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ON THE ERRORS OF ONE METHOD OF SOLUTION OF A STATIC EQUATION FOR A STRING

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Abstract. The paper deals with the accuracy of the method of solution of a nonlinear integro-differential equation for a string.

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1. Statement of the Problem

Let us consider the boundary value problem

$$\varphi\Big(\int_0^1 w_x^2\,dx\Big)w_{xx}=f(x),\quad 0<x<1, \tag{1}$$

$$w(0)=w(1)=0, \tag{2}$$

characterizing the static state of a string. Here $w=w(x)$ is the displacement function we have to define, while of the given functions $f=f(x)$ and $\varphi=\varphi(z)$ the first one corresponds to the acting force and the second one is described by a stress-strain relation. It is assumed that $\varphi(z)$, $0\le z<\infty$, is a continuously differentiable function that satisfies the condition

$$\varphi(z)>\alpha>0,\quad 0\le z<\infty, \tag{3}$$

and the function $f(x)$ is twice continuously differentiable, $0<x<1$.

When $\varphi(z)$ is a linear function, (1) is obtained from Kirchhoff's string oscillation equation by eliminating the time argument t [1]. The introduction of the function $\varphi(z)$ enables us not to restrict the consideration to Hooke's law in the stress-strain relation [2].

2. Method of Solution

To find $w(x)$, we will use M. Chipot's approach [3], [4]. The function $w(x)$ is written in the form

$$w(x)=\lambda v(x) \tag{4}$$

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where λ and $v(x)$ are respectively the parameter and the function to be found. Substituting (4) into (1) we obtain

$$\lambda\varphi\biggl(\int_0^1 (\lambda v')^2\,dx\biggr)v'' = f. \tag{5}$$

Without loss of generality equality (5) is replaced by the system of equations

$$v'' = f,$$
$$\lambda\varphi\biggl(\int_0^1 (\lambda v')^2\,dx\biggr) = 1.$$

As follows from (2) and (4), the function $v(x)$ vanishes on the boundary. Therefore for this function we have the boundary value problem

$$v'' = f, \tag{6}$$
$$v(0) = v(1) = 0, \tag{7}$$

while the parameter λ is defined as a solution of the equation

$$\lambda\varphi(s\lambda^2) = 1, \tag{8}$$

where

$$s = \int_0^1 (v')^2\,dx. \tag{9}$$

A solution of problem (6), (7) is written as

$$v(x) = (x-1)\int_0^x \xi f(\xi)\,d\xi + x\int_x^1 (\xi-1)f(\xi)\,d\xi. \tag{10}$$

As for the parameter s which participates in the derivation of equation (8), its value is calculated on the strength of (9) and (10) by the formula

$$s = \int_0^1 \biggl[\int_0^x \xi f(\xi)\,d\xi + \int_x^1 (\xi-1)f(\xi)\,d\xi\biggr]^2 dx. \tag{11}$$

3. Errors of the Method and Their Estimates

Let us investigate the errors of the method. They may arise for two reasons. Firstly, it is natural to assume that integrals (10) and (11) are defined by means of quadrature formulas, i.e. approximately. So, instead of exact values of integrals (10) and (11) we obtain respectively the function $\overline{v}(x)$ and a value $\overline{s} > 0$. Substituting the value $\overline{s}$ in equation (8) instead of s, we see that an exact solution of this equation is not λ but $\overline{\lambda}$. Secondly, since in the general case the nonlinear equation $\lambda\varphi(\overline{s}\lambda^2) = 1$ cannot be solved exactly, we have to use the iteration method which will give an iteration approximation $\overline{\lambda}_n$, where n is the number of an iteration step, $n = 0, 1, \ldots$. Thus instead of $w(x) = \lambda v(x)$, an exact solution of problem (1), (2), we have $\overline{w}_n(x) = \overline{\lambda}_n \overline{v}(x)$. Let us estimate the error of the method $\overline{w}_n(x) - w(x)$ using values one part of which are obtained by the application of the method, while the other part can be calculated by means of the initial data of the problem.

Since

$$|\overline{w}_n(x) - w(x)| \leq |\overline{\lambda}_n - \lambda|\, |\overline{v}(x)| + |\lambda|\, |\overline{v}(x) - v(x)|$$

and

$$|\overline{\lambda}_n - \lambda| \leq |\overline{\lambda}_n - \overline{\lambda}| + |\overline{\lambda} - \lambda|,$$

for the error of the method we get the estimate

$$|\overline{w}_n(x) - w(x)| \leq \\ \leq \left(|\overline{\lambda} - \lambda| + |\overline{\lambda}_n - \overline{\lambda}|\right)(|\overline{v}(x)| + \Delta v(x)) + |\overline{\lambda}_n| \Delta v(x), \qquad (12)$$

where $\Delta v(x) > 0$ is the upper boundary of the error $|\overline{v}(x) - v(x)|$, derived by an expression for the remainder term of the quadrature formula which is used for calculating $v(x)$ according to (10). Furthermore, the right-hand part of (12) contains the values $\overline{v}(x)$ and $\overline{\lambda}_n$. We can define them since they are results of the counting we perform. As for $|\overline{\lambda} - \lambda|$ and $|\overline{\lambda}_n - \overline{\lambda}|$, these values are unknown. Let us describe the technique of obtaining their upper boundary and also show how we can obtain an upper boundary for $\Delta v(x)$.

