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FREDHOLMNESS OF SINGULAR INTEGRAL OPERATORS IN WEIGHTED MORREY SPACES

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Abstract. We study the well known singular integral equations with Cauchy kernel, in weighted Morrey spaces $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$ on a curve Γ on the complex plane. We give conditions for these operators to be Fredholm and give the formula for their Fredholm index in these non-separable spaces. Also, in relation with the wide usage of singular integral equations in applications, we show how the known formulas for the solution, adjusted for Morrey spaces, allow to easily control these or those desired properties of solutions by means of parameters p and λ and parameters of the weight ρ.

რეზიუმე. სტატიაში შესწავლილია კარგად ცნობილი სინგულარული ინტეგრალური განტოლებები კოშის გულით კომპლექსური სიბრტყის Γ წირზე განსაზღვრულ წონიან მორის $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$ სივრცეებში. დადგენილია ამ ოპერატორების ფრედჰოლმურობის პირობები და ამ არასეპარაბელურ სივრცეებში მიღებულია ფრედჰოლმურობის ინდექსის ფორმულა. გამოყენებებში სინგულარული ინტეგრალური განტოლებების მნიშვნელოვან როლთან დაკავშირებით, ნაჩვენებია თუ მორის სივრცეებზე მორგებული ამონახსნების ცნობილი ფორმულები როგორ იძლევიან მათი ამა თუ იმ თვისებების გაკონტროლების საშუალებას p და λ პარამეტრებისა და ρ წონის მეშვეობით.

1. Introduction

Singular integral operators

$$\mathfrak{A}u := a(t)u(t) + b(t)(Su)(t), \qquad (Su)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{u(\tau)}{\tau - t}d\tau, \quad t \in \Gamma \tag{1.1}$$

are well studied in various function spaces, for instance in the spaces $L_p(\Gamma,\varrho)$ or $H^\lambda(\Gamma,\varrho)$, or in Orlicz spaces, see, for instance, [3], [8], [12], [13], [16], [26]. Recently there appeared investigations of singular integral equations in

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some non-standard spaces such as generalized Lebesgue spaces with variable exponent, see [15], [17], [18], [19], [27], [28].

Meanwhile, they exists a scale of rather "standard" spaces, the scale of Morrey spaces $\mathcal{L}^{p,\lambda}(\Gamma),\ 0 \leq \lambda < 1$, which never were used for the study of normal solvability of singular integral equations, although there are widely known and intensively used in Partial Differential Equations, see for instance [9]. The modification of the Morrey spaces for $\lambda \geq 1$, known as Morrey-Campanato spaces, in the case $1 < \lambda \leq 1 + p$ is isomorphic to the Hölder space $H^{\frac{\lambda-1}{p}}(\Gamma)$ of order $\frac{\lambda-1}{p}$, so Morrey-Campanato spaces, with λ in this interval, do not give a new scale for the theory of singular integral equations. This is not the case in the interval $0 < \lambda < 1$ we deal with in this paper. Within this interval, the Morrey space is a subspace of p-integrable functions, whose regularity is measured not in uniform terms like in Hölder spaces, but in mean value terms.

As is well known, the nature of the solvability of the singular integral equation $\mathfrak{N}u = f$ does not depend much on the choice of the space for solutions u, provided that the curve Γ is closed, the singular operator S is bounded and the coefficients are continuous functions and pointwise multipliers in the chosen space. This is no more true when the coefficients $a(t)$ and $b(t)$ are discontinuous or the curve is open, since solutions in general have singularities at the end-points of the curve and at the points of discontinuity of the coefficients. In this case, the existence of various scales of function spaces becomes more important: making use of the parameters which define the space, one can control these or other properties of solutions. The usage of the weighted Lebesgue spaces $L^p(\Gamma, \varrho)$ as in [12], [13] allows to control the integrability properties of solutions, which in application is usually reduced to controlling the order of singularities of solution at end points of the curve. When singular equations are studied in weighted generalized Lebesgue spaces $L^{p(\cdot)}(\Gamma, \varrho)$ with variable exponent $p(t)$, then in the control of singularities of solutions, one should can restrict the values of $p(t)$ only at a given set of points, see [19], where there was also given a certain general scheme of investigation of singular integral equations in Banach function spaces. Looking for solutions in Hölder spaces with power type singularities as in [8], [26] or in weighted Hölder spaces $H^\lambda(\Gamma, \varrho)$ as in [4], [5], [6], allows to obtain solutions continuous everywhere except for a certain set of points, with controlled singularities at those points and a given order of smoothness (the Hölder exponent λ). An extension to the case of generalized Hölder spaces $H^\omega(\Gamma, \varrho)$ with a prescribed majorant ω of continuity modulus, may be found in [30], [29], [32], see also [33] for the realization of this extension in cases important for applications.

In this paper we consider singular integral equations in weighted Morrey spaces $\mathcal{L}^{p,\lambda}(\Gamma, \varrho)$. The nature of the solvability of singular integral equations in Morrey spaces was never explicitly observed, although the research follows more or less the same lines as for other spaces used in the theory of singular integral equations. Observe that the Morrey spaces $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$ are non-separable in the case $\lambda \neq 0$, which causes some known difficulties in the study of their solvability in these spaces, thus the case of Morrey spaces having common features with the case of other non-separable spaces, such as for instance Hölder or generalized Hölder spaces. In this case the behaviour of solutions is controlled by the parameters p,λ and also by the parameters of the weight. We admit power or, more generally, weights of Bary-Stechkin type. The exponent p and the weight allow to locally control properties of solutions, while the parameter λ in a certain sense is more related to properties expressed in the terms of mean value behaviour.

2. Preliminaries

2.1. **On weighted Morrey spaces on Carleson curves.** Let Γ be a bounded rectifiable curve on the complex plane $\mathbb{C}$. We denote $\tau = t(\sigma), \quad t = t(s)$ where σ and s stand for the arc abscissas of the points τ and t, and $d\mu(\tau) = d\sigma$ will stand for the arc-measure on Γ. We also use the notation

$$\Gamma(t,r) = \{\tau \in \Gamma : |\tau - t| < r\} \quad \text{and} \quad \Gamma_*(t,r) = \{\tau \in \Gamma : |\sigma - s| < r\},$$

so that $\Gamma_*(t,r) \subseteq \Gamma(t,r)$, and denote $\ell = \mu\Gamma =$ length of Γ.

Definition 2.1. A a curve Γ is said to be a *Carleson curve*, if $\mu\Gamma(t,r) \leq Cr$ for all $t \in \Gamma$ and $r > 0$, where $C > 0$ does not depend on t and r. A curve Γ is said to satisfy *the arc-chord condition at a point* $t_0 = t(s_0) \in \Gamma$, if there exists a constant $k > 0$, not depending on t such that

$$|s - s_0| \leq k|t - t_0|, \quad t = t(s) \in \Gamma. \tag{2.1}$$

Finally, a curve Γ is said to satisfy the *(uniform) arc-chord condition*, if

$$|s - \sigma| \leq k|t - \tau|, \quad t = t(s), \tau = t(\sigma) \in \Gamma. \tag{2.2}$$

In the sequel Γ is always assumed to be a Carleson curve.

