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UPPER BOUND AND LOWER BOUND FOR INTEGRAL OPERATORS ON WEIGHTED SPACES

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Abstract. The purpose of this paper is to study the problem of finding an upper bound and a lower bound of norms of certain kernel integral operators on weighted spaces. In fact, we consider Averaging, Copson and Hilbert operators on weighted Lorentz space $\Lambda(w,p)$. Also, we study such constants on conjugate space $M(w)$, of $\Lambda(w,1)$ with decreasing non-negative weight functions.

რეზიუმე. ნაშრომში გამოკვლეულია გარკვეული გულიანი ინტეგრალური ოპერატორების ნორმები წონიან სივრცეებში. შესწავლილია გასაშუალების, კოპსონისა და ჰილბერტის ოპერატორები წონიან ლორენცის $\Lambda(w,p)$ სივრცეებში. ანალოგიური პრობლემა გამოკვლეულია აგრეთვე, $\Lambda(w,1)$ სივრცის შეუღლებული $M(w)$ სივრცისათვის კლებადი არაუარყოფითი წონითი ფუნქციების შემთხვევაში.

1. Introduction

Suppose that $1 \le p < \infty$ and $w = w(x)$ is a decreasing non-negative function on $(0,1)$. We assume that $W(x) = \int_0^x w(t)dt$ is finite for each x and $\lim_{x\to\infty} w(x) = 0$ and also $\int_0^\infty w(x)dx = \infty$. Also the Lorentz space $\Lambda(w,p)$ is defined as follows:

$$\Lambda(w,p) = \left\{ f : \int_0^\infty w(x) f^*(x)^p dx < \infty \right\},$$

where f is real valued function on $(0,\infty)$ and $f^*(x)$ is the decreasing rearrangement of $|f(x)|$. Moreover, we define norm of $\Lambda(w,p)$ by

$$\|f\|_{\Lambda(w,p)} = \left(\int_0^\infty w(x) f^*(x)^p dx \right)^{1/p}.$$

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We write $\Lambda(w)$ instead of $\Lambda(w,1)$ and $\|\cdot\|_{\Lambda(w)}$ instead of $\|\cdot\|_{\Lambda(w,1)}$. The conjugate space of $\Lambda(w)$ is $M(w)$, see [6]. In fact,

$$M(w)=\left\{f:\sup_{E}\frac{\int_E|f(x)|dx}{\int_0^{meas\,E}w(x)dx}<\infty\right\},$$

with $\int_0^{meas\,E}w(x)dx>0$ and its norm is defined by

$$\|f\|_{M(w)}=\sup_{E}\frac{\int_E|f(x)|dx}{\int_0^{meas\,E}w(x)dx}\,.$$

Here E signifies an arbitrary measurable subset of $(0,\infty)$. If $f\in M(w)$ is non-negative decreasing function, then

$$\|f\|_{M(w)}=\sup_{l}\frac{\int_0^l|f(x)|dx}{\int_0^l w(x)dx}\,.$$

We write $\|B\|_{\Lambda(v,w)}$ for the norm of B as an operator from $\Lambda(v)$ into $\Lambda(w)$, and $\|B\|_{\Lambda(w)}$ for the norm of B as an operator from $\Lambda(w)$ into itself. Also, we write $\|B\|_{M(v,w)}$ for the norm of B as an operator from $M(v)$ into $M(w)$, and $\|B\|_{M(w)}$ for the norm of B as an operator from $M(w)$ into itself. Our second concern is to settle lower bounds of the form

$$\|B\|_{\Lambda(w)}\geq L\|f\|_{\Lambda(v)},\quad \big(\|Bf\|_{M(w)}\geq L\|f\|_{M(v)}\big),$$

valid for every non-negative decreasing function f (In fact, we restrict ourselves to monotone decreasing functions) and L is a constant not depending on f. We seek the largest possible value of the constant L, and denote the best lower bound by $L_{\Lambda(v,w)}$ for operators from $\Lambda(v)$ into $\Lambda(w)$. Also it is denoted by $L_{\Lambda(w)}$ on $\Lambda(w)$. Moreover, we denote lower bounds of operators from $M(v)$ into $M(w)$ by $L_{M(v,w)}$ and $L_{M(w)}$ for operators on $M(w)$.

