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ON THE NUMERICAL INVESTIGATION OF INSTABILITY AND TRANSITION IN FLOW BETWEEN TWO POROUS ROTATING CYLINDERS WITH A TRANSVERSE PRESSURE GRADIENT

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Abstract. The stability of a viscous flow in a space between two rotating porous cylinders in the presence of a constant transverse pressure gradient is studied. The methods of the theory of bifurcation together with the numerical calculations used in a wide class of problems possessing symmetry, allowed to reveal in the problem under consideration the transitions to rather complicated regimes of liquid motion.

რეზიუმე. შესწავლილია ორ მბრუნავ ფოროვან ცილინდრს შორის სითხის დინების მდგრადობის ამოცანა, როდესაც დინებაზე მოქმედებს მუდმივი ტრანსვერსალური წნევის გრადიენტი. ჰიდროდინამიკური დინებების ბიფურკაციის თეორია რიცხვით მეთოდებთან ერთად, რომელიც გამოიყენება სიმეტრიის მქონე დინებების ფართო კლასისათვის, იძლევა საშუალებას მოცემული ამოცანისათვის გამოკვლეულ იქნას გადასვლები სითხის დინების რთული რეჟიმებისაკენ.

The first theoretical results dealing with the investigations of stability of a liquid flow between two non-porous rotating cylinders under the action of a constant transverse pressure gradient were obtained as far back as in the 50th-60th ([1–4]). In these works, when studying the problem on the first loss of stability of the main flow, the investigations were based on the assumption that the first loss of stability takes place under axisymmetric disturbances in a form of vortices which was observed experimentally ([4]). In 1971, it was shown in [5] that the instability can be initiated by oscillatory modes, as well. In all the above-mentioned works, the investigations were carried out for a linear case and the results were obtained by using approximation for close cylinders and under rotation of only inner cylinder.

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In [6], the author considers the problem of stability of a liquid flow for an arbitrary gap between two non-porous cylinders with a constant transverse pressure gradient It was shown that when the main stationary flow losses its stability, there may arise, along with the axisymmetric motions, the secondary auto-oscillatory modes connected with azimuthal waves of different periods. In addition, neutral curves responding to those two types of stability loss may, unlike the Couette flow [7], intersect not only when the cylinders rotate in different directions, but when they rotate in one and the same direction, which leads near the point of intersection to more complicated regimes of liquid motion. For the investigation of the above-mentioned modes, the use was made of the theory of hydrodynamical flows bifurcation with cylindrical symmetry developed by V. Yudovich in Russia and by G. Iooss and P. Chossat in France.

In the present work we consider the problem of stability of liquid flow between rotating cylinders with a transverse pressure gradient when the cylinders are porous. The bifurcation sequences resulting in chaotic modes in a small neighborhood of a point of intersection of neutral curves are studied.

1. Let a viscous incompressible liquid fill up the space between two porous cylinders of radii R_1, R_2, rotating with angular velocities Ω_1, Ω_2, respectively. It is assumed that mass forces are absent, fluid inflow s through one cylinder is equal to the fluid outflow through the other and simultaneously constant pressure gradient acts on the flow in the azimuthal direction.

We use the Navier-Stokes and continuity equations in the cylindrical coordinates r, θ, z with the z-axis directed along the axis of cylinders:

$$\frac{d\vec{V}'}{dt} = -\frac{1}{\rho}' \nabla \Pi' - \nu \operatorname{rot} \operatorname{rot} \vec{V}', \quad \operatorname{div} \vec{V}' = 0, \tag{1.1}$$

where $\vec{V}'(t, v_r', v_\theta', v_z')$ is the velocity vector, Π' is pressure, ν is the kinematic coefficient of viscosity, ρ' is density, t is time,

$$\nabla = \Big\{ \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \Big\},$$

$$\frac{d\vec{V}'}{dt} = \frac{\partial \vec{V}'}{\partial t} + (\vec{V}', \nabla)\vec{V}' + \Big\{ -\frac{1}{r^2} v_\theta'^2, \frac{1}{r^2} v_r' v_\theta', 0 \Big\}.$$