The Morrey spaces $\mathcal{L}^{p,\lambda}(\Gamma)$ on Γ are defined in the usual way, via the norm

$$\|f\|_{p,\lambda} := \|f\|_{\mathcal{L}^{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma, r>0} \left\{ \frac{1}{r^\lambda} \int\limits_{\Gamma(t,r)} |f(\tau)|^p \, d\mu(\tau) \right\}^{\frac{1}{p}}, \tag{2.3}$$

where $1 \leq p < \infty$ and $0 \leq \lambda \leq 1$. For a non-negative weight function $\varrho(t)$ the weighted space Morrey space is introduced as

$$\mathcal{L}^{p,\lambda}(\Gamma,\varrho) = \{f : \varrho f \in \mathcal{L}^{p,\lambda}(\Gamma)\} \tag{2.4}$$

with

$$\|f\|_{p,\lambda;\varrho} := \|f\|_{\mathcal{L}^{p,\lambda}(\Gamma,\varrho)} = \|\varrho f\|_{\mathcal{L}^{p,\lambda}(\Gamma)}. \tag{2.5}$$

Embeddings of non-weighted Morrey spaces ([22], Theorem 4.3.6 or [9], Chapter III, Proposition 1.1) known for bounded domains in $\mathbb{R}^n$, hold also on Carleson curves, the proof being the same, based only on the usage of the Hölder inequality and the fact that $\mu\Gamma(t,r) \leq Cr$. Hence we have the following embedding for weighted spaces as well.

Lemma 2.2. *Let Γ be bounded Carleson curve, ϱ a weight function, let $1 \leq p \leq q < \infty$, $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$. If*

$$\frac{1-\lambda}{p} \geq \frac{1-\mu}{q} \tag{2.6}$$

then

$$L^{q,\mu}(\Gamma,\varrho) \hookrightarrow L^{p,\lambda}(\Gamma,\varrho). \tag{2.7}$$

Corollary 2.3. *Let Γ be bounded Carleson curve, ϱ a weight function, $1 \leq p < \infty$, $0 \leq \lambda \leq 1$. Then*

$$L^{p,\lambda}(\Gamma,\varrho) \hookrightarrow L^{p_1}(\Gamma,\varrho) \quad \text{for any} \quad 1 \leq p_1 \leq \frac{p}{1-\lambda}, \tag{2.8}$$

where $\hookrightarrow$ stands for the continuous embedding.

2.2. **On admissible weight functions.** In the sequel, we will deal with weights of the form

$$\varrho(t) = \prod_{k=1}^{N} \varphi_k(|t-t_k|), \quad t_k \in \Gamma. \tag{2.9}$$

Although the functions φ_k should be defined only on $[0,d]$, where $d = \operatorname{diam}\Gamma = \sup_{t,\tau\in\Gamma}|t-\tau| < \ell$, everywhere below we consider them as defined on $[0,\ell]$.

Let W be the class of continuous functions φ on $[0,\ell]$, positive on $(0,\ell]$, and let W_0 be the subclass of almost increasing functions $\varphi \in W$ vanishing at the origin: $\varphi(0) = 0$. By $\widetilde{W}$ we denote the class of functions $\varphi \in W$ such $x^\alpha\varphi(x) \in W_0$ for some $\alpha = \alpha(\varphi) > 0$.

Let $x, y \in (0,\ell]$ and $x_+ = \max(x,y)$, $x_- = \min(x,y)$. By $\mathbf{V}_{++}$ and V_{-+} we denote the classes of functions $\varphi \in W$ defined by the following conditions

$$\mathbf{V}_{++}: \qquad \left|\frac{\varphi(x)-\varphi(y)}{x-y}\right| \leq C\frac{\varphi(x_+)}{x_+}, \tag{2.10}$$

$$\mathbf{V}_{-+}: \qquad \left|\frac{\varphi(x)-\varphi(y)}{x-y}\right| \leq C\frac{\varphi(x_-)}{x_+}. \tag{2.11}$$

Example 2.4. Let $\alpha, \beta \in \mathbb{R}^1$ and $A > \ell$. Then

$$x^\alpha\left(\ln\frac{A}{x}\right)^\beta \in \begin{cases} \mathbf{V}_{++}, & \text{if } \alpha > 0, \beta \in \mathbb{R}^1 \text{ or } \alpha = 0 \text{ and } \beta \leq 0 \\ \mathbf{V}_{-+}, & \text{if } \alpha < 0, \beta \in \mathbb{R}^1 \text{ or } \alpha = 0 \text{ and } \beta \geq 0. \end{cases}$$

For brevity we denote $\|f\|_p = \left(\int\limits_\Gamma |f(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}}$, so that

$$\|f\|_{p,\lambda} = \sup_{t,r} \left\| \frac{\chi_{\Gamma(t,r)}(\cdot)}{r^{\frac{\lambda}{p}}} f(\cdot) \right\|_p . \tag{2.12}$$

2.3. **Matuszewska-Orlicz type indices of weights.** It is known that the property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so called Matuszewska-Orlicz indices. We refer to [14], [20, 21], [23] (p. 20), [24], [25], [30], [31] for the properties of the indices of such a type. For a function $\varphi \in W_0$, the numbers

$$m(\varphi) = \sup_{0<x<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}$$

and

$$M(\varphi) = \sup_{x>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x\to \infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}$$

are known as *the Matuszewska-Orlicz type lower and upper indices* of the function $\varphi(r)$. Note that in this definition $\varphi(x)$ need not to be an N-function: only its behaviour at the origin is of importance. Observe that

$$0 \le m(\varphi) \le M(\varphi) \le \infty \quad \text{for} \quad \varphi \in W_0.$$

It is obvious that

$$m[x^\lambda \varphi(x)] = \lambda + m(\varphi), \quad M[x^\lambda \varphi(x)] = \lambda + M(\varphi), \qquad \lambda \in \mathbb{R}^1. \tag{2.13}$$

Consequently,

$$-\infty < m(\varphi) \le M(\varphi) \le \infty \quad \text{for} \quad \varphi \in \widetilde{W}.$$

Definition 2.5. We say that a function $\varphi \in W_0$ belongs to the Zygmund class $\mathbb{Z}^\beta$, $\beta \in \mathbb{R}^1$, if $\int_0^h \frac{\varphi(x)}{x^{1+\beta}} dx \le c\frac{\varphi(h)}{h^\beta}$, and to the Zygmund class $\mathbb{Z}_\gamma$, $\gamma \in \mathbb{R}^1$, if $\int_h^\ell \frac{\varphi(x)}{x^{1+\gamma}} dx \le c\frac{\varphi(h)}{h^\gamma}$. We also denote

$$\mathbb{Z}^\beta_\gamma := \mathbb{Z}^\beta \bigcap \mathbb{Z}_\gamma,$$

the latter class being also known as Bary-Stechkin-Zygmund class [2].

The following statement is known, see [14], Theorems 3.1 and 3.2.

