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EQUIVALENT CONDITIONS FOR HARDY INEQUALITIES

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Abstract. A result is proved which gives the equivalence of five higher dimensional integral conditions. As a consequence, new characterizations for higher dimensional Hardy inequalities are obtained.

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1. Introduction

The celebrity Hardy inequality proved by G. H. Hardy in 1920 [4], in its modern form, reads as

$$\left(\int_a^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}}, \tag{1.1}$$

where $-\infty \le a < b \le \infty$, $1 < p$, $q < \infty$, u, v are weight functions and f is a non-negative measurable function defined on (a, b). In this paper, we are concerned with the case $p \le q$. Many different conditions are available proved by different people at different times characterizing the inequality (1.1). In particular, we mention the names of Muckenhoupt [10], Kokilashvili [5], Persson and Stepanov [12], Wedestig [13]. Since all these conditions are characterizations of (1.1), they all are equivalent, however, via (1.1). The up to date development of Hardy inequality has been described in the monographs [6, 7, 9, 11].

In [2], Gogatishvili, Kufner, Persson and Wedestig proved a result which gives, as a special case, the equivalence of the conditions under discussion without involving the inequality (1.1). Precisely, the following was proved.

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Theorem A. *For $-\infty \le a < b \le \infty$, α, β and s positive numbers and f, g measurable functions positive a.e. in (a,b), let*

$$F(x) := \int_x^b f(t)dt, \quad G(x) := \int_a^x g(t)dt$$

and

$$A_1(x;\alpha,\beta) := F^\alpha(x) G^\beta(x),$$
$$A_2(x;\alpha,\beta,s) := \left(\int_x^b f(t) G^{\frac{\beta-s}{\alpha}}(t)dt\right)^\alpha G^s(x),$$
$$A_3(x;\alpha,\beta,s) := \left(\int_a^x g(t) F^{\frac{\alpha-s}{\alpha}}(t)dt\right)^\beta F^s(x),$$
$$A_4(x;\alpha,\beta,s) := \left(\int_a^x f(t) G^{\frac{\beta+s}{\alpha}}(t)dt\right)^\alpha G^{-s}(x),$$
$$A_5(x;\alpha,\beta,s) := \left(\int_x^b g(t) F^{\frac{\alpha+s}{\alpha}}(t)dt\right)^\beta F^{-s}(x).$$

The numbers $A_1 := \sup_{a<x<b} A_1(x;\alpha,\beta)$ and $A_i := \sup_{a<x<b} A_i(x;\alpha,\beta,s)$ $(i = 2,3,4,5)$ are mutually equivalent. The constants in the equivalence relations can depend on α, β and s.

The aim of this paper is to prove the higher dimensional analogue of Theorem A which would give, as a special case, the equivalent conditions characterizing the higher dimensional version of the inequality (1.1). Also, this theorem would yield the equivalent conditions for the higher dimensional version of the conjugate of (1.1).

By weight functions on $\mathbb{R}^n$, we shall mean positive locally integrable functions on $\mathbb{R}^n$. The symbol B_x denotes a ball in $\mathbb{R}^n$ with radius $|x|$, $x \in \mathbb{R}^n$ and centered at origin. By $\sum_{n-1}$, we shall denote the surface of B_1, i.e., unit ball in $\mathbb{R}^n$. By $A \approx B$, we shall mean that A and B are equivalent, i.e., there exist constants c_1 and c_2 such that $A \le c_1 B$ and $B \le c_2 A$.

2. Equivalence Result

Our main result, giving equivalence of several quantities, is the following:

Theorem 1. *Let α, β, s be positive real numbers and f, g be measurable functions positive a.e. in $\mathbb{R}^n$.*

Denote

$$F(x) := \int\limits_{\mathbb{R}^n \setminus B_x} f(y)dy, \quad G(x) = \int\limits_{B_x} g(y)dy\,.$$

The following quantities are equivalent:

(i) $\mathcal{B}_1 := \sup\limits_{|x|>0} B_1(x;\alpha,\beta) := \sup\limits_{|x|>0} F^{\alpha}(x)G^{\beta}(x)$,

(ii) $\mathcal{B}_2 := \sup\limits_{|x|>0} B_2(x;\alpha,\beta,s) := \sup\limits_{|x|>0} \left(\int\limits_{\mathbb{R}^n \setminus B_x} f(t)G^{\frac{\beta-s}{\alpha}}(t)dt \right)^{\alpha} G^{s}(x)$,