Equations (1.1) are considered under the condition

$$\int_0^{2\pi} \int_{R_1}^{R_2} \rho' v_z' r dr d\theta = 0, \tag{1.2}$$

which ensures the absence of the flux of velocity through the cross-section of cylinders spaces. At the same time we require that the boundary conditions

$$\begin{gathered} R_1 v'_r|_{r=R_1} = R_2 v'_r|_{r=R_2} = s, \\ v'_\theta|_{r=R_i} = \Omega_i R_i, \quad v'_z|_{r=R_i} = 0 \quad (i = 1, 2). \end{gathered} \tag{1.3}$$

be fulfilled.

The problem (1.1)–(1.3) possesses a group of symmetry $\mathbf{G} = SO(2) * O(2)$ which is invariant with respect to the rotations L_θ^δ near the cylinder axis, to shifts L_z^h along that axis, inversion J, acting onto the velocity field by the rules

$$\begin{aligned} &(L_\theta^\delta v)(t, r, \theta, z) = v(t, r, \theta + \delta, z) \\ &(L_z^h v)(t, r, \theta, z) = v(t, r, \theta, z + h) \\ &(Jv)(t, r, \theta, z) = (v_r(t, r, \theta, -z), v_\theta(t, r, \theta, -z), -v_z(t, r, \theta, -z) \end{aligned} \tag{1.4}$$

for any real δ and h.

We choose the scales R_1,$\Omega_1 R_1$, $1/\Omega_1$, $\nu\rho'\Omega_1$ respectively for length, velocity, time and pressure.

The problem (1.1)–(1.3) admits an exact solution with the velocity vector $\vec{V}_0 = \{v_{0r}, v_{0\theta}, v_{0z}\}$ and pressure Π_0 ([8]):

$$\begin{gathered} v_{0r} = \frac{\varkappa_0}{r}, \\ v_{0\theta} = \begin{cases} \dfrac{1}{2\varkappa} P_\theta \Big(-r + A_1 r^{\varkappa+1} + \dfrac{B_1}{r} \Big) + A r^{\varkappa+1} + \dfrac{B}{r}, & \varkappa \neq -2 \\ \dfrac{1}{4} P_\theta \Big(r - \dfrac{A'_1 \ln r + 1}{r} \Big) + \dfrac{A' \ln r + 1}{r}, & \varkappa = -2, \end{cases} \\ \frac{1}{Re} \frac{\partial \Pi_0}{\partial r} = \frac{v_{0\theta}}{r} + \frac{\varkappa_0^2}{r^3} \end{gathered} \tag{1.5}$$

where

$$\begin{gathered} A_1 = \frac{R^2 - 1}{R^{\varkappa+2} - 1}, \quad B_1 = \frac{R^2 (R^\varkappa - 1)}{R^{\varkappa+2} - 1}, \quad A = \frac{\Omega R^2 - 1}{R^{\varkappa+2} - 1}, \\ B = \frac{R^2 (R^\varkappa - \Omega)}{R^{\varkappa+2} - 1}, \quad A'_1 = \frac{R^2 - 1}{\ln R}, \quad A' = \frac{\Omega R^2 - 1}{\ln R}, \\ R = \frac{R_2}{R_1}, \quad \Omega = \frac{\Omega_2}{\Omega_1}, \quad \varkappa_0 = \frac{s}{\Omega_1 R_1^2}, \mathrm{Re} = \frac{\Omega_1 R_1^2}{\nu}, P_\theta = \frac{\partial \Pi_0}{\partial \theta}. \end{gathered} \tag{1.6}$$

Here $\varkappa = s/\nu$ is the radial Reynolds number (for $\varkappa > 0$, the liquid flows through the inner cylinder, while for $\varkappa < 0$ through the external one), P_θ=constant, $\mathrm{Re} = \frac{\Omega_1 R_1^2}{\nu}$ is Reynolds number.