Theorem 2.6. *Let $\varphi \in \widetilde{W}$ and $\beta, \gamma \in \mathbb{R}^1$. Then*

$$\varphi \in \mathbb{Z}^\beta \Longleftrightarrow m(\varphi) > \beta \quad \textit{and} \quad \varphi \in \mathbb{Z}_\gamma \Longleftrightarrow M(\varphi) < \gamma.$$

Besides this

$$m(\varphi) = \sup\left\{\delta > 0 : \quad \frac{\varphi(x)}{x^\delta} \quad \textit{is almost increasing}\right\}, \tag{2.14}$$

and

$$M(\varphi) = \inf\left\{\lambda > 0 : \quad \frac{\varphi(x)}{x^\lambda} \quad \textit{is almost decreasing}\right\}. \tag{2.15}$$

Remark 2.7. Theorem 2.6 was formulated in [14] for $\beta \geq 0, \gamma > 0$ and $\varphi \in W_0$. It is evidently true when these exponents are negative, and for $\varphi \in \widetilde{W}$, the latter being obvious by the definition of the class $\widetilde{W}$ and formulas (2.13).

2.4. **On the boundedness of the singular operator in weighted Morrey spaces.** The following theorem was proved in [34].

Theorem 2.8. *Let the curve* Γ *satisfy the arc-chord condition,* $1 < p < \infty, 0 \leq \lambda < 1$*, and* ϱ *be weight* (2.9).
I) *The operator* S_Γ *is bounded in the Morrey space* $\mathcal{L}^{p,\lambda}(\Gamma, \varrho)$, *if*

$$\frac{\lambda - 1}{p} < m(\varphi_k) \leq M(\varphi_k) < \frac{\lambda}{p} + \frac{1}{p'}, \quad k = 1, 2, \ldots, N. \tag{2.16}$$

II) *in case of power weights* $\varphi_k(r) = r^{\alpha_k}$, *under the additional assumption that* Γ *is smooth in neighborhoods of the nodes* t_k *of the weight, the conditions* $\frac{\lambda-1}{p} < \alpha_k < \frac{\lambda}{p} + \frac{1}{p'}$, $\quad k = 1, 2, \ldots, N$, *are also necessary for the boundedness.*

2.5. **On some basics related to Fredholmness.** We recall that a linear operator $\mathfrak{A}$ in a Banach space X is called Fredholm if its kernel $\ker \mathfrak{A} := \{u \in X : \mathfrak{A}u = 0\}$ has a finite dimension $\alpha := \dim(\ker \mathfrak{A}) < \infty$, the range $R(\mathfrak{A}) := \{f \in X : f = \mathfrak{A}u, u \in X\}$ is closed in X and has a finite dimension $\beta := dim(R(\mathfrak{A})) < \infty$. Following [10], the ordered pair (α, β) will be referred to as d-characteristic of the operator $\mathfrak{A}$. The difference $Ind_X \mathfrak{A} := \alpha - \beta$ is called the index of $\mathfrak{A}$.

We will need some notions known in the Fredholm theory of singular integral operators with piecewise coefficients and presented in the form most convenient for our goals in [19]. The following definitions are some modifications of notions taken from [19], the main modification being related to omission of the assumption that "nice" functions are dense in the space X. What is exposed below, will be used in Section 3 for the case $X = \mathcal{L}^{p,\lambda}(\Gamma, \varrho)$.

Let $X = X(\Gamma)$ be any Banach space of functions on a closed simple Jordan rectifiable curve Γ satisfying the following assumptions:

(1) $C(\Gamma) \subset X(\Gamma) \subset L^1(\Gamma)$,
(2) $\|a\, f\|_X \leq \sup\limits_{t\in\Gamma} |a(t)| \cdot \|f\|_X$ for every $a \in L^\infty(\Gamma)$,
(3) the operator S is bounded in $X(\Gamma)$,

(4) for the space $X(\Gamma)$ there exist two functions $\alpha(t)$ and $\beta(t)$, $0 < \alpha(t) < 1$, $0 < \beta(t) < 1$, such that the operator $|t-t_0|^{\gamma(t_0)}S|t-t_0|^{-\gamma(t_0)}I$ with $t_0 \in \Gamma$ is bounded in the space $X(\Gamma)$ for all $\gamma(t_0) \in (-\alpha(t_0), 1-\beta(t_0))$.

Let $D^+ = int\ \Gamma$, $D^- = ext\ \Gamma$, $z_0 \in D^+$. For a function $a \in PC(\Gamma)$ with jumps at t_k, $k = 1, 2, \ldots, N$, we put

$$\gamma(t) = \frac{1}{2\pi i}\, ln\, \frac{a(t-0)}{a(t+0)} \quad \text{and} \quad \omega(t) = \prod_{k=1}^{n} (t-z_0)_k^{\gamma(t_k)} \tag{2.17}$$

where the functions $\omega_k(z) = (z-z_0)_k^{\gamma(t_k)}$ stand for univalent analytic functions in the complex plane with the cut passing from z_0 to infinity through the point $t_k \in \Gamma$ and which crosses Γ at t_k only. The function $a_1(t) = \frac{a(t)}{\omega(t)}$ is continuous on Γ independently of the choice of the branch for

$$\Re\gamma(t_k) = \frac{1}{2\pi}\ \arg \frac{a(t_k-0)}{a(t_k+0)}. \tag{2.18}$$

Definition 2.9. ([19]) Let $X(\Gamma)$ be a Banach function space satisfying the above assumptions 1.-3. A function $a \in PC(\Gamma)$ is called X-nonsingular if $\inf\limits_{t\in\Gamma} |a(t)| > 0$ and

$$\frac{1}{2\pi}\ \arg \frac{a(t_k-0)}{a(t_k+0)} \notin [\alpha(t_k), \beta(t_k)] + \mathbb{Z} \tag{2.19}$$

where $[\cdots] + \mathbb{Z}$ stands for the set $\bigcup_{\xi\in[\cdots]}\{\xi, \xi \pm 1, \xi \pm 2, \ldots\}$. When a is X-nonsingular, the integer

$$ind_X\ a \ = \ \sum_{k=1}^{n} [\theta(t_k) - \Re\gamma(t_k)] \tag{2.20}$$

is called X-index of the function a, where $\theta(t_k) = \frac{1}{2\pi} \int\limits_{t_k+0}^{t_{k+1}-0} d\ arg\ a(t)$ are increments of the argument and $\Re\gamma(t_k)$ are chosen in the interval $\beta(t_k) - 1 < \Re\gamma(t_k) < \alpha(t_k)$.

In the case $X = \mathcal{L}^{p,\lambda}(\Gamma, \varrho)$, according to Theorem 2.8 we have

$$\alpha(t) = \frac{1-\lambda}{p} + \begin{cases} m(\varphi_k), & t = t_k \\ 0, & t \neq t_k \end{cases}, \quad \beta(t) = \frac{1-\lambda}{p} + \begin{cases} M(\varphi_k), & t = t_k \\ 0, & t \neq t_k. \end{cases} \tag{2.21}$$

Observe that in the case where $\varphi_k(r) = r^{\alpha_k}$ is a power function, the values

$$\alpha(t) = \frac{1-\lambda}{p} + \begin{cases} \alpha_k, & t = t_k \\ 0, & t \neq t_k \end{cases}, \quad \beta(t) = \frac{1-\lambda}{p} + \begin{cases} \alpha_k, & t = t_k \\ 0, & t \neq t_k. \end{cases} \tag{2.22}$$

are exact, that is, the operator $|t-t_0|^{\gamma(t_0)}S|t-t_0|^{-\gamma(t_0)}I$ is not bounded in the Morrey space $X=\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$ when $\gamma(t_0)\notin(-\alpha(t_0),1-\beta(t_0))$, see Theorem 2.8.