2. Integral operators on $\Lambda(w)$

We study the upper and lower bounds of certain operator from $\Lambda(v)$ into $\Lambda(w)$ which is recently considered in [1]–[5] on weighted sequence spaces for certain matrix operators such as Cesaro, Copson, Hausdorff and Hilbert operators.

We begin with some definitions and lemmas which will be useful in the sequel.

Lemma 2.1. *Suppose that f, g are non-negative functions on $(0,\infty)$ and g is deceasing on $(0,\infty)$. Assume that $\lim_{x\to\infty}g(x)=0$. Then*

$$\int_0^\infty f(x)\,g(x)\,dx=\int_0^\infty\left(\int_0^x f(t)dt\right)d(-g(x)).$$

The proof is trivial, therefore it is omitted.

Lemma 2.2. *Let f, g and w be non-negative functions on $(0,\infty)$. If w is decreasing and $\lim_{x\to\infty} w(x) = 0$, and for each $0 < x < \infty$, we have*

$$\int_0^x f(t)dt \le \int_0^x g(t)dt,$$

then

$$\int_0^\infty w(t)\, f(t)\, dt \le \int_0^\infty w(t)\, g(t)\, dt.$$

Proof. Applying Lemma 2.1, we have the statement. □

Let B be an integral operator which is defined by

$$(Bf)(x) = \int_0^\infty b(x,y)\, f(y)\, dy,$$

where $b(x,y)$ is a non-negative and measurable function. We define

$$r(x,y) = \int_0^y b(x,t)\, dt, \quad c(x,y) = \int_0^x b(t,y)\, dt.$$

Consider the following conditions:

(1) For each x and y (x, y are non-negative), $b(x,y) \ge 0$.
(2) $r(x,y)$ decreases with x for each y.
(2*) $b(x,y)$ decreases with x for each y.
(3) $c(x,y)$ decreases with y for each x.
(3*) $b(x,y)$ decreases with y for each x.
(4) For each x, $\lim_{y\to\infty} b(x,y) = 0$.

Condition (1) implies that for each $x \ge 0$, $|Bf(x)| \le (B|f|)(x)$ and hence non-negative functions are sufficient for determine norm of B. Condition (2*) is stronger than (2), and (3*) is also stronger than (3). Condition (2*) clearly implies that Bf is decreasing for any non-negative function f, while (2) implies that Bf is decreasing for decreasing, non-negative function $f \in \Lambda(w)$, since by Lemma 2.1

$$Bf(x) = \int_0^\infty b(x,y)\, f(y)\, dy = \int_0^\infty r(x,y)\, d(-f(y)).$$

The following statement deduce that non-negative decreasing functions are sufficient to be determined norm of integral operators on $\Lambda(w)$.

Proposition 2.1. *Suppose that B is an integral operator with conditions* (1), (3) *and* (4). *Then*

$$\|Bf\|_{\Lambda(w)} \le \|Bf^*\|_{\Lambda(w)}$$

for all non-negative functions f belong to $\Lambda(w)$.

Proof. By Lemma 2.1, for each $0 < l < \infty$, we have

$$\int_0^l B\,f(x)\,dx = \int_0^l \int_0^\infty b(x,y)\,f(y)\,dy\,dx = \int_0^\infty c(l,y)\,f(y)\,dy =$$
$$= \int_0^\infty \Big(\int_0^y f(t)\,dt \Big) d(-c(l,y)).$$

Similarly, we deduce that

$$\int_0^l B\,f^*(x)\,dx = \int_0^l \int_0^\infty b(x,y)\,f^*(y)\,dy\,dx = \int_0^\infty c(l,y)\,f^*(y)\,dy =$$
$$= \int_0^\infty \Big(\int_0^y f^*(t)\,dt \Big) d(-c(l,y)).$$