A solution (1.5)–(1.6) is by itself a stationary Couette flow when the liquid flows through the surfaces of cylinders with a constant pressure gradient along the cylinder circumference.

Assuming $\vec{\mathbf{V}}' = \vec{\mathbf{V}}_0 + \vec{\mathbf{V}}$, $\Pi' = \Pi_0 + \Pi$. For the disturbances $\vec{\mathbf{V}}$ and Π we obtain the following system of equations:

$$\begin{gathered}\frac{\partial \vec{\mathbf{V}}}{\partial t} + N\vec{\mathbf{V}} - \frac{1}{Re} M\vec{\mathbf{V}} + \frac{1}{Re}\nabla\Pi = -\mathcal{L}(\vec{\mathbf{V}}, \vec{\mathbf{V}}), \\ (\nabla, (r\vec{\mathbf{V}})) = 0, \quad \vec{\mathbf{V}}\big|_{r=1,R} = 0,\end{gathered} \tag{1.7}$$

where

$$\begin{aligned}
&M\vec{\mathbf{V}} = \Big\{\Delta v_r - \frac{1-\varkappa}{r^2} v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta},\ \Delta v_\theta - \frac{1+\varkappa}{r^2} v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}, \Delta v_z\Big\}, \\
&N\vec{\mathbf{V}} = \omega\frac{\partial \vec{\mathbf{V}}}{\partial \theta} + \big\{-2\omega v_\varphi, -g v_r, 0\big\}, \\
&\mathcal{L}(\vec{\mathbf{V}}, \vec{\mathbf{U}}) = \Big\{(\vec{\mathbf{V}}, \nabla)u_r - \frac{v_\theta u_\theta}{r}, (\vec{\mathbf{V}}, \nabla)u_\theta + \frac{v_r u_\theta}{r}, (\vec{\mathbf{V}}, \nabla)u_z\Big\}, \\
&\Delta = \frac{\partial^2}{\partial r^2} + \frac{1-\varkappa}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \\
&\omega = \frac{v_{0\theta}}{r}, \quad g = \frac{dv_{0\theta}}{dr} + \frac{v_{0\theta}}{r}.
\end{aligned}$$

It is assumed that the velocity components and pressure are periodic in azimuthal and axial directions, of periods $2\pi/\alpha$ and $2\pi/m$ respectively. Note that the wave numbers α and m are assumed to be fixed parameters of the problem.

As the Reynolds number Re growths, the flow under consideration losses its stability and, as a result, there may arise (depending on the parameters of the problem) axisymmetric or auto-oscillatory modes in the form of a running azimuthal wave of periods 2π and π in azimuthal direction(see [8]). We have to investigate those oscillatory modes which arise in a small neighborhood of the point of intersection of neutral curves responding for axisymmetric and oscillatory instability of flow (1.5)–(1.6).

2. The use is made of three complex amplitude equations describing the modes which arise in a small neighborhood of the point of intersection of neutral curves of monotone axisymmetric and oscillatory nonaxisymmetric loss of stability. This system obtained by V. Yudovich by means of the method of the theory of bifurcations together with application of the theorem on a neutral manifold, is a generalization of the well-known Landau's amplitude equation ([9]). An analogous amplitude system corresponding to the intersection of bifurcations producing azimuthal waves with different wave numbers has been considered in [10]–[12].

Let $(\Omega_*, \mathrm{Re}_*)$ be a point on a plane of parameters (Ω, Re) responding for the intersection of neutral curves of axisymmetric and oscillatory three-dimensional stability loss of the flow (1.5)–(1.6). Solutions of the nonlinear problem for disturbances are sought in the form ([13–16])