A function $a\in PC(\Gamma), \inf_\Gamma|a(t)|>0$, is X-nonsingular with $X=\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$, if

$$\frac{1}{2\pi}\ \arg\frac{a(t_k-0)}{a(t_k+0)}-\frac{\lambda}{p}\notin\left[-\frac{1}{p}-m(\varphi_k),\frac{1}{p'}-M(\varphi_k)\right]+\mathbb{Z} \tag{2.23}$$

which turns into

$$\frac{1}{2\pi}\ \arg\frac{a(t_k-0)}{a(t_k+0)}\neq\frac{\lambda-1}{p}-\alpha_k \pmod 1 \tag{2.24}$$

in the case $\varphi_k(r)=r^{\alpha_k}$.

3. Formulations of the Main Result

We assume that $a(t),b(t)\in PC(\Gamma)$ and $t_k, k=1,2,\dots,m$, are points of discontinuity of $a(t),b(t)$. When saying in the following theorem that a point $t_k\in\Gamma$ is not a *whirling point* of Γ, we mean that

$$\sup_{\substack{t\in\Gamma\\ t\neq t_0}}|\arg(t-t_0)|<\infty.$$

Theorem A. *Let a closed curve Γ satisfy the arc-chord condition, let $a(t),b(t)\in PC(\Gamma)$ and $\varrho(t)$ be weight* (2.9). *Suppose also that the points t_k are not whirling points of Γ. The operator $\mathfrak{A}$ is Fredholm in the space $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$, $1<p<\infty, 0\le\lambda<1$, if $\inf_{t\in\Gamma}|a(t)\pm b(t)|\neq 0$, and*

$$\frac{1}{2\pi}\arg\frac{g(t_k-0)}{g(t_k+0)}-\frac{1-\lambda}{p}\notin[m(\varphi_k),M(\varphi_k)]+\mathbb{Z},\quad k=1,2,\dots,n, \tag{3.1}$$

where $g(t)=\frac{a(t)+b(t)}{a(t)-b(t)}$. The index of the operator $\mathfrak{A}$ in the space $L^{p,\lambda}(\Gamma,\varrho)$ is equal to the $L^{p,\lambda}$-index of the function g:

$$Ind_{L^{p,\lambda}(\Gamma,\varrho)}\mathfrak{A}=-ind_{L^{p,\lambda}(\Gamma,\varrho)}g(t):=\varkappa$$

(*and the d-characteristic is equal to $(\varkappa,0)$, if $\varkappa\ge 0$ and $(0,|\varkappa|)$, if $\varkappa\le 0$*).

4. Proof of Theorem A

Observe, that the statement of Theorem A is like a particular case of general Theorem A in [19] for Banach spaces $X(\Gamma)$. However, Theorem C in [19] was proved for separable spaces $X(\Gamma)$, which is not the case when $X(\Gamma)$ is Morrey space. It may be also observed that the proof of the sufficiency part in [19] after some modifications is valid for non-separable spaces X as well. For the completeness of the presentation we present a direct short proof of our Theorem A without the usage of dense sets in $X=\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$.

4.1. **On compactness of the commutator $Sg - gS$ in $\mathcal{L}^{p,\lambda}(\Gamma, \varrho)$.**

Lemma 4.1. *Let the curve Γ satisfy the arc-chord condition, let $g \in C(\Gamma)$ and ϱ - weight* (2.9)*. Under the assumptions $1 < p < \infty, 0 \leq \lambda < 1$ and condition* (2.16)*, the commutator $S_\Gamma g - gS_\Gamma$ is compact in the weighted Morrey space $\mathcal{L}^{p,\lambda}(\Gamma, \varrho)$, with weight* (2.9)*.*

Proof. The statement of the lemma follows from Theorem 2.8 and the fact that continuous functions may be uniformly approximated on Γ by rational functions. The latter is valid for an arbitrary Jordan curve Γ, as is known from the famous Mergelyan's result, see for instance [7], p. 169. This fact is known, having being observed in [19], p. 71, for an arbitrary Banach function space $X(\Gamma)$ on an arbitrary curve Γ in which the singular operator S is bounded. $\square$

4.2. **Standard lemmas.** For a function $G(t) = \frac{a(t)-b(t)}{a(t)+b(t)}$ we put

$$G(t) = W(t)G_0(t), \quad W(t) = \prod_{k=1}^{n} (t - z_0)^{\gamma_k} = \prod_{k=1}^{n} \frac{W_k^+(t)}{W_k^-(t)},$$

where $W_k^\pm(t)$, $t \in \Gamma$, are the limiting values of the functions $W_k^\pm(z) = (z - t_k)^{\gamma_k}$, $W_k^-(z) = \left(\frac{z-t_k}{t-z_0}\right)^{\gamma_k}$ analytical in D^+ and D^-, resp. As is known [8], $G_0(t) \in C(\Gamma)$ under the choice $\gamma_k = \frac{1}{2\pi i} \ln \frac{G(t_k-0)}{G(t_k+0)}$ independently of the choice of a branch of the logarithm. (Recall that t_k are assumed to be different from whirling points of the curve). In the operator form we have

$$\mathfrak{A}u = A(t)\left(a_0 I + b_0 W^+ S \frac{1}{W^+}\right)(P_+ + WP_-)u \tag{4.1}$$

where $P_\pm = \frac{1}{2}(I \pm S)$, $W^+(t) = \prod_{k=1}^{n} W_k^+(t)$, $a_0(t) = \frac{1+G_0(t)}{2}$, $b_0(t) = \frac{1-G_0(t)}{2}$, $A(t) = a(t) + b(t)$. The following lemma justifies the usage of (4.1) for $u \in \mathcal{L}^{p,\lambda}(\Gamma, \varrho)$.

Lemma 4.2. *Let Γ be a curve satisfying the arc-chord condition, ϱ be weight* (2.9) *and $a(t), b(t) \in PC(\Gamma)$. Under condition* (2.16)*, representation* (4.1) *is valid for $u \in \mathcal{L}^{p,\lambda}(\Gamma, \varrho)$, $1 < p < \infty, 0 \leq \lambda < 1$, under the choice*

$$\frac{\lambda - 1}{p} - m(\varphi_k) < \Re\gamma_k < \frac{\lambda}{p} + \frac{1}{p'} - M(\varphi_k), \ k = 1, 2, \ldots, N. \tag{4.2}$$

Proof. To verify (4.1) on $\mathcal{L}^{p,\lambda}(\Gamma, \varrho)$, we first note that it is valid, as is known, on "nice" functions $u(t)$, and the operators $P_\pm$ and $W^+ S \frac{1}{W^+}$ involved in representation (4.1) are bounded in $\mathcal{L}^{p,\lambda}(\Gamma, \varrho)$, which follows from Theorem 2.8.