Since $\int_0^y f(t)dt \le \int_0^y f^*(t)dt$ for all y, we have $\int_0^l Bf(x)dx \le \int_0^l Bf^*(x)dx$. Then Lemma 2.2 implies that $\|Bf\|_{\Lambda(w)} \le \|Bf^*\|_{\Lambda(w)}$, and so we have the statement. □

Theorem 2.1. *Suppose that B satisfies conditions* (1), (2), (3) *and* (4). *Let $u(y) = \int_0^\infty w(x)b(x,y)dx$, $U(y) = \int_0^y u(t)dt$ and $V(y) = \int_0^y v(t)dt$. If*

$$M = \sup_{y>0} \frac{U(y)}{V(y)} < \infty,$$

then B is a bounded integral operator from $\Lambda(v)$ into $\Lambda(w)$ and

$$\|B\|_{Lb(v,w)} = \sup_{y>0} \frac{U(y)}{V(y)}\,, \quad L_{\Lambda(v,w)}(B) = \inf_{y>0} \frac{U(y)}{V(y)}\,.$$

Proof. Denote the stated infimum by m and let $f \in \Lambda(v)$ be a decreasing non-negative function.

Since $\lim_{x\to\infty} f(x) = 0$, applying Lemma 2.1 we have

$$\|Bf\|_{\Lambda(w)} = \int_0^\infty w(x) \Big(\int_0^\infty b(x,y)\,f(y)\,dy \Big) dx = \int_0^\infty f(y)\,u(y)\,dy =$$
$$= \int_0^\infty U(y)\,d(-f(y)).$$

Thus

$$m\int_0^\infty V(y)\,d(-f(y)) \le \|Bf\|_{\Lambda(w)} \le M\int_0^\infty V(y)\,d(-f(y)).$$

Since

$$\|f\|_{\Lambda(v)} = \int_0^\infty v(y)\,f(y)\,dy = \int_0^\infty V(y)\,d(-f(y)),$$

we have

$$m\|f\|_{\Lambda(v)} \le \|Bf\|_{\Lambda(w)} \le M\|f\|_{\Lambda(v)}.$$

Hence

$$\|B\|_{\Lambda(v,w)} \le M, \quad L_{\Lambda(v,w)}(B) \ge m.$$

Further, for each $y > 0$ suppose that f is the characteristic function of $[0, y]$. Then

$$\|f\|_{\Lambda(v)} = V(y), \quad \|Bfk\|_{\Lambda(w)} = U(y).$$

Therefore

$$\|B\|_{\Lambda(v,w)} \ge M, \quad L_{\Lambda(v,w)}(B) \le m.$$

This complete the proof of the theorem. □

It is also essentially a smoother version of the above proof for the discrete case, see [1]. In the sequel we assume that the integral operator B satisfies conditions (1), (2), (3) and (4).

Proposition 2.2. *Suppose that $v(x)$ is an arbitrary weight function and $w(x) = 1/x^\alpha$, where $0 < \alpha < 1$ and $b(x, y)$ satisfies*

$$b(\lambda x, \lambda y) = \frac{1}{\lambda}\,b(x, y)$$

for all $x, y, \lambda > 0$. Then $U(y) = (1-\alpha)U(1)W(y)$ and

$$\|B\|_{\Lambda(v,w)} = (1-\alpha)\,U(1)\,\sup_{y>0}\frac{W(y)}{V(y)}\,,$$

$$L_{\Lambda(v,w)}(B) = (1-\alpha)\,U(1)\,\inf_{y>0}\frac{W(y)}{V(y)}\,.$$

Proof. We have

$$r(\lambda x, \lambda y) = \int_0^{\lambda y} b(\lambda x, t)\,dt = \int_0^{y} b(\lambda x, \lambda u)\,\lambda\,du = \int_0^{y} b(x, u)du = r(x, y).$$