$$\begin{aligned}
&\vec{\mathbf{V}} = \sqrt{|\delta_1|}(\vec{\mathbf{U}} + \vec{\mathbf{U}}^*), \quad \Pi = \sqrt{|\delta_1|}(p + p^*),\\
&\vec{\mathbf{U}} = \eta_0(t)\vec{\Phi}_0(r,z) + e^{ic_0 t}\big[\eta(t)\vec{\Phi}(r,\theta,z) + \eta_1(t)\vec{\Phi}_1(r,\theta,z)\big] + \cdots,\\
&p = \eta_0(t)p_0(r,z) + e^{ic_0 t}\big[\eta(t)p(r,\theta,z) + \eta_1(t)p_1(r,\theta,z)\big] + \cdots,
\end{aligned}$$

where $\delta_1 = \mathrm{Re} - \mathrm{Re}_*$, η_0, η, η_1 are unknown complex amplitudes, c_0 is an unknown cyclic frequency of neutral azimuthal waves; $\vec{\Phi}_0, p_0$ and $\vec{\Phi}, p$ are eigensolutions of linearized problems corresponding to (1.7) for rotationally symmetric ($m = 0$), as well as for three-dimensional oscillatory disturbances, where the vector $\vec{\Phi}_1 = J\vec{\Phi}$. For unknown complex amplitudes we obtain the following dynamical system [11-13]

$$\begin{aligned}
&\dot{\eta}_0 = \sigma\eta_0 + A|\eta_0|^2\eta_0 + B|\eta|^2\eta_0 + B^*|\eta_1|^2\eta_0 + D\eta_0^*\eta^*\eta_1,\\
&\dot{\eta} = \mu\eta + P|\eta_0|^2\eta + Q|\eta|^2\eta + R|\eta_1|^2\eta + S(\eta_0^*)^2\eta_1,\\
&\dot{\eta}_1 = \mu\eta_1 + P|\eta_0|^2\eta_1 + R|\eta|^2\eta_1 + Q|\eta_1|^2\eta_1 + S\eta_0^2\eta,
\end{aligned} \tag{2.1}$$

where A, D are the real and B, P, Q, R, S - the complex coefficients. The $\mathbf{G} = SO(2) * O(2)$ - symmetry enables to reduce the six-dimensional amplitude system to the four-dimensional system, where complex amplitudes are represented in a polar form:

$$\eta_0 = \rho_0 e^{i\psi_0}, \quad \eta = \rho e^{i\psi}, \quad \eta_1 = \rho_1 e^{i\psi_1}.$$

Thus we have obtained the system which is called a motor subsystem of amplitude equations for modules ρ_0, ρ, ρ_1 and phase invariant $\beta = 2\psi_0 + \psi - \psi_1$,

$$\begin{aligned}
&\dot{\rho}_0 = \big[\sigma + A\rho_0^2 + B_r(\rho^2 + \rho_1^2)\big]\rho_0 + D\rho_0\rho\rho_1\cos\beta,\\
&\dot{\rho} = (\mu_r + P_r\rho_0^2 + Q_r\rho^2 + R_r\rho_1^2)\rho + (S_r\cos\beta + S_i\sin\beta)\rho_0^2\rho_1,\\
&\dot{\rho}_1 = (\mu_r + P_r\rho_0^2 + R_r\rho^2 + Q_r\rho_1^2)\rho_1 + (S_r\cos\beta - S_i\sin\beta)\rho_0^2\rho,\\
&\dot{\beta} = C(\rho^2 - \rho_1^2) - 2D\rho\rho_1\sin\beta -\\
&\qquad -\big[S_i(\rho^2 - \rho_1^2)\cos\beta + S_r(\rho^2 + \rho_1^2)\sin\beta\big]\rho_0^2/\rho\rho_1,\\
&C = 2B_i + Q_i - R_i,
\end{aligned} \tag{2.2}$$

r denote real and i imaginary part of complex value, σ and μ_r are free parameters which define position of the point (Ω_*, Re_*) at which we construct decompositions (2.1). Equations for the phases have the form

$$\begin{aligned}
&\dot{\psi}_0 = B_i(\rho^2 - \rho_1^2) - D\rho\rho_1\sin\beta,\\
&\dot{\psi} = \mu_i + P_i\rho_0^2 + Q_i\rho^2 + R_i\rho_1^2 + (S_i\cos\beta - S_r\sin\beta)\rho_0^2\rho_1/\rho,\\
&\dot{\psi}_1 = \mu_i + P_i\rho_0^2 + R_i\rho^2 + Q_i\rho_1^2 + (S_i\cos\beta + S_r\sin\beta)\rho_0^2\rho_1/\rho.
\end{aligned} \tag{2.3}$$

