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ON THE MUCKENCHAUPT CONDITION IN VARIABLE LEBESGUE SPACES

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Abstract. Let $\mathcal{P}(\mathbb{R}^n)$, $(n \geq 2)$ be the class of exponents p for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. We construct an example showing that $\sup\limits_Q \frac{1}{|Q|}\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} < \infty$ does not imply $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

რეზიუმე. ვთქვათ $\mathcal{P}(\mathbb{R}^n)$, $(n \geq 2)$ წარმოადგენს ყველა იმ p ექსპონენტთა ერთობლიობას, რომელთათვისაც ჰარდი-ლიტვუდის მაქსიმალური ოპერატორი M შემოსაზღვრულია $L^{p(\cdot)}(\mathbb{R}^n)$ სივრცეში. ნაშრომში აგებულია ექსპონენტა, რომელიც არ ეკუთვნის $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ კლასს, მაგრამ ის აკმაყოფილებს მაკენჰაუპტის ტიპის პირობას $\sup\limits_Q \frac{1}{|Q|}\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} < \infty$.

1. Introduction

Let $p(\cdot) : \mathbb{R}^n \mapsto [1, +\infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of functions f such that for some $\lambda > 0$,

$$\int\limits_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{p(\cdot)} = \left\{\lambda > 0; \int\limits_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx \leq 1\right\}.$$

The Hardy-Littlewood maximal function is defined for $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing the point x.

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Assume that $p_- = \operatorname{essinf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ = \operatorname{esssup}_{x\in\mathbb{R}^n} p(x)$. Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all functions p $(1 < p_- \leq p_+ < \infty)$ for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In [1] L. Diening showed that $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if there exists $c > 0$ such that for any family of pairwise disjoint cubes π and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\Big\| \sum_{Q\in\pi} (|f_Q|)\chi_Q \Big\|_{p(\cdot)} \leq c\|f\|_{p(\cdot)}, \tag{1.1}$$

where $f_Q = \frac{1}{|Q|}\int_Q f$.

We say that dx satisfies the condition $A_{p(\cdot)}$, $(dx \in A_{p(\cdot)})$ if $p_- > 1$, $p_+ < \infty$ and there exists $c > 0$ such that for any cube Q and $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$|f|_Q \|\chi_Q\|_{p(\cdot)} \leq c\|f\chi_Q\|_{p(\cdot)}.$$

It is easy to see that $dx \in A_{p(\cdot)}$ if and only if $\sup_Q A(Q) < \infty$, where

$$A(Q) = \frac{1}{|Q|}\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)},$$

and $p'(x) = \dfrac{p(x)}{p(x)-1}$. (Note that this condition would be a full analogue of the classical Muckenhoupt A_p condition in the context of variable Lebesgue spaces.)

It is natural to ask whether (1.1) can be replaced by $dx \in A_{p(\cdot)}$. In [2] was prowed that if $p(\cdot)$ is a constant outside some ball, then $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if and only if $dx \in A_{p(\cdot)}$. A. Lerner prowed([3]) that following

Theorem 1.1. *Let $dx \in A_{p(\cdot)}$, and let $E \subset \mathbb{R}^n$ be a measurable set of positive measure. Then there exists a constant $c > 0$ depending on $p(\cdot), n$ and E such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|(Mf)\chi_E\|_{p(\cdot)} \leq c\|f\chi_E\|_{p(\cdot)}.$$

Given a function $p(\cdot)$, we say that M is weak type $(p(\cdot), p(\cdot))$ if there exists $c > 0$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\sup_{\alpha>0} \alpha\|\chi_{x:|Mf(x)>\alpha}\|_{p(\cdot)} \leq c\|f\|_{p(\cdot)}$$

It is easy to see that weak $(p(\cdot), p(\cdot))$ property of M implies $dx \in A_{p(\cdot)}$. A. Lerner [3] prowed following

Theorem 1.2. *Let $p(\cdot)$ be a radially decreasing function with $p_- > 1$ and $p_+ < \infty$. Then M is of weak $(p(\cdot), p(\cdot))$ if and only if $dx \in A_{p(\cdot)}$.*

We prove following

Theorem 1.3. *Let $n \geq 2$. There exists a exponent $p(\cdot)$ such that:*
1) $dx \in A_{p(\cdot)}$;
2) $p(\cdot)\overline{\in}\mathcal{P}(\mathbb{R}^n)$;

3) *Operator M is not weak $(p(\cdot), p(\cdot))$ type;*
4) $dx\overline{\in}A_{\alpha p(\cdot)}$ *for any α where $1/p_- < \alpha < 1$;*
5) $dx\overline{\in}A_{p(\cdot)-\alpha}$ *for any α where $0 < \alpha < p_- - 1$.*

Remark. Note that the following are equivalent:
1)$p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and
2) $\alpha p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for some $1/p_- < \alpha < 1$ (see [1]).