(iii) $\mathcal{B}_3 := \sup\limits_{|x|>0} B_3(x;\alpha,\beta,s) := \sup\limits_{|x|>0} \left(\int\limits_{B_x} g(t)F^{\frac{\alpha-s}{\beta}}(t)dt \right)^{\beta} F^{s}(x)$,

(iv) $\mathcal{B}_4 := \sup\limits_{|x|>0} B_4(x;\alpha,\beta,s) := \sup\limits_{|x|>0} \left(\int\limits_{B_x} f(t)G^{\frac{\beta+s}{\alpha}}(t)dt \right)^{\alpha} G^{-s}(x)$,

(v) $\mathcal{B}_5 := \sup\limits_{|x|>0} B_5(x;\alpha,\beta,s) := \sup\limits_{|x|>0} \left(\int\limits_{\mathbb{R}^n \setminus B_x} g(t)F^{\frac{\alpha+s}{\beta}}(t)dt \right)^{\beta} F^{-s}(x)$.

Proof. We shall show that the quantity $\mathcal{B}_1$, is equivalent to $\mathcal{B}_i$, $i = 2, \dots, 5$.

$\mathcal{B}_1 \approx \mathcal{B}_2$

First consider the case when$s \le \beta$.

As $\frac{\beta-s}{\alpha} \ge 0$ and therefore for $|z| > |x|$, we have

$$G^{\frac{\beta-s}{\alpha}}(z) \ge G^{\frac{\beta-s}{\alpha}}(x), \tag{2.1}$$

since G is increasing. Therefore

$$\begin{aligned} B_2(x;\alpha,\beta,s) &= \left(\int\limits_{R^n \setminus B_x} f(y)G^{\frac{\beta-s}{\alpha}}(y)dy \right)^{\alpha} G^{s}(x) \ge \\ &\ge \left(\int\limits_{\mathbb{R}^n \setminus B_x} f(y)dy \right)^{\alpha} G^{\beta}(x) = F^{\alpha}(x)G^{\beta}(x)\,, \end{aligned}$$

and so

$$\mathcal{B}_2 \ge \mathcal{B}_1\,. \tag{2.2}$$

On the other hand if we write

$$W(x) = \int\limits_{\mathbb{R}^n \setminus B_x} f(y) G^{\frac{\beta-s}{\alpha}}(y) dy =$$

$$= \int\limits_{|x|}^{\infty} \int\limits_{\sum_{n-1}} t^{n-1} f(t\tau) \tilde{G}_n^{\frac{\beta-s}{\alpha}}(t) dt d\tau = \int\limits_{|x|}^{\infty} \tilde{W}(t) dt = \tilde{W}_n(|x|) \tag{2.3}$$

we have

$$G^s(x) W^{\alpha}(x) = G^s(x) \Bigg(\int\limits_{\mathbb{R}^n \setminus B_x} f(y) G^{\frac{\beta-s}{\alpha}}(y) F^{\frac{\beta-s}{\beta}}(y) F^{\frac{s-\beta}{\beta}}(y) dy \Bigg)^{\alpha} =$$

$$= G^s(x) \Bigg(\int\limits_{|x|}^{\infty} \int\limits_{\sum_{n-1}} t^{n-1} f(t\tau) \Bigg(\int\limits_{0}^{t} \int\limits_{\sum_{n-1}} r^{n-1} g(r\rho) dr d\rho \Bigg)^{\frac{\beta-s}{\alpha}} \times$$

$$\times \tilde{F}_n^{\frac{\beta-s}{\beta}}(t) \tilde{F}_n^{\frac{s-\beta}{\beta}}(t) dt d\tau \Bigg)^{\alpha} =$$

$$= G^s(x) \Bigg(\int\limits_{|x|}^{\infty} \int\limits_{\sum_{n-1}} t^{n-1} f(t\tau) \tilde{G}_n^{\frac{\beta-s}{\alpha}}(t) \tilde{F}_n^{\frac{\beta-s}{\alpha}}(t) \tilde{F}_n^{\frac{s-\beta}{\beta}}(t) dt d\tau \Bigg)^{\alpha},$$

where

$$\tilde{F}_n(t) = \int\limits_{t}^{\infty} \int\limits_{\sum_{n-1}} s^{n-1} f(s\tau) ds d\tau = \int\limits_{t}^{\infty} \tilde{F}(s) ds \tag{2.4}$$

and

$$\tilde{G}_n(t) = \int\limits_{0}^{t} \int\limits_{\sum_{n-1}} r^{n-1} g(r\rho) dr d\rho = \int\limits_{0}^{t} \tilde{G}(r) dr. \tag{2.5}$$