This is not yet sufficient for the justification, since the space $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$ is not separable, no set of "nice" functions being dense in $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$. To justify the validity of (4.1) for all $u \in \mathcal{L}^{p,\lambda}(\Gamma,\varrho)$, it suffices to observe that

$$\mathcal{L}^{p,\lambda}(\Gamma,\varrho) \subset L^{p_1}(\Gamma,\varrho), \quad p_1 = \frac{p}{1-\lambda} \tag{4.3}$$

according to embedding (2.8) and inequalities (4.2) may be rewritten as

$$-\frac{1}{p_1} - m(\varphi_k) < \Re\gamma_k < \frac{1}{p_1'} - M(\varphi_k),$$

which also guarantees the boundedness of the operators $P_\pm$ and $W^+S\frac{1}{W^+}$ in the space $L^{p_1}(\Gamma,\varrho)$ according to the same Theorem 2.8 with $\lambda = 0$ and p replaced by p_1. □

Lemma 4.3. *Let Γ be a curve satisfying the arc-chord condition, ϱ be weight* (2.9) *satisfying assumption* (2.16)*, and $a(t), b(t) \in PC(\Gamma)$. Under condition* (4.2)*, the operator $P_+ + WP_-$ is invertible in $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$.*

Proof. The equalities

$$W^+\left(P_+ + \frac{1}{W}P_-\right)\frac{1}{W^+}\left(P_+ + WP_-\right)u =$$
$$= (P_+ + WP_-)W^+\left(P_+ + \frac{1}{W}P_-\right)\frac{1}{W^+}u = u$$

are well known, see for instance [11], p.352, or [13], being easily verified on nice function. Extension of these equalities to functions in $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$ can be done in the same way as in the proof of Theorem 10.1, basing on imbedding (4.3). □

Lemma 4.4. *Let Γ be a curve satisfying the arc-chord condition and $a, b \in C(\Gamma)$, $a(t) \pm b(t) \neq 0$. The operator $\mathfrak{A} = aI + bS$ is Fredholm in the space $\mathcal{L}^{p,\lambda}(\Gamma,\varrho)$ with weight* (2.9) *satisfying assumption* (2.16)*, its index is equal to $\varkappa = ind\, G(t)$, where $ind\, G(t)$ is the winding number of $G(t)$, and its d-characteristics is $(\varkappa, 0)$ when $\varkappa \geq 0$ and $(0, |\varkappa|)$ otherwise.*

Proof. Similar statements in case of continuous coefficients are well known for a more general setting of a class of Banach function spaces, the Fredholmness of the operator not depending much on the choice of the space when coefficients of are continuous, see for instance [19], p. 71. Observe that the Fredholmness itself is a consequence of the existence of the regularizer $\frac{1}{a^2-b^2}(aI - bS)$, which in checked directly thanks to the compactness of the commutator provided by Lemma 4.1. The statement on the d-characteristic follows from the embeddings

$$H^\lambda(\Gamma) \subset \mathcal{L}^{p,\lambda}(\Gamma,\varrho) \subset L^p(\Gamma,\varrho)$$

and the fact that the d-characteristic of the operator $aI + bS$ in the spaces $H^\lambda(\Gamma)$ and $L^p(\Gamma, \varrho)$ is exactly the same. □

4.3. **Proof itself of Theorem A.** This proof is prepared by Lemmas 4.2-4.4. By condition (5.4), we can choose the numbers $\Re\gamma_k$ in the interval (4.2). Then by Lemma 4.2, the representation (4.1) is valid. For Fredholmness of the operator $\mathfrak{A} = aI + bS$ it remains to refer to Lemmas 4.2-4.4. As regards the index, by Lemmas 4.3 and 4.4, we have $Ind\,\mathfrak{A} = ind\,G_0(t)$ and it remains to note that $ind\ G_0(t)$ is nothing else but $-ind_{L^{p,\lambda}} \frac{1}{G}$, by the construction of $G_0(t)$.

5. The Case of an Interval of the Real Line

Because of applications, the case of an interval of the real line is of especial interest. We consider equations over the interval $\Gamma = [0, 1]$ of the real line:

$$A\varphi(x) := a(x)\varphi(x) + \frac{b(x)}{\pi} \int\limits_0^1 \frac{\varphi(t)\,dt}{t - x} = f(x), \qquad 0 < x < 1 \tag{5.1}$$

and assume that $a(x)$ and $b(x)$ are continuous real-valued functions.

For this case we derive the corresponding corollary to Theorem A. So we may apply Theorem A and deal with two points $t = 0$ and $t = 1$ of discontinuities of the coefficients and the weight with nodes at the end-points of the interval:

$$\rho(x) = \varphi_0(x)\varphi_1(1 - x). \tag{5.2}$$

5.1. **Conditions of normal solvability and formula for the index.** Let

$$\theta(x) : = \arg \frac{a(x) - ib(x)}{a(x) + ib(x)} \tag{5.3}$$

with the choice $\theta(0) \in [0, 2\pi)$. From Theorem A we obtain the following statement, accompanied also by formulas for the general solution in the next subsection.

Corollary 5.1. *Let $a, b \in C([0, 1])$ and $\varrho(t)$ be weight* (5.2)*. The operator $\mathfrak{A}$ is Fredholm in the space $\mathcal{L}^{p,\lambda}([0, 1], \varrho)$, $1 < p < \infty, 0 \le \lambda < 1$, if*

$$\inf_{0<x<1}(|a(x)| + |b(x)|) \neq 0,$$

and

$$\begin{aligned} \frac{\theta(0)}{2\pi} - \frac{1-\lambda}{p} &\notin [m(\varphi_0), M(\varphi_0)] + \mathbb{Z}, \\ \frac{\theta(1)}{2\pi} + \frac{1-\lambda}{p} &\notin [-M(\varphi_1), -m(\varphi_1)] + \mathbb{Z}. \end{aligned} \tag{5.4}$$

Under these conditions, the Fredholm index of equation (5.1) *in the space* $\mathcal{L}^{p,\lambda}([0,1],\rho)$ *is given by the formula*

$$\varkappa = \frac{1}{2\pi}\Delta\theta(x)\Big|_{[0,1]} + \frac{\theta(1)}{2\pi} - \frac{\theta(0)}{2\pi} \tag{5.5}$$

where $\Delta\theta(x)|_{[0,1]}$ *is the increment of* $\theta(x)$ *along* $[0,1]$ *and the values of* $\theta(0)$ *and* $\theta(1)$ *are chosen by the rules*

$$\begin{aligned} M(\varphi_0) - 1 &< \frac{\theta(0)}{2\pi} - \frac{1-\lambda}{p} < m(\varphi_0), \\ m(\varphi_1) &< \frac{\theta(1)}{2\pi} + \frac{1-\lambda}{p} < 1 - M(\varphi_1). \end{aligned} \tag{5.6}$$