To find coefficients of the amplitude system for the given values of cylinder radii R_1, R_2, axial α and azimuthal m numbers and parameters $\varkappa$ and P_θ, we use a computing algorithm which was realized for the Couette flow at the Rostov State University by V. Kolesov ([7]). Applying this algorithm, we seek for a point of intersection of neutral curves $(\Omega_*, \mathrm{Re}_*)$, perform numerical integration of the linearized problem corresponding to (1.7) and its conjugate. We seek also for solutions of inhomogeneous systems of ordinary differential equations with right-hand sides which are expressed explicitly in terms of $\vec{\Phi}_0, \vec{\Phi}, \vec{\Phi}_1$. Next, the amplitude coefficients are calculated by means of the explicit formulas and already found solutions of linear boundary value problems. For brevity, these formulas are omitted here. Analogous calculations were performed for porous cylinders in [14], [15] and for isothermal Couette flows in [16].

In Table, we present the results of calculations when radius of the outer cylinder is two times greater than that of the inner cylinder ($R = 2$), azimuthal wave number $m = 1$ for different values of axial wave number α, a transverse gradient of pressure P_θ, and radial Reynolds number $\varkappa$.

Parameters	$\varkappa = 1$, $P_\theta = 1$ $\alpha = 3$	$\varkappa = 1$, $P_\theta = 4$ $\alpha = 4$	$\varkappa = 1$, $P_\theta = 2$ $\alpha = 3$	$\varkappa = -1$, $P_\theta = 6$ $\alpha = 5$	$\varkappa = -1$, $P_\theta = 6$ $\alpha = 3$	$\varkappa = 1$, $P_\theta = 1$ $\alpha = 3$
Ω_*	-0,3222	-0,2277	-0,3429	0,024025	0,0401	-0,4065
Re_*	103.058	111.8	100.921	163.929	185.2375	93.998
C_*	0.26529	0.27066	0.26531	0.2607	0.2579	0.26935
A	-848.385	-142.806	688.069	-767.817	941.953	1221.998
B_r	-444.129	-132.701	719.6768	-753.646	-583.007	689.446
B_i	-3139.638	-116.736	-249.465	860.1278	97.31918	513.5-1
D	-585.391	-222.947	332.1889	-1189.6	-1777.568	928.4678
P_r	-6913.774	-839.261	-8.7173	-313.569	973.748	893.099
P_i	6441.846	2070.683	1322.423	-1018.7	-3152.432	334.527
Q_z	-3241.75	-149.757	-21.7877	-76.858	-26.858	12.1604
Q_i	-727.027	29.749	-60.282	-1132.64	-1239.23	-156.0902
R_r	-411.272	-1660.465	-767.8816	-169.51	118.399	142.176
R_i	11200.149	1757.373	1888.165	532.665	369.577	894.875
S_r	-5087.4	-488.798	-18.0144	-125.517	1040.671	618.629
S_i	18784.199	2077.353	2685.4861	-71.17518	157.403	1084.624

As is seen, the neutral curves corresponding to the axisymmetric and three-dimensional oscillatory loss of stability may intersect at one or several points (for example, for $\varkappa = 1, P_\theta = 1, \alpha = 3$). In addition, neutral curves intersect both when cylinders rotate in different directions and when they rotate in one and the same direction. Thus we can conclude that for a transverse pressure gradient, the interaction of rotationally symmetric and tree-dimensional modes between two porous cylinders may produce complicated modes of liquid motion even when cylinders rotate in one and the same direction.