Thus, using (2.5) we have

$$G^s(x) W^{\alpha}(x) \le \left(\sup_{t > |x|} \tilde{G}_n^{\frac{\beta-s}{\alpha}}(t) \tilde{F}_n^{\frac{\beta-s}{\alpha}}(t) \right)^{\alpha} \times$$

$$\times\ G^s(x) \Bigg(\int\limits_{|x|}^{\infty} \tilde{F}(t) \Bigg(\int\limits_{t}^{\infty} \tilde{F}(y) dy \Bigg)^{\frac{s-\beta}{\beta}} dt \Bigg)^{\alpha} =$$

$$= \left(\sup_{t > |x|} \tilde{G}_n^{\beta}(t) \tilde{F}_n^{\alpha}(t) \right)^{\frac{\beta-s}{\beta}} \left(\frac{\beta}{s} \right)^{\alpha} G^s(x) F^{\frac{\alpha s}{\beta}}(x) \le$$

$$\le \left(\sup_{|y| > |x|} G^{\beta}(y) F^{\alpha}(y) \right)^{\frac{\beta-s}{\beta}} \left(\frac{\beta}{s} \right)^{\alpha} \left(\sup_{|x| > 0} G^{\beta}(x) F^{\alpha}(x) \right)^{\frac{s}{\beta}} \le$$

$$\le \left(\frac{\beta}{s}\right)^{\alpha} \sup_{|x|>0} B_1(x;\alpha,\beta),$$

which gives that

$$\mathcal{B}_2 \le \left(\frac{\beta}{s}\right)^{\alpha} \mathcal{B}_1$$

and combining with (2.2) yields that $\mathcal{B}_1 \approx \mathcal{B}_2$ in the case $s \le \beta$.

Next, we consider the case when $s > \beta$. Using W from above and (2.3), we consider ,

$$F^{\alpha}(x)G^{\beta}(x) = G^{\beta}(x)\Bigg(\int\limits_{\mathbb{R}^n\setminus B_x} f(y)G^{\frac{\beta-s}{\alpha}}(y)G^{\frac{s-\beta}{\alpha}}(y)W^{\frac{s-\beta}{\alpha}}(y)W^{\frac{\beta-s}{\alpha}}(y)dy\Bigg)^{\alpha} =$$

$$= G^{\beta}(x)\Bigg(\int\limits_{|x|}^{\infty}\int\limits_{\sum_{n-1}} t^{n-1}f(t\tau)\tilde{G}_n^{\frac{\beta-s}{\alpha}}(t)\tilde{G}_n^{\frac{s-\beta}{\alpha}}(t)\tilde{W}_n^{\frac{s-\beta}{s}}(t)\tilde{W}_n^{\frac{\beta-s}{s}}(t)dtd\tau\Bigg)^{\alpha} \le$$

$$\le \left(\sup_{t>|x|} \tilde{G}_n^{s-\beta}(t)\tilde{W}_n^{\frac{(s-\beta)\alpha}{s}}(t)\right)G^{\beta}(x)\Bigg\{\int\limits_{|x|}^{\infty}\tilde{W}(t)\Bigg(\int\limits_{t}^{\infty}\tilde{W}(s)ds\Bigg)^{\frac{\beta-s}{s}}dt\Bigg\}^{\alpha} =$$

$$= \left(\sup_{|y|>|x|} G^{s}(y)W^{\alpha}(y)\right)^{\frac{s-\beta}{s}}\left(\frac{s}{\beta}\right)^{\alpha}G^{\beta}(x)W^{\frac{\beta\alpha}{s}}(x) \le$$

$$\le \left(\frac{s}{\beta}\right)^{\alpha}\left(\sup_{|y|>|x|} G^{s}(y)W^{\alpha}(y)\right)^{1-\frac{\beta}{s}}\left(\sup_{|x|>0} G^{s}(x)W^{\alpha}(x)\right)^{\frac{\beta}{s}} \le$$

$$\le \left(\frac{s}{\beta}\right)^{\alpha}\left(\sup_{|x|>0} B_2(x;\alpha,\beta,s)\right),$$

which gives that

$$\mathcal{B}_1 \le \left(\frac{s}{\beta}\right)^{\alpha}\mathcal{B}_2\,. \tag{2.6}$$