2. Proof of Theorem 1.3

We consider the case $n = 2$; the same proof with trivial modifications works for $n > 2$.

Let p_1, p_2 are positive constants satisfying the conditions $1<p_1<p_2<\infty$, and $\frac{2}{p_1} - \frac{1}{p_2} \le 1$ $(\Leftrightarrow \frac{2}{p_1'} \ge \frac{1}{p_2'})$. Fix a function $\widetilde{p} \in C^\infty(\mathbb{R})$ such that $\widetilde{p}(x) = p_1$ if $x \in \mathbb{R}\backslash[0;3]$; $\widetilde{p}(x) = p_2$ if $x \in [1,2]$; and $p_1 \le \widetilde{p}(x) \le p_2$ for $x \in [0;1] \cup [2;3]$.

Define $p(x,y) = \widetilde{p}(x)$; $(x,y) \in \mathbb{R}^2$. We are going to show that $dx \in A_{p(\cdot)}$.

Note that if $|Q \cap ([0;3] \times \mathbb{R})| = 0$ then $A(Q) = 1$. Let $Q \cap ([0;3] \times \mathbb{R}) \neq \emptyset$ and $|Q| > 100$. Next, let $4Q \cap ([1;2] \times \mathbb{R}) = M$, $4Q \cap ((\mathbb{R} \setminus [0,3]) \times \mathbb{R}) = N$ and $4Q \setminus (M \cup N) = K$. Note That $|M| = 2\sqrt{|Q|}$, $|N| = 2\sqrt{|Q|} \times (2\sqrt{|Q|} - 3)$ and $|K| = 4\sqrt{|Q|}$.

It is easy to see that $\|\chi_K\|_{p(\cdot)} \le |K|^{1/p_1} \le c|Q|^{1/p_1}$ and we get $\|\chi_{4Q}\|_{p(\cdot)} \asymp \|\chi_M\|_{p(\cdot)} + \|\chi_N\|_{p(\cdot)} + \|\chi_K\|_{p(\cdot)} \asymp (\sqrt{|Q|})^{1/p_2} + |Q|^{1/p_1} + (\sqrt{|Q|})^{1/p_1} \asymp |Q|^{1/p_1}$.

Analogously $\|\chi_{4Q}\|_{p'(\cdot)} \asymp \|\chi_M\|_{p'(\cdot)} + \|\chi_N\|_{p'(\cdot)} + \|\chi_K\|_{p'(\cdot)} \asymp (\sqrt{|Q|})^{1/p_2'} + (|Q|)^{1/p_1'} + (\sqrt{|Q|})^{1/p_2'} = (\sqrt{|Q|})^{1/p_2'} + (\sqrt{|Q|})^{1/p_2'} + (\sqrt{|Q|})^{2/p_1'} \asymp (\sqrt{|Q|})^{2/p_1'} = |Q|^{1/p_1'}$.

Combining these estimates yields $A(Q) \le cA(4Q) \le c$ if $|Q| > 100$. If $\frac{1}{4} \le |Q| \le 100$ then the estimate $A(Q) \le c$ is trivial. It remains to consider the case when $0 < |Q| < \frac{1}{4}$ and $|Q \cap ([0;3] \times \mathbb{R})| > 0$. In this case we have (see [1])

$$\|\chi_Q\|_{p(\cdot)} \asymp |Q|^{\frac{1}{p_Q}} \quad \text{and} \quad \|\chi_Q\|_{p'(\cdot)} \asymp |Q|^{\frac{1}{p_Q'}}$$

where $\frac{1}{p_Q} = \frac{1}{|Q|} \int_Q \frac{1}{p(x)}\, dx$ and $\frac{1}{p_Q'} = \frac{1}{|Q|} \int_Q \frac{1}{p'(x)}\, dx$, and therefore $A(Q) \le c$.

Fix a sequence $a_i \ge 0$, $i \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} a_i^{p_2} < \infty$ and $\sum_{i=1}^{\infty} a_i^{p_1} = \infty$. Define the function $f(x,y) = \sum_{i=1}^{\infty} a_i \chi_{[1;2]\times[i;i+1]}$. Note that $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

It is obvious $Mf(x,y) \geq \frac{1}{9}a_i$ for $(x,y) \in ([3,4] \times [i,i+1]; i \in \mathbb{N}$ and consequently $Mf\overline{\in}L^{p(\cdot)}(\mathbb{R}^2)$. Consequently $p(\cdot)\overline{\in}\mathcal{P}(\mathbb{R}^2)$.