3. As is known ([7]), the motor subsystem has on invariant planes the following equilibria:

The main stationary flow:

$$\rho_0 = \rho = \rho_1 = 0, \ \forall \ \beta; \tag{3.1}$$

Vortices:

$$\rho_0^2 = -\sigma/A, \quad \rho = \rho_1 = 0, \ \forall \ \beta; \tag{3.2}$$

Purely azimuthal waves:

$$\rho_0 = 0, \quad \rho^2 = \rho_1^2 = -\mu_r/(Q_r + R_r), \ \forall \ \beta; \tag{3.3}$$

Spiral waves:

$$\rho_0 = \rho_1 = 0, \quad \rho^2 = -\mu_r/Q_r, \quad \forall, \quad \beta \tag{3.4}$$

$$\rho_0 = \rho = 0, \quad \rho_1^2 = -\mu_r/Q_r, \quad \forall \quad \beta; \tag{3.5}$$

A pair of mixed azimuthal waves

$$\rho_0^2 = \Delta_1^+/\Delta^+, \quad \rho^2 = \rho_1^2 = \Delta_2^+/\Delta^+, \quad \beta = 0, \tag{3.6}$$

$$\rho_0^2 = \Delta_1^-/\Delta^-, \quad \rho^2 = \rho_1^2 = \Delta_2^-/\Delta^-, \quad \beta = \pi, \tag{3.7}$$

$$\Delta_1^+ = \sigma(Q_r + R_r) - \mu_r(2B_r + D), \quad \Delta_2^+ = \mu_r A - \sigma(P_r + S_r),$$

$$\Delta_1^- = \sigma(Q_r + R_r) - \mu_r(2B_r - D), \quad \Delta_2^+ = \mu_r A - \sigma(P_r - S_r),$$

$$\Delta^+ = (2B_r + D)(P_r + S_r) - A(Q_r + R_r),$$

$$\Delta^- = (2B_r - D)(P_r - S_r) - A(Q_r + R_r).$$

Besides equilibria (3.1)–(3.7), the motor subsystem may have up to four pairs of generic equilibria not lying on the invariant planes.

As is known, to equilibria of the motor subsystem lying on the invariant planes there correspond stationary or periodic modes of liquid motion, while to generic equilibria there correspond two-frequency quasi-periodic liquid motions [7].

As our calculations performed for the flow (1.5)–(1.6) for certain value of parameters of the problem has shown, all possible cases are realized: the motor subsystem may have no one equilibrium, or there may exist equilibria of all the above-mentioned types. They may be stable or unstable. We have revealed the case of oscillatory loss of equilibrium stability which produces limit cycles, i.e., isolated periodic solutions of the motor subsystem ([17]).

Calculations were performed for $R = 2$, $m_* = 0, 1$ for different values of parameters P_θ , $\varkappa$, α. As is mentioned in [7], the scale change of unknowns and time upon investigation of the motor subsystem allows us to restrict ourselves to the consideration of only three values of parameters σ, in particular, $\sigma = 0, \sigma = \pm 10$. The programs realizing these calculations were composed algorithmically in Turbo-Pascal language. Analysis of limit cycles of the motor subsystem was carries out by program of Cyclop developed by V. Kolesov at the Rostov State University.

Here we present the results of calculations allowing us to judge about transitions, natural for the problem under consideration.