Hence

$$U(y) = \int_0^\infty \frac{1}{x^\alpha}\,r(x, y)^p\,dx = \int_0^\infty \frac{1}{y^\alpha t^\alpha}\,r(yt, y)^p y\,dt = y^{1-\alpha}\int_0^\infty \frac{1}{t^\alpha}\,r(t, 1)^p\,dt,$$

so that $U(y) = (1-\alpha)U(1)W(y)$. Theorem 2.1 completes the proof. □

Corollary 2.1. *Suppose that* $V(x) = \frac{1}{1-\alpha}\frac{x^{1-\alpha}(1+x)}{2x+1}$ *and* $w(x) = 1/x^{\alpha}$, *where* $0 < \alpha < 1$ *and* $b(x, y)$ *satisfies*

$$b(\lambda x, \lambda y) = \frac{1}{\lambda}\, b(x, y)$$

for all $x, y, \lambda > 0$. *Then*

$$\|B\|_{\Lambda(v,w)} = 2(1-\alpha)\, U(1), \quad L_{\Lambda(v,w)}(B) = (1-\alpha)\, U(1).$$

Proof. Since $\frac{W(y)}{V(y)} = \frac{2y+1}{1+y}$ and it is an increasing function with respect y, and moreover, $\lim_{y\to\infty} \frac{2y+1}{1+y} = 2$, we deduce

$$\sup_{y>0} \frac{W(y)}{V(y)} = 2, \quad \inf_{y>0} \frac{W(y)}{V(y)} = 1. \qquad \square$$

Corollary 2.2 ([1], Proposition 3)**.** *Suppose that* $w(x) = 1/x^{\alpha}$, *where* $0 < \alpha < 1$ *and* $b(x, y)$ *satisfies*

$$b(\lambda x, \lambda y) = \frac{1}{\lambda}\, b(x, y)$$

for all $x, y, \lambda > 0$. *Then*

$$\|B\|_{\Lambda(w)} = L_{\Lambda(w)}(B) = (1-\alpha)\, U(1).$$

Proof. Applying Proposition 2.2 for $v(x) = w(x) = 1/x^{\alpha}$, we get the proof. $\square$

The Hilbert operator H is given by the kernel $b(x, y) = 1/(x + y)$, this kernel satisfies all conditions mentioned in Proposition 2.2 and so we obtain the following result.

Proposition 2.3. *Let* $V(x) = \frac{1}{1-\alpha}\frac{x^{1-\alpha}(1+x)}{2x+1}$ *and* $w(x) = 1/x^{\alpha}$ *where* $0 < \alpha < 1$ *and* $b(x, y) = 1/(x + y)$. *Then*

$$\|H\|_{\Lambda(v,w)} = 2(1-\alpha) \int_0^{\infty} \frac{1}{x^{\alpha}} \log\Big(1 + \frac{1}{x}\Big) dx,$$

$$L_{\Lambda(v,w)}(H) = (1-\alpha) \int_0^{\infty} \frac{1}{x^{\alpha}} \log\Big(1 + \frac{1}{x}\Big) dx.$$

Proof. We have

$$U(1) = \int_0^{\infty} \frac{1}{x^{\alpha}}\, r(x, 1)\, dx = \int_0^{\infty} \frac{1}{x^{\alpha}} \log\Big(1 + \frac{1}{x}\Big) dx.$$

Applying Corollary 2.1, we have the statement. $\square$

Also, if we have same weight function in above statement, we have

Proposition 2.4 ([1], Proposition 4)**.** . *Let* $w(x) = 1/x^\alpha$ *where* $0 < \alpha < 1$ *and* $a(x, y) = 1/(x + y)$. *Then*

$$\|H\|_{\Lambda(w)} = L_{\Lambda(w)}(H) = (1-\alpha)\int\limits_0^\infty \frac{1}{x^\alpha}\,\log\Big(1+\frac{1}{x}\Big)dx.$$

The averaging operator A is given by $(Af)(x) = \frac{1}{x}\int_0^x f(y)dy$, so that

$$b(x,y) = \begin{cases} 1/x & \text{for} \quad y \le x, \\ 0 & \text{for} \quad y > x. \end{cases}$$

This function satisfies all conditions mentioned in Proposition 2.2 and so we have the following result.