If $\varkappa \geq 0$*, the number of linearly independent solutions in* $\mathcal{L}^{p,\lambda}([0,1],\rho)$ *of the homogeneous singular integral equation is equal to* $\varkappa$*, and the non-homogeneous equation is unconditionally solvable. If* $\varkappa < 0$*, then for the non-homogeneous equation there exist* $|\varkappa|$ *linearly independent solvability conditions.*

Remark 5.2. Formula for the Fredholm index $\varkappa$ may be recalculated in terms more convenient for applications:

$$\varkappa = \left[\xi_0 - \frac{\theta(0)}{2\pi}\right] + \left[\xi_1 + \frac{\theta(1)}{2\pi}\right] \tag{5.7}$$

where the brackets denote the integer part of a number, $\theta(0)$ and $\theta(1)$ are the end-point values of an arbitrary, but the same branch of $\theta(x) = \arg\frac{a(x)-ib(x)}{a(x)+ib(x)}$, while ξ_0 and ξ_1 are arbitrarily chosen points of the intervals

$$\begin{aligned} &\left[\frac{1-\lambda}{p} + m(\varphi_0), \frac{1-\lambda}{p} + M(\varphi_0)\right], \\ &\left[\frac{1-\lambda}{p} + m(\varphi_1), \frac{1-\lambda}{p} + m(\varphi_1)\right], \end{aligned} \tag{5.8}$$

respectively.

Remark 5.3. The right-hand side of formula (5.7) does not depend on the choice of auxiliary parameters ξ_0 and ξ_0, because by conditions (5.4) the numbers $\frac{\theta(0)}{2\pi}$ and $-\frac{\theta(1)}{2\pi}$ are not allowed to take values from the intervals where ξ_0 and ξ_1 are chosen. In the case where the weight functions have coinciding index numbers: $m(\varphi_0) = M(\varphi_0)$ and $m(\varphi_1) = M(\varphi_1)$, the "prohibited" intervals (5.8) degenerate to the points $\frac{1-\lambda}{p} + m(\varphi_0)$ and $\frac{1-\lambda}{p} + m(\varphi_1)$, and the formula for the Fredholm index $\varkappa$ takes the form

$$\varkappa = \left[\frac{1-\lambda}{p} + m(\varphi_0) - \frac{\theta(0)}{2\pi}\right] + \left[\frac{1-\lambda}{p} + m(\varphi_1) + \frac{\theta(1)}{2\pi}\right]. \tag{5.9}$$

5.2. **Formulas for solutions.**

Theorem 5.4. *Let* $1 < p < \infty,\ 0 \le \lambda < 1$, *let* ϱ *be weight* (5.2) *and*

$$\frac{1-\lambda}{p} + m(\varphi_j) > 0, \qquad \frac{1-\lambda}{p} + M(\varphi_j) < 1, \quad j = 0,1,$$

be satisfied. Let also $\min_{x\in[0,1]}(|a(x)| + |b(x)|) > 0$, *conditions* (5.4) *be fulfilled and* $\varkappa$ *given by* (5.5) *or* (5.7). *In the case* $\varkappa \ge 0$, *the general solution of equation* (5.1) *in the space* $\mathcal{L}^{p,\lambda}([0,1],\varrho)$ *is given by the formula*

$$\varphi(x) = \frac{b(x)}{a^2(x)+b^2(x)} x^{\mu_0}(1-x)^{\mu_1}\mathcal{Z}(x)P_{\varkappa-1}(x) + \frac{a(x)f(x)}{a^2(x)+b^2(x)} -$$

$$- \frac{b(x)}{a^2(x)+b^2(x)} \cdot \frac{x^{\mu_0}(1-x)^{\mu_1}\mathcal{Z}(x)}{\pi}\int\limits_0^1 \frac{f(t)\,dt}{t^{\mu_0}(1-t)^{\mu_1}\mathcal{Z}(t)(t-x)} \tag{5.10}$$

where $P_{\varkappa-1}(x) = \sum\limits_{k=0}^{\varkappa-1} c_k x^k$ *is a polynomial of degree* $\varkappa - 1$ *with arbitrary coefficients,*

$$\mathcal{Z}(x)=\exp\Psi(x), \quad \Psi(x)=\frac{1}{2\pi}\int\limits_0^1 \frac{\theta(t)dt}{t-x} + \frac{\theta(0)}{2\pi}\ln x - \frac{\theta(1)}{2\pi}\ln(1-x)\in C([0,1])$$

and the exponents μ_0 *and* μ_1 *are given by the formulas*

$$\mu_0 = \begin{cases} -\frac{\theta(0)}{2\pi}, & \text{if} \quad 0 \le \frac{\theta(0)}{2\pi} < \frac{1-\lambda}{p} + m(\varphi_0) \\ \\ 1-\frac{\theta(0)}{2\pi}, & \text{if} \quad \frac{1-\lambda}{p} + M(\varphi_0) < \frac{\theta(0)}{2\pi} < 1 \end{cases} \tag{5.11}$$

and

$$\mu_1 = \begin{cases} \left\{\frac{\theta(1)}{2\pi}\right\}, & \text{if} \quad 0 \le \left\{\frac{\theta(1)}{2\pi}\right\} < 1 - M(\varphi_1) - \frac{1-\lambda}{p} \\ \\ \left\{\frac{\theta(1)}{2\pi}\right\} - 1, & \text{if} \quad 1 - M(\varphi_1) - \frac{1-\lambda}{p} < \left\{\frac{\theta(1)}{2\pi}\right\} < 1 \end{cases} \tag{5.12}$$

where $\left\{\frac{\theta(1)}{2\pi}\right\} = \frac{\theta(1)}{2\pi} - \left[\frac{\theta(1)}{2\pi}\right]$ *is the fractional part of* $\frac{\theta(1)}{2\pi}$.

Proof. Formulas of such a kind for the general solution are well known in the theory of singular integral equations. The main point now is of technical nature - to calculate the exponents μ_0, μ_1 in dependence on the parameters $p, \lambda, m(\varphi_0), M(\varphi_0), m(\varphi_1), M(\varphi_1)$ of the space $\mathcal{L}^{p,\lambda}([0,1],\varrho)$. We recall that under the initial choice of $\arg\theta(0) \in [0, 2\pi)$, the exponents $\mu_0 \in (-1,1)$ and $\mu_1 \in (-1,1)$ always - for any space - have the form

$$\mu_0 = -\frac{\theta(0)}{2\pi} + k, \qquad \mu_1 = \frac{\theta(1)}{2\pi} - \left[\frac{\theta(1)}{2\pi}\right] + \ell \tag{5.13}$$

where k and ℓ are integers depending on the space of solutions, and they may have the only values $k = 0, 1$ and $\ell = 0, -1$ and the choice of the values of these integers is determined by the fact that the singular operator $x^{\mu_0}(1-x)^{\mu_1} \int\limits_0^1 \frac{f(t)\,dt}{t^{\mu_0}(1-t)^{\mu_1}(t-x)}$ involved in (5.4), should be continuous in the space $\mathcal{L}^{p,\lambda}([0,1], \varrho)$. According to Theorem 2.8, we arrive at the conditions

$$\frac{\lambda-1}{p} - m(\varphi_j) < \mu_j < \frac{\lambda-1}{p} - M(\varphi_j), \quad j = 0, 1.$$