On the other hand, since $s > \beta$. The inequality (2.1) holds in the reverse direction and therefore

$$B_2(x;\alpha,\beta,s) = G^{s}(x)\Bigg(\int\limits_{\mathbb{R}^n\setminus B_x} f(y)G^{\frac{\beta-s}{\alpha}}(y)dy\Bigg)^{\alpha} \le$$

$$\le G^{s}(x)\Bigg(\int\limits_{\mathbb{R}^n\setminus B_x} f(y)dy\Bigg)^{\alpha}G^{\beta-s}(x) = F^{\alpha}(x)G^{\beta}(x)$$

i.e.,

$$\mathcal{B}_2 \le \mathcal{B}_1$$

which when combined with (2.6), yields that $\mathcal{B}_1 \approx \mathcal{B}_2$ in the case $s > \beta$ too. Thus the equivalence of $\mathcal{B}_1$ and $\mathcal{B}_2$ is proved completely.

$\mathcal{B}_1 \approx \mathcal{B}_3$

It follows similarly as the case of $\mathcal{B}_1 \approx \mathcal{B}_2$. One needs to change the roles of F and G.

$\mathcal{B}_1 \approx \mathcal{B}_4$

Let

$$W_0(x) = \int\limits_{B_x} f(y) G^{\frac{\beta+s}{\alpha}}(y)dy = \\ = \int\limits_0^{|x|} \int\limits_{\sum_{n-1}} t^{n-1} f(t\tau) G^{\frac{\beta+s}{\alpha}}(t\tau) dt d\tau = \int\limits_0^{|x|} \tilde{W}_0(t), \tag{2.7}$$

where

$$\tilde{W}_0(t) = \int\limits_{\sum_{n-1}} t^{n-1} f(t\tau) G^{\frac{\beta+s}{\alpha}}(t\tau) d\tau,$$
$$B_4(x;\alpha,\beta,s) = G^{-s}(x) W_0^{\alpha}(x),$$

and, using (2.5) and (2.7), we have

$$B_1(x;\alpha,\beta) = G^{\beta}(x)\Bigg(\int\limits_{\mathbb{R}^n\setminus B_x} f(x) G^{\frac{\beta+s}{\alpha}}(x) G^{-\frac{\beta+s}{\alpha}}(x)dx\Bigg)^{\alpha} =$$
$$= G^{\beta}(x)\Bigg(\int\limits_{|x|}^{\infty}\int\limits_{\sum_{n-1}} t^{n-1} f(t\tau) G^{\frac{\beta+s}{\alpha}}(t\tau) G^{-(\frac{\beta+s}{\alpha})}(t\tau) dt d\tau\Bigg)^{\alpha} =$$
$$= G^{\beta}(x)\Bigg(\int\limits_{|x|}^{\infty} \tilde{G}_n^{-(\frac{\beta+s}{\alpha})}(t) d\Bigg(\int\limits_0^t \tilde{W}_0(s)ds\Bigg)dt\Bigg)^{\alpha} \le$$
$$\le G^{\beta}(x)\Bigg(\tilde{G}_n^{-(\frac{\beta+s}{\alpha})}(\infty) W_0(\infty) + \frac{\beta+s}{\alpha}\times \\ \times \int\limits_{|x|}^{\infty} \tilde{G}(t)(\tilde{G}_n(t))^{-(\frac{\beta+s}{\alpha})-1} W_0(t)dt\Bigg)^{\alpha} \le$$
$$\le G^{\beta}(x) \sup_{|y|>|x|} G^{-s}(y) W_0^{\alpha}(y)\Bigg(\tilde{G}_n^{-\frac{\beta}{\alpha}}(\infty) + \frac{\beta+s}{\alpha}\times$$

$$\times \int\limits_{|x|}^{\infty} \tilde{G}_n^{-\frac{\beta}{\alpha}-1}(t) d\bigg(\int\limits_0^t \tilde{G}(s)ds \bigg) dt \bigg)^{\alpha} \leq$$

$$\leq G^{\beta}(x) \sup_{|y|>0} B_4(y;\alpha,\beta,s) \times$$

$$\times \left(\tilde{G}_n^{-\frac{\beta}{\alpha}}(\infty) + \frac{\beta+s}{\alpha} (\tilde{G}_n^{-\frac{\beta}{\alpha}}(|x|) - \tilde{G}_n^{-\frac{\beta}{\alpha}}(\infty)) \right)^{\alpha} =$$

$$= \sup_{|y|>0} B_4(y;\alpha,\beta,s) \left[\frac{\beta+s}{\alpha} + \left(1 - \frac{\beta+s}{\alpha}\right) \left(\frac{G(x)}{G(\infty)} \right)^{\frac{\beta}{\alpha}} \right]^{\alpha} \leq$$

$$\leq \left(1 + \frac{s}{\beta}\right)^{\alpha} \sup_{|y|>0} B_4(y;\alpha,\beta,s).$$