1. Estimation of $|\overline{\lambda} - \lambda|$. Let us find out how the replacement of the parameter s by $\overline{s}$ influences the solution λ of equation (8). We rewrite this equation in the form

$$\lambda = \frac{1}{\varphi(s\lambda^2)} \qquad (13)$$

and apply the Taylor formula. As a result we have

$$|\overline{\lambda} - \lambda| \leq \frac{M_1}{m_0^2} \max{}^2(\overline{\lambda}, \lambda)\, |\overline{s} - s|,$$
$$m_0 = \min_z |\varphi(z)|, \;\; M_1 = \max_z |\varphi'(z)|,$$
$$0 \leq z \leq \max(\overline{s}, s) \max{}^2(\overline{\lambda}, \lambda). \qquad (14)$$

Let us replace (3) by an expression that would allow us to estimate $|\overline{\lambda} - \lambda|$ by means of calculated values. Using (13) and (3), we write the sought inequality as

$$|\overline{\lambda} - \lambda| \leq \frac{M_1}{\alpha^2 m_0^2} \Delta s,$$
$$m_0 = \min_z |\varphi(z)|, \quad M_1 = \max_z |\varphi'(z)|, \quad 0 \leq z \leq \frac{1}{\alpha^2} (\overline{s} + \Delta s). \qquad (15)$$

Here $\Delta s > 0$ denotes the upper boundary of the error $|\overline{s} - s|$ which is obtained by means of an expression for the remainder term of the quadrature formula used for calculating s according to (11). In particular, if we apply the generalized formula of trapezoids with step $h = \dfrac{1}{N}$ and nodes $x_i = ih$, $i = 0, 1, \dots, N$, then in (15) we can set [5]

$$\Delta s \le \frac{h^2}{6}\left[\frac{1}{2}\max_x |f(x)| \max_x |f'(x)| + \Big(\max_x |f(x)|\Big)^2\right] +$$
$$+2\sum_{p=1}^{2}\Big(\max_x |f(x)|\Big)^{2-p} \times$$
$$\times\left[\frac{h^2}{12}\Big(2\max_x |f'(x)| + \max_x |f''(x)|\Big)\right]^p, \quad 0 < x < 1. \tag{16}$$

2. Estimation of $|\overline{\lambda}_n - \overline{\lambda}|$. Let us consider one of simple methods of construction of iteration approximations $\overline{\lambda}_n$ and estimate the corresponding error $|\overline{\lambda}_n - \overline{\lambda}|$ [6]. Thus we solve the equation

$$\lambda\varphi(\overline{s}\lambda^2) = 1 \tag{17}$$

an exact solution of which is $\overline{\lambda}$. Squaring equality (17) and multiplying by $\overline{s}$, for $\mu = \overline{s}\lambda^2$ we come to an equation of the form

$$g(\mu) = 0, \tag{18}$$

where $g(\mu) = \mu\varphi^2(\mu) - \overline{s}$. As follows from (3), at the ends of the interval $\left[0, \dfrac{\overline{s}}{\alpha^2}\right]$ the function $g(\mu)$ changes its sign, which implies that equation (18) has a solution on this interval. We apply the bisection method to (18) and construct a sequence of embedded into each other intervals $[a_0, b_0] \supset [a_1, b_1] \supset \cdots \supset [a_n, b_n] \supset \cdots$ such that $g(a_n)g(b_n) \le 0$, $b_n - a_n = \dfrac{1}{2^n}(b_0 - a_0)$, $n = 0, 1, \dots$, $a_0 = 0$, $b_0 = \dfrac{\overline{s}}{\alpha^2}$. The sequence $\{a_n\}$, $n = 0, 1, \dots$, will tend to a solution of equation (18). Take into account the type of relationship between λ and μ, we conclude that $\lim\limits_{n\to\infty} \overline{\lambda}_n = \overline{\lambda}$, where $\overline{\lambda}_n = \left(\dfrac{a_n}{\overline{s}}\right)^{1/2}$. The rate of convergence is characterized by the relation

$$|\overline{\lambda}_n - \overline{\lambda}| \le \left(\frac{a_n}{\overline{s}} + \frac{1}{2^n\alpha^2}\right)^{\frac{1}{2}} - \left(\frac{a_n}{\overline{s}}\right)^{\frac{1}{2}}. \tag{19}$$

3. estimation of $\Delta v(x)$. Let us assume that the values of $v(x)$ are calculated by formula (10) at the nodes $x_i = ih$, $i = 1, 2, \dots, N-1$, $h = \dfrac{1}{N}$ and

also that, as above, the integrals are calculated by the generalized formula of trapezoids. Then, as shown in [5], we have

$$\Delta v(x_i) \le \frac{x_i(1-x_i)h^2}{12} \times$$
$$\times \Big(\max_{0\le x\le x_i} |2f'(x) + xf''(x)| + \max_{x_i\le x\le 1} |2f'(x) + (x-1)f''(x)| \Big). \qquad (20)$$

Using formulas (15), (16), (19) and (20), together with $\overline{v}(x_i)$ and $\overline{\lambda}_n$ in (12), we obtain the numerical value of the method error boundary at the nodal points x_i, $i = 1, 2, \ldots, N-1$.

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