Fix $\varepsilon > 0$ such that $\frac{1}{p_1} - \frac{1}{p_2} - \varepsilon > 0$. Let $a_i = i^{-\frac{1}{p_2}-\varepsilon}$; $i \in \mathbb{N}$. Then $f \in L^{p(\cdot)}(\mathbb{R}^2)$. Let $E_n = \{Mf(x,y) > \frac{1}{10}a_n\}$; $n \in \mathbb{N}$. Note that $[3,4]\times[1,n]\subset E_n$ and $\|\chi_{E_n}\|_{p(\cdot)} \geq n^{1/p_1}$. We have $a_n\|\chi_{E_n}\|_{p(\cdot)} \geq n^{1/p_1-1/p_2-\varepsilon} \to \infty$ when $n \to \infty$. Consequently operator M not have weak $(p(\cdot), p()\cdot)$ type.

Note that if $1 < p_1 < p_2 < \infty$, $\frac{2}{p_1} - \frac{1}{p_2} > 1$ and $|Q| > 1$ then $\|\chi_{4Q}\|_{p'(\cdot)} \asymp |Q|^{1/p_1}$ and $\|\chi_{4Q}\|_{p'(\cdot)} \asymp (\sqrt{|Q|})^{1/p_2'}$. In this case we have

$$A_{4Q} \asymp \frac{1}{|Q|}|Q|^{1/p_1}(\sqrt{|Q|})^{1/p_2'} = |Q|^{1/p_1+1/2p_2'-1} = (\sqrt{|Q|})^{2/p_1-1/p_2-1}.$$

Since $A_{4Q} \to \infty$, when $|Q| \to \infty$, we obtain $dx\overline{\in}A_{p(\cdot)}$.

Assume that $1 < p_1 < p_2 < \infty$ and $\frac{2}{p_1} - \frac{1}{p_2} = 1$. In this case we have $dx \in A_{p(\cdot)}$. If $1/p_- < \alpha < 1$ then $\frac{2}{\alpha p_1} - \frac{1}{\alpha p_2} > 1$ and consequently $dx\overline{\in}A_{\alpha p(\cdot)}$. Analogously if $0 < \alpha < p_1 - 1$ then $\frac{2}{p_1 - \alpha} - \frac{1}{p_2 - \alpha} > 1$ and $dx\overline{\in}A_{p(\cdot)-\alpha}$. $\square$

Remark. Lerner showed in [4] that if the exponent $p(\cdot) : \mathbb{R}^n \to [1, +\infty)$, $(1 < p_- \leq p_+ < \infty)$ satisfies

$$M_Q\Big(\frac{1}{p}\Big) \leq \frac{c}{\log(e + \max(|Q|, |Q|^{-1}, |\mathrm{center}Q|)}$$

for all cubes Q, where

$$M_Q\Big(\frac{1}{p}\Big) = \frac{1}{|Q|}\int\limits_Q \left|\frac{1}{p(x)} - \frac{1}{p_Q}\right| dx, \quad \frac{1}{p_Q} = \frac{1}{|Q|}\int\limits_Q \frac{1}{p(x)}dx,$$

then for some large α, $\alpha + p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Note also that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ then

$$M_Q\Big(\frac{1}{p}\Big) \leq \frac{c}{\log(e + \max(|Q|, |Q|^{-1}))} \tag{2.1}$$

(see [5]). Let $p(\cdot)$ exponent is from theorem 1.3. For any cube $Q \subset \mathbb{R}^2$, $Q = I \times J$ we have

$$M_Q\Big(\frac{1}{p}\Big) = \frac{1}{|Q|}\int\limits_Q \left|\frac{1}{p(x,y)} - \frac{1}{p_Q}\right| dxdy = \frac{1}{|I|}\int\limits_I \left|\frac{1}{\widetilde{p}(x)} - \frac{1}{\widetilde{p}_Q}\right| dx.$$

It is not difficult to show that

$$M_I\Big(\frac{1}{\widetilde{p}}\Big) \leq \frac{c}{\log(e + \max(|I|, |I|^{-1}))}$$

and for exponent $p(\cdot)$ we have (2.1) estimate, but for any $\alpha > 0$ $\alpha + p(\cdot)\overline{\in}\mathcal{P}(\mathbb{R}^n)$.

Note that if for exponent $p(\cdot)$ we have (2.1) estimate and it is constant outside some ball, then $\alpha + p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for some $\alpha > 0$ (see [5]).

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