Proposition 2.5. *Suppose that* $V(x) = \frac{1}{1-\alpha}\,\frac{x^{1-\alpha}(1+x)}{2x+1}$ *and* $w(x) = 1/x^\alpha$, *where* $0 < \alpha < 1$ *and* A *is the averaging operator. Then*

$$\|A\|_{\Lambda(v,w)} = \frac{2}{\alpha}\,, \quad L_{v,w}(A) = \frac{1}{\alpha}\,.$$

Proof. We have

$$r(x,1) = \begin{cases} 1/x & \text{for} \quad x > 1; \\ 1 \quad \text{for} \quad x \le 1; \end{cases}$$

so that

$$U(1) = \int\limits_0^\infty \frac{1}{x^\alpha}\,r(x,1)\,dx = \frac{1}{\alpha(1-\alpha)}\,.$$

This completes the proof of the statement. □

In the same way, one can show the following result.

Proposition 2.6 ([1], Proposition 6)**.** *Let* $w(x) = 1/x^\alpha$, *where* $0 < \alpha < 1$ *and let* A *be the averaging operator. Then*

$$\|A\|_{\Lambda(w)} = L_{\Lambda(w)}(A) = \frac{1}{\alpha}\,.$$

The Copson operator C is given by $(Cf)(x) = \int_x^\infty \frac{f(y)}{y}dy$, so that

$$b(x,y) = \begin{cases} 1/y & \text{for} \quad x \le y, \\ 0 & \text{for} \quad x > y. \end{cases}$$

We define the 1-regularity constant of $w(x)$ to be

$$r_1(w) = \sup_{x>0}\frac{W(x)}{xw(x)}\,,$$

and say that $w = w(x)$ is 1-regular if this is finite.

Proposition 2.7. *If w is 1-regular, then the Copson operator C maps $\Lambda(w)$ into itself. Also we have*

$$\|C\|_{\Lambda(w)} \leq r_1(w).$$

Proof. Since

$$u(y) = \frac{W(y)}{y} \leq r_1(w)\, w(y) \quad (\forall\, y > 0),$$

then

$$U(y) = \int_0^y u(t)\, dt \leq \int_0^y r_1(w)\, w(t)\, dt = r_1(w)\, W(y).$$

Hence

$$\|C\|_{\Lambda(w)} = \sup_{y>0} \frac{U(y)}{W(y)} \leq r_1(w). \qquad \square$$

Proposition 2.8. *If*

$$\sup_{y>0} \frac{1}{W(y)} \int_0^y \frac{W(t)}{t}\, dt < \infty,$$

then the Copson operator C is a bounded operator from $\Lambda(w)$ into itself. Also we have

$$\|C\|_{\Lambda(w)} = \sup_{y>0} \frac{1}{W(y)} \int_0^y \frac{W(t)}{t}\, dt.$$

Proof. Since

$$u(y) = \int_0^y w(x)\, a(x, y)\, dx = \frac{W(y)}{y}\,,$$

then

$$\|C\|_w = \sup_{y>0} \frac{U(y)}{W(y)} = \sup_{y>0} \frac{1}{W(y)} \int_0^y \frac{W(t)}{t}\, dt. \qquad \square$$

Proposition 2.9. *Let C be the Copson operator, and $w(y) = 1/y^\alpha$, where $0 < \alpha < 1$. Then the Copson operator C is a bounded operator from $\Lambda(w)$ into itself and*

$$\|C\|_{\Lambda(w)} = \frac{1}{1-\alpha}\,.$$

Proof. With our standing notation,

$$\frac{u(y)}{w(y)} = \frac{W(y)}{y\, w(y)} = \frac{1}{1-\alpha}\,,$$

hence $\frac{U(y)}{W(y)} = \frac{1}{1-\alpha}$ and

$$\|C\|_{\Lambda(w)} = \frac{1}{1-\alpha}\,. \qquad \square$$

3. Integral operators on $M(w)$

In this section, we study the upper and lower bounds of certain integral operator from $M(w)$ into $M(v)$.