Then according to (5.13) we easily arrive at formulas (5.11)-(5.12). $\square$

It remains to mention that in the case $\varkappa < 0$, the equation is not always solvable in the corresponding space and the necessary and sufficient condition on the right-hand side $f(x)$ in this case is the following:

$$\int\limits_0^1 \frac{f(t)t^{k-1}\,dt}{t^{\mu_0}(1-t)^{\mu_1}\mathcal{Z}(t)} = 0, \qquad k = 1, 2, \ldots, |\varkappa|.$$

6. The Case of Constant Coefficients

6.1. **Formulas for solutions.** Let $a(x) = a = const$ and $b(x) = b = const$, $a^2 + b^2 \neq 0$. We denote

$$\theta = \arg \frac{a - bi}{a + bi} \in [0, 2\pi) \quad \text{and} \quad \alpha = \frac{\theta}{2\pi} \in [0, 1).$$

For constant coefficients, conditions (5.4) of normal solvability take the form

$$\alpha \notin [u_0, v_0] \cup [1 - v_1, 1 - u_1], \tag{6.1}$$

where

$$u_j = \frac{1-\lambda}{p} + m(\varphi_j), \ v_j = \frac{1-\lambda}{p} + M(\varphi_j), \quad 0 < u_j \le v_j < 1, \quad j = 0, 1.$$

The index $\varkappa$ is calculated by the formula

$$\varkappa = [u_0 - \alpha] + [u_1 + \alpha],$$

following from (5.7), where the brackets stand as usual for the entire part of number. This index $\varkappa$ may take only three values $1, 0, -1$.

In the next corollary, derived from Theorem 5.4, we explicitly present these values in dependence on the parameters of the space, supposing for simplicity that the weight functions $\varphi_j(x), j = 0, 1$, have coinciding indices:

$$m(\varphi_j) = M(\varphi_j), \quad j = 0, 1.$$

so that $u_j = v_j$, $j = 0, 1$ in this case.

Corollary 6.1. *Let* $1 < p < \infty,\ 0 \le \lambda < 1$, *let* $\rho(x)$ *be weight* (5.2) *with* $m(\varphi_j) = M(\varphi_j)$, *and*

$$0 < \frac{1-\lambda}{p} + m(\varphi_j) < 1, \quad j = 0, 1.$$

Equation (5.1) *is normally solvable, if* $\alpha \ne u_0$ *and* $\alpha \ne 1 - u_1$. *The index* $\varkappa$ *is equal to*

$$\varkappa = 0, \quad \textit{if} \quad 0 \le \alpha < \min(u_0, 1-u_1) \quad \textit{and} \quad \max(u_0, 1-u_1) < \alpha < 1$$

and

$$\varkappa = \begin{cases} -1, & \textit{if } u_0 < 1-u_1 \textit{ and } u_0 < \alpha < 1-u_1 \\ +1, & \textit{if } 1-u_1 < u_0 \textit{ and } 1-u_1 < \alpha < u_0 \end{cases}$$

in the case $u_0 \ne 1 - u_1$. *Solutions are given by formula* (5.4) *with*

$$\mu_0 = \begin{cases} -\alpha, & \textit{if } 0 \le \alpha < u_0 \\ 1-\alpha, & \textit{if } u_0 < \alpha < 1, \end{cases} \qquad \mu_1 = \begin{cases} \alpha, & \textit{if } 0 \le \alpha < 1-u_1 \\ \alpha - 1, & \textit{if } 1-u_1 < \alpha < 1. \end{cases}$$

6.2. **The case of the Sönghen equation.** The equation

$$\frac{1}{\pi}\int_0^1 \frac{\varphi(t)\,dt}{t-x} = f(x) \tag{6.2}$$

referred sometimes as Sönghen equation, is known in Aerodynamics, due to the papers by H.Sönghen, see for instance [35]; as is welll known, this equation is an essential tool in various problems of Aerodynamics, see for instance the paper by A.V.Balakrishnan [1]. For equation (6.2) we have $\alpha = \frac{1}{2}$. From Corollary 6.1 by direct calculations we obtain the following statement, which we also give for the case $m(\varphi_j) = M(\varphi_j), j = 0, 1$.

Corollary 6.2. *Equation* (6.2) *is normally solvable in the space* $\mathcal{L}^{p,\lambda}([0,1], \varrho)$, $1 < p < \infty,\ 0 \le \lambda < 1,\ \varrho(x) = \varphi_0(x)\varphi_1(1-x),\ \varphi_0, \varphi_1 \in \Phi_2$, *if*

$$m(\varphi_i) \ne \frac{1}{2} - \frac{1-\lambda}{p},\ j = 0, 1.$$

The index $\varkappa$ *of the equation in this space is calculated by the following formula*

$$\varkappa = \begin{cases} 0, & \textit{if } \left(m(\varphi_0) + \frac{1-\lambda}{p} - \frac{1}{2}\right)\left(m(\varphi_1) + \frac{1-\lambda}{p} - \frac{1}{2}\right) < 0 \\ -1, & \textit{if } m(\varphi_j) + \frac{1-\lambda}{p} - \frac{1}{2} < 0,\ j = 0, 1, \\ 1, & \textit{if } m(\varphi_j) + \frac{1-\lambda}{p} - \frac{1}{2} > 0,\ j = 0, 1. \end{cases}$$

Correspondingly to the cases $\varkappa - 0, -1, 1$, *the solution is given by the following familiar formulas*

$$\varphi(x) = -\frac{1}{\pi}\sqrt{\frac{1-x}{x}}\int_0^1 \sqrt{\frac{t}{1-t}}\frac{f(t)\,dt}{t-x},$$

$$\text{if } \ m(\varphi_0) > \frac{1}{2} - \frac{1-\lambda}{p}, \ m(\varphi_1) < \frac{1}{2} - \frac{1-\lambda}{p}, \tag{6.3}$$

$$\varphi(x) = -\frac{1}{\pi}\sqrt{\frac{x}{1-x}}\int_0^1 \sqrt{\frac{1-t}{t}}\frac{f(t)\,dt}{t-x},$$

$$\text{if } \quad m(\varphi_0) < \frac{1}{2} - \frac{1-\lambda}{p}, \ m(\varphi_1) > \frac{1}{2} - \frac{1-\lambda}{p}, \tag{6.4}$$

($\varkappa = 0$: *the cases of unique and unconditional solvability*),

$$\varphi(x) = -\frac{\sqrt{x(1-x)}}{\pi}\int_0^1 \frac{f(t)\,dt}{\sqrt{t(1-t)}(t-x)},$$

$$\text{if } \quad m(\varphi_j) < \frac{1}{2} - \frac{1-\lambda}{p}, \quad j = 0, 1, \tag{6.5}$$

($\varkappa = -1$: *the case of unique but conditional solvability:* $\int_0^1 \frac{f(t)\,dt}{\sqrt{t(1-t)}} = 0$),

$$\varphi(x) = \frac{C}{\sqrt{x(1-x)}} - \frac{1}{\pi\sqrt{x(1-x)}}\int_0^1 \frac{\sqrt{t(1-t)}f(t)\,dt}{t-x},$$

$$\text{if } \quad m(\varphi_j) > \frac{1}{2} - \frac{1-\lambda}{p}, \quad j = 0, 1, \tag{6.6}$$

($\varkappa = 1$: *the case of unconditional but non-unique solvability:* C *is arbitrary*).