Thus

$$\sup_{|x|>0} B_1(x;\alpha,\beta) \leq \left(1 + \frac{s}{\beta}\right)^{\alpha} \sup_{|x|>0} B_4(x;\alpha,\beta,s).$$

To prove the opposite inequality, we assumes that

$$\sup_{|x|>0} B_1(x;\alpha,\beta,s) < \infty,$$

$$B_4(x;\alpha,\beta,s) = G^{-s}(x) \bigg(\int\limits_{B_x} G^{\frac{\beta+s}{\alpha}}(x) f(x) dx \bigg)^{\alpha} =$$

$$= G^{-s}(x) \bigg(\int\limits_0^{|x|} \tilde{G}_n^{\frac{\beta+s}{\alpha}}(t) d\bigg(- \int\limits_t^{\infty} \tilde{F}(s) ds \bigg) dt \bigg)^{\alpha} =$$

$$= G^{-s}(x) \bigg(\tilde{G}_n^{\frac{\beta+s}{\alpha}}(t) \tilde{F}_n(t)\big|_{|x|}^{0} + \frac{\beta+s}{\alpha} \times$$

$$\times \int\limits_0^{|x|} \tilde{F}_n(t) \tilde{G}_n^{\frac{\beta+s}{\alpha}-1}(t) d\bigg(\int\limits_0^t \tilde{G}(r) dr \bigg) dt \bigg)^{\alpha} \leq$$

$$\leq G^{-s}(x) \bigg(\sup_{0<t<|x|} \tilde{G}_n^{\beta}(t) \tilde{F}_n^{\alpha}(t) \bigg) \times$$

$$\times \bigg(\frac{\beta+s}{\alpha} \int\limits_0^{|x|} \tilde{G}_n^{\frac{s}{\alpha}-1}(t) d\bigg(\int\limits_0^t \tilde{G}(r) dr \bigg) dt \bigg)^{\alpha} \leq$$

$$\leq \left(\frac{\beta+s}{\alpha} \right)^{\alpha} \sup_{|y|>0} G^{\beta}(y) F^{\alpha}(y) G^{-s}(x) \bigg(\frac{\alpha}{s} G^{\frac{s}{\alpha}}(x) \bigg)^{\alpha} =$$

$$= \left(\frac{\beta+s}{\alpha}\right)^{\alpha} \sup_{|x|>0} B_1(x;\alpha,\beta).$$

Hence, we have

$$\sup_{|x|>0} B_4(x;\alpha,\beta,s) \le \left(1+\frac{s}{\beta}\right)^{\alpha} \sup_{|x|>0} B_1(x;\alpha,\beta).$$

It follows

$$\mathcal{B}_1 \approx \mathcal{B}_4.$$

$\mathcal{B}_1 \approx \mathcal{B}_5$

It follows similarly as the case of $\mathcal{B}_1 \approx \mathcal{B}_4$. One needs to change the roles of F and G. □

3. Equivalence Conditions of Hardy Inequalities

In this section, first we give several conditions all are necessary and sufficient for the higher dimensional Hardy inequality.

Theorem 2. *Let $1 < p \le q < \infty$, $s \in (0,\infty)$ and u, v be weight functions defined on $\mathbb{R}^n$. Let*

$$U(x) = \int_{\mathbb{R}^n\setminus B_x} u(y)dy \quad \text{and} \quad V(x) = \int_{B_x} v^{1-p'}(y)dy.$$

Then the inequality

$$\left(\int_{\mathbb{R}^n}\left(\int_{B_x} f(y)dy\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left\{\int_{\mathbb{R}^n} f^p(x)v(x)dx\right\}^{\frac{1}{p}} \tag{3.1}$$

holds for all non-negative measurable functions f if and only if any one of the following equivalent conditions holds

(i) $\mathcal{D}_1 := \sup_{|x|>0}\left\{U^{\frac{1}{q}}(t)V^{\frac{1}{p'}}(t)\right\} < \infty$,