For $P_\theta = 1$ (a pressure gradient along the rotating inner cylinder), $\varkappa = 1$ (liquid injection through the inner cylinder), $\alpha = 3$ (disturbances $\frac{2\pi}{3}$, periodic in the axial direction), $\Omega_* = -0.322$ (intersection of neutral curves under the rotation of cylinders in different directions), $\sigma = 10, \mu_r > 0$ (the point $(\Omega_*, \mathrm{Re}_*)$ lies above the neutral curves corresponding to the rotationally symmetric and oscillatory loss of stability in the main stationary flow), we have the following results: the main stationary flow (1.5)–(1.6) exists for any values of free parameter μ_r, is unstable, and there appear Teylor's vortices which are stable for $\mu_r < 25.12$, loss their stability for $\mu_r > 25.12$ and produce steady pair of mixed azimuthal waves (3.6). For $\mu_r = 50.975$, a connected pair of generic equilibria branching off from the mixed azimuthal waves exists in the range $25.12 < \mu_r < 119.2102$. At the point of bifurcation $\mu_r = 37.939$, a limit cycle is branching off from the mixed azimuthal waves (3.6). This cycle exists for $37.939 < \mu_r < 50.2$. For $\mu_r = 43.8$ there occurs bifurcation of a period doubling from which a connected pair of stable 4-rotational limit cycles branches off. Further increase of μ_r leads to a number of doublings of connected pairs of cycles which in turn produce a connected pair of chaotic attractors and then their destruction. The results of these calculations are given in Figures 1-4 in the form of phase trajectories onto the coordinate planes. Numbers $0, 1, 2, 3$ on the coordinate axes denote respectively modules of amplitudes ρ_0, ρ_1, ρ_2 and phase invariant β.

For $\sigma = 0$ and any $\mu_r > 0$, the unstable spiral waves are branching off from the main flow, for $\sigma = -10$ and any $\mu_r > 0$ there exist only stable purely azimuthal and unstable spiral waves. When P_θ =2, $\varkappa = 1$, $\Omega_* = -0.34296$, $\alpha = 3$, $\sigma = -10$, we have: in the range $0 < \mu_r < 30$ there exists a connected pair of unstable generic equilibria produced from mixed azimuthal waves (3.6)–(3.7) for $\mu_r = 4.5676$ and from spiral waves for $\mu = 30.273$. At the point $\mu_r = 0.241$ a symmetric limit cycle is branching off from mixed azimuthal waves and exists in the range $0.241 < \mu_r < 0.372$ (Fig.5).For $\mu_r = 0.372$ it losses its symmetry, but remains stable in the range $0.372 < \mu_r < 0.38$ and for $0.38 < \mu_r < 0.41$ stable limit cycles transfer into unstable limit cycles (Fig.6). Thus we have bifurcation of the symmetry loss.

In Fig.7 we can see a case of a nonsymmetric limit cycle ($P_\theta = 4$, $\varkappa = 1$, $\Omega_* = -0.2772$, $\alpha = 4$) for $\sigma = 10$ and $\mu_r > 0$ which is branching off from azimuthal waves for $\mu_r = 98.54506$. This cycle in the range $98.9 < \mu_r < 104.1$ is stable, and for $\mu_r = 104.1$ it losses stability. For $104.1 < \mu_r < 106.19$, this cycle exists, bifurcates, an again we obtain symmetric cycle (Fig.8) which is stable in the range $106.1 < \mu_r < 106.7$, and then it losses its stability.

Consequently, as our calculation show in the presence of a transverse pressure gradient in the viscous liquid flows between two rotating porous cylinders there take place bifurcation sequences and as a result in the neighborhood of the point of intersection of neutral curves there occur chaotic, complete regimes.