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References

1. A. V. Balakrishnan, Possio Integral Equation of Aeroelasticity Theory. *J. Aerosp. Engrg.*, **16**(2003), 139–154.
2. N. K. Bari and S. B. Stechkin, Best approximations and differential properties of two conjugate functions. (Russian) *Trudy Moskov. Mat. Obšč*, **5**(1956), 483–522.
3. A. Böttcher and Yu. Karlovich, Carleson curves, Muckenhoupt weights, and Toeplitz operators. *Progress in Mathematics*, 154. *Birkh:̈auser Verlag, Basel*, 1997.

4. R. V. Duduchava, On singular integral equations in weighted Hölder spaces. (Russian) *Dokl. Akad. nauk SSSR, ser. matem.*, **191(13)**(1970), 16–19.
5. R. V. Duduchava, Singular integral equations in Hölder spaces with weight. I. Hölder coefficients. *Mat. Issled.*, **5(2(16))**(1970), 104–124.
6. R. V. Duduchava, Singular integral equations in Hölder spaces with weight. II. Piecewise Hölder coefficients. *Mat. Issled.*, **5(3(17))**(1970), 58–82.
7. D. Gaier, Lectures on approximation in the complex domain. (German) *Birkhäuser Verlag, Basel-Boston, Mass.*, 1980.
8. F. D. Gakhov, Boundary value problems. (Russian) *Nauka, Moscow*, 1977.
9. M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems. Annals of Mathematics Studies, 105. *Princeton University Press, Princeton, NJ*, 1983.
10. I. Gohberg and M. Krein, Fundamental aspects of defect numbers, root numbers and indexes of linear operators. (Russian) *Uspehi Mat. Nauk (N.S.)* **12**(1957), No. 2(74), 43–118.
11. I. Gohberg and N. Krupnik, The spectrum of singular integral operators in L_p spaces. (Russian) *Studia Math.* **31**(1968), 347–362.
12. I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations. I. Introduction. Translated from the 1979 German translation by Bernd Luderer and Steffen Roch and revised by the authors. Operator Theory: Advances and Applications, 53. *Birkhäuser Verlag, Basel*, 1992.
13. I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations. Vol. II. General theory and applications. Translated from the 1979 German translation by S. Roch and revised by the authors. Operator Theory: Advances and Applications, 54. *Birkhäuser Verlag, Basel*, 1992.
14. N. K. Karapetiants and N. G. Samko, Weighted theorems on fractional integrals in the generalized Hölder spaces via indices m_ω and M_ω. *Fract. Calc. Appl. Anal.*, **7**(2004), No. 4, 437–458.
15. A. Yu. Karlovich, Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces. *J. Integral Equations Appl.*, **15**(2003), No. 3, 263–320.
16. G. Khuskivadze, V. Kokilashvili and V. Paatashvili, Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14**(1998).
17. V. Kokilashvili and V. Paatashvili, The Riemann-Hilbert problem in weighted classes of Cauchy type integrals with density from $L^{p(\cdot)}(\Gamma)$. *J. Complex Anal. and Oper. Theory*, 2008, DOI 10.1007/s11785-008-0067-9.
18. V. Kokilashvili, V. Paatashvili and S. Samko, Boundary value problems for analytic functions in the class of Cauchy-type integrals with density in $L^{p(\cdot)}(\Gamma)$. *Bound. Value Probl.*, 2005, No. 1, 43–71.
19. V. Kokilashvili and S. Samko, Singular integral equations in the Lebesgue spaces with variable exponent. *Proc. A. Razmadze Math. Inst.* **131**(2003), 61–78.
20. S. G. Krein, Yu. I. Petunin and E. M. Semenov, Interpolation of linear operators. (Russian) *Nauka, Moscow*, 1978.
21. S. G. Krein, Yu. I. Petunin and E. M. Semenov, Interpolation of linear operators. Translated from the Russian by J. Szucs. Translations of Mathematical Monographs, 54. *American Mathematical Society, Providence, R.I.*, 1982.
22. A. Kufner, O. John and S. Fučik, Function spaces. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. *Noordhoff International Publishing, Leyden; Academia, Prague*, 1977.

23. Lech Maligranda, Indices and interpolation. *Dissertationes Math. (Rozprawy Mat.)* **234**(1985).
24. Lech Maligranda, Orlicz spaces and interpolation. Seminars in Mathematics, 5. *Universidade Estadual de Campinas, Departamento de Matemática, Campinas*, 1989.
25. W. Matuszewska and W. Orlicz, On some classes of functions with regard to their orders of growth. *Studia Math.* **26**(1965), 11–24.
26. N. I. Muskhelishvili, Singular integral equations. Boundary problems of function theory and their application to mathematical physics. *Translation by J. R. M. Radok. P. Noordhoff N. V., Groningen*, 1953.
27. V. S. Rabinovich and S. G. Samko, Essential spectra of pseudodifferential operators in Sobolev spaces with variable smoothness and variable Lebesgue indices. *Doklady Akademii Nauk*, 417(2):167–170, 2007. *transl. in Doklady Mathematics*, 2007, Vol. 76, No. 3, 835–838.
28. V. S. Rabinovich and S. G. Samko, Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces. *Integral Equations Operator Theory* **60**(2008), No. 4, 507–537.
29. N. G. Samko, Criterion of Fredholmness of singular operators with piece-wise continuous coefficients in the generalized Hölder spaces with weight. page 363. *Proceedings of IWOTA* 2000, *Setembro* 12-15, *Faro, Portugal*, *Birkhäuser*, In: *Operator Theory: Advances and Applications*, Vol. 142.
30. N. G. Samko, Singular integral operators in weighted spaces with generalized Hölder condition. *Proc. A. Razmadze Math. Inst.* **120**(1999), 107–134.
31. N. G. Samko, On non-equilibrated almost monotonic functions of the Zygmund-Bary-Stechkin class. *Real Anal. Exch.*, 30(2):727, 2004/2005.
32. N. G. Samko, Singular integral operators in weighted spaces of continuous functions with non-equilibrated continuity modulus. *Math. Nachr.* **279**(2006), No. 12, 1359–1375.
33. N. G. Samko, Solutions of singular integral equations in function spaces with weights oscillating at end points. *Sixth International Conference on Mathematical Problems in Engineering and Aerospace Sciences*, 687–694, *Camb. Sci. Publ., Cambridge*, 2007.
34. N. G. Samko, Weighted Hardy and singular operators in Morrey spaces, *J. Math. Anal. and Appl.*, 2008. doi: 10.1016/j.jmaa.2008.09.021.
35. H. Söhngen, Die Lösungen der Integralgleichung $g(x) = \frac{1}{2\pi}\int_{-a}^{a} \frac{f(\xi)d\xi}{x-\xi}$ und deren Anwendung in der Tragfl"ugeltheorie. (German) *Math. Z.* **45**(1939), No. 1, 245–264.

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