(ii) $\mathcal{D}_2 := \sup_{|x|>0}\left(\int_{\mathbb{R}^n\setminus B_x} u(t)V^{q(\frac{1}{p'}-s)}(t)dt\right)^{\frac{1}{q}} V^s(x) < \infty$,

(iii) $\mathcal{D}_3 := \sup_{|x|>0}\left(\int_{B_x} u(t)V^{q(\frac{1}{p'}+s)}(t)dt\right)^{\frac{1}{q}} V^{-s}(x) < \infty$,

(iv) $\mathcal{D}_4 := \sup_{|x|>0}\left(\int_{B_x} v^{1-p'}(t)U^{p'(\frac{1}{q}-s)}(t)dt\right)^{\frac{1}{p'}} U^s(x) < \infty$,

(v) $\mathcal{D}_5 := \sup\limits_{|x|>0} \left(\int\limits_{\mathbb{R}^n \setminus B_x} v^{1-p'}(t) U^{p'(\frac{1}{q}+s)}(t)dt \right)^{\frac{1}{p'}} U^{-s}(x) < \infty.$

Proof. In Theorem 1, if we take $f(x) = u(x)$, $g(x) = v^{1-p'}(x)$, $\alpha = \frac{1}{q}$, $\beta = \frac{1}{p'}$, then we find that

$$\mathcal{D}_1 \approx \mathcal{D}_2 \approx \mathcal{D}_3 \approx \mathcal{D}_3 \approx \mathcal{D}_5 .$$

But it is known, see [1] that $\sup \mathcal{D}_1 < \infty$ is a necessary and sufficient condition for the inequality (3.1) to hold. Hence the assertion follows. $\square$

Now, we give several equivalent conditions corresponding to the higher dimensional conjugate Hardy operator.

Theorem 3. *Let $1 < p \le q < \infty$, $s \in (0, \infty)$ and u, v be weight functions on $\mathbb{R}^n$. Let*

$$U(x) = \int\limits_{B_x} u(y)dy \quad \text{and} \quad V(x) = \int\limits_{\mathbb{R}^n \setminus B_x} v^{1-p'}(y)dy.$$

Then the inequality

$$\left\{ \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n \setminus B_x} f(y)dy \right)^q u(x)dx \right\}^q \le C \left\{ \int\limits_{\mathbb{R}^n} f^p(x)v(x)dx \right\}^{\frac{1}{p}} \tag{3.2}$$

holds for all non-negative measurable functions f if and only if any one of the following equivalent conditions holds

(i) $\mathcal{D}_1^* := \sup\limits_{|x|>0} \left\{ U^{\frac{1}{q}}(t) V^{\frac{1}{p'}}(t) \right\} < \infty$

(ii) $\mathcal{D}_2^* := \sup\limits_{|x|>0} \left\{ \int\limits_{B_x} u(t) V^{q(\frac{1}{p'}-s)}(t)dt \right\}^{\frac{1}{q}} V^s(x) < \infty$

(iii) $\mathcal{D}_3^* := \sup\limits_{|x|>0} \left(\int\limits_{\mathbb{R}^n \setminus B_x} u(t) V^{q(\frac{1}{p'}+s)}(t)dt \right)^{\frac{1}{q}} V^{-s}(x) < \infty$

(iv) $\mathcal{D}_4^* := \sup\limits_{|x|>0} \left\{ \int\limits_{\mathbb{R}^n \setminus B_x} v^{1-p'}(t) U^{p'(\frac{1}{q}-s)}(t)dt \right\}^{\frac{1}{p'}} U^s(x) < \infty$

(v) $\mathcal{D}_5^* := \sup\limits_{|x|>0} \left\{ \int\limits_{B_x} v^{1-p'}(t) U^{p'(\frac{1}{q}+s)}(t)dt \right\}^{\frac{1}{p'}-s} U^{-s}(x) < \infty$

Proof. In Theorem 1, if we take $f(x) = v^{1-p'}(x)$, $g(x) = u(x)$, $\alpha = \frac{1}{p'}$, $\beta = \frac{1}{q}$, we find that

$$\mathcal{D}_1^* \approx \mathcal{D}_2^* \approx \mathcal{D}_3^* \approx \mathcal{D}_4^* \approx \mathcal{D}_5^* .$$

But it is known, see [1] that $\sup \mathcal{D}_1^* < \infty$ is a necessary and sufficient condition for the inequality (3.2) to hold. Hence the assertion follows. □

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