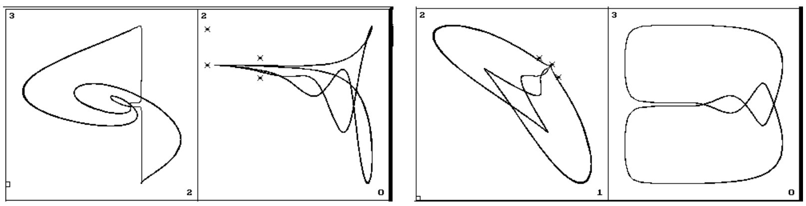

Fig. 1

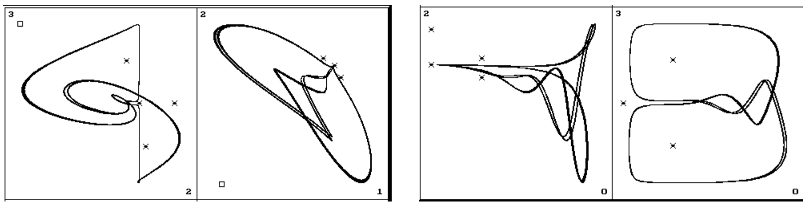

Fig. 2

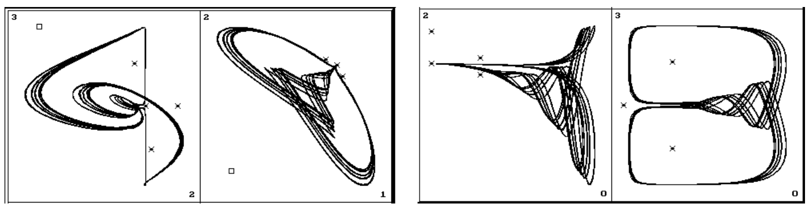

Fig.3

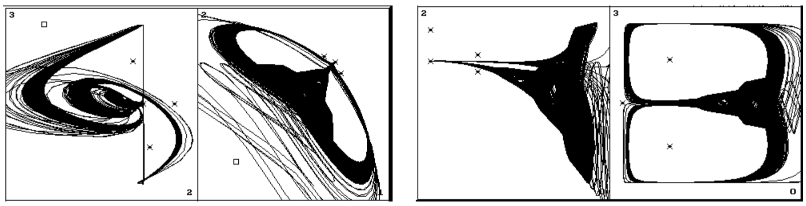

Fig. 4

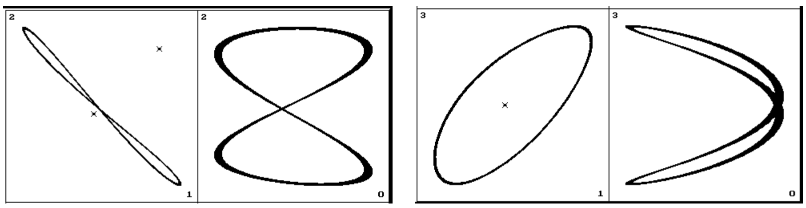

Fig. 5

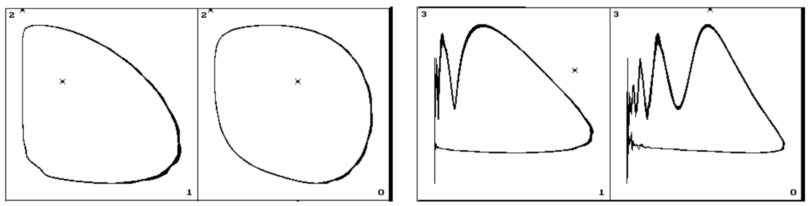

Fig. 6

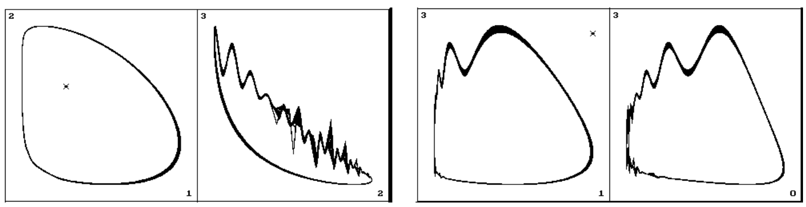

Fig. 7

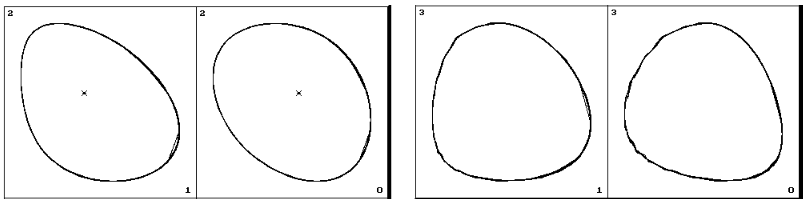

Fig. 8

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