU_\infty}{d\xi}ze^{-z},\end{aligned} \tag{23}$$

$$
\begin{aligned}
W(x,y) \approx & \varphi_0(\xi,z) + \frac{1}{N}\varphi_1(\xi,z) = W_\infty(\xi)(1-e^{-z}) + \\
& + \frac{1}{6N}U_\infty \frac{dW_\infty}{d\xi}(-e^{-z}+e^{-2z}) + \\
& + \frac{1}{N}W_\infty \frac{dU_\infty}{d\xi}\Big(-\frac{1}{2}ze^{-z} + \frac{1}{6}e^{-2z} - \frac{1}{6}e^{-z}\Big).
\end{aligned} \tag{24}
$$

For the skin friction we have

$$
\begin{aligned}
\tau = & \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \sqrt{\frac{N}{\nu}}\mu \frac{\partial}{\partial z}\Big(f_0 + \frac{1}{N}f_1\Big) = \\
& = \sqrt{N}\nu\mu\Big[U_\infty - \frac{1}{2N}U_\infty \frac{dU_\infty}{d\xi}\Big],
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\tau = & \mu \frac{\partial W}{\partial y}\Big|_{y=0} = \sqrt{\frac{N}{\nu}}\mu \frac{\partial}{\partial z}\Big(\varphi_0 + \frac{1}{N}\varphi_1\Big) = \\
& = \sqrt{\frac{N}{\nu}}\Big[W_\infty - \frac{1}{6N}U_\infty \frac{dW_\infty}{d\xi} - \frac{2}{3}\frac{1}{N}W_\infty \frac{dU_\infty}{d\xi}\Big].
\end{aligned} \tag{26}
$$

If $U_\infty = \text{const}$ and $W_\infty = \text{const}$, then the above relations yield

$$
\tau\Big|_{U_\infty=\text{const}} = \sqrt{\frac{N}{\nu}}U_\infty, \quad \tau\Big|_{U_\infty=\text{const}, W_\infty=\text{const}} = \sqrt{\frac{N}{\nu}}W_\infty.
$$

We can see from (25) that the skin friction $\tau = \mu\frac{\partial u}{\partial y}|_{y=0}$ does not depend on the circumferential constituent W_∞, and (26) shows that the skin friction $\tau = \mu\frac{\partial w}{\partial y}|y=0$ depends not only on the axial constituent, but also on the circumferential constituent of the external velocity and their derivatives. If these values are constant, then the skin frictions depend on u_∞ and W_∞ only in the first degree and also on the parameter of the external magnetic field $N^{-1/2}$. For strong magnetic fields the skin frictions grow just as $N^{-1/2}$.

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