Theorem 3.1. *Suppose that B satisfies conditions (1), (2), (3) and (4). Let $S(x) = \int_0^\infty w(y)c(x,y)dy$ and*

$$M = \sup_{x>0} \frac{S(x)}{V(x)} < \infty.$$

Then B is a bounded operator from $M(w)$ into $M(v)$ and

$$\|B\|_{M(w,v)} = \sup_{x>0} \frac{S(x)}{V(x)}\,.$$

Proof. Let $f \in M(w)$ be a decreasing non-negative function and $\|f\|_{M(w)} = 1$. Hence

$$\int_0^l f \le \int_0^l w, \quad (\forall\, l).$$

Applying Lemma 2.1, we have

$$\int_0^x B\,f(t)\,dt = \int_0^\infty c(x,y)\,f(y)\,dy = \int_0^\infty \Big(\int_0^y f(t)\,dt\Big) d(-c(x,y)) \le$$

$$\le \int_0^\infty \Big(\int_0^y w(t)\,dt\Big) d(-c(x,y)) = S(x) \le M\,V(x);$$

hence $\|Bf\|_{M(v)} \le M$, and $\|B\|_{M(w,v)} \le M$. If $f = w$, then

$$\|f\|_{M(w)} = 1, \quad \|Bf\|_{M(v)} = M.$$

Therefore

$$\|B\|_{M(w,v)} \ge M.$$

This completes the proof of the theorem. $\square$

Proposition 3.1. *Suppose that $v(x)$ is an arbitrary weight function and $w(x) = 1/x^\alpha$ where $0 < \alpha < 1$ and $b(x,y)$ satisfies*

$$b(\lambda x, \lambda y) = \frac{1}{\lambda}\, b(x,y),$$

for all $x, y, \lambda > 0$. Then $S(x) = (1-\alpha)S(1)W(x)$ and

$$\|B\|_{M(w,v)} = (1-\alpha)\,S(1)\,\sup_{x>0} \frac{W(x)}{V(x)}\,,$$

Proof. In the same way as in Proposition 2.2, We have $c(\lambda x, \lambda y) = c(x, y)$. Hence

$$S(x) = \int_0^\infty \frac{1}{y^\alpha} c(x, y)\, dy = \int_0^\infty \frac{1}{x^\alpha t^\alpha} c(x, xt) x\, dt = x^{1-\alpha} \int_0^\infty \frac{1}{t^\alpha} c(1, t)\, dt,$$

so that $S(x) = (1-\alpha)S(1)W(x)$. □

Corollary 3.1. *Suppose that* $V(x) = \frac{1}{1-\alpha} \frac{x^{1-\alpha}(1+x)}{2x+1}$ *and* $w(x) = 1/x^\alpha$ *where* $0 < \alpha < 1$ *and* $b(x, y)$ *satisfies*

$$b(\lambda x, \lambda y) = \frac{1}{\lambda} b(x, y),$$

for all $x, y, \lambda > 0$. *Then*

$$\|B\|_{M(w,v)} = 2(1-\alpha)\, S(1).$$

Corollary 3.2. *Let* $w(x) = 1/x^\alpha$ *where* $0 < \alpha < 1$ *and* $a(x, y)$ *satisfy*

$$b(\lambda x, \lambda y) = \frac{1}{\lambda} b(x, y),$$

for all $x, y, \lambda > 0$. *Then*

$$\|B\|_{M(w)} = (1-\alpha)\, S(1).$$

Proposition 3.2. *Suppose that* $V(x) = \frac{1}{1-\alpha} \frac{x^{1-\alpha}(1+x)}{2x+1}$ *and* $w(x) = 1/x^\alpha$, *where* $0 < \alpha < 1$ *and* $b(x, y) = 1/(x + y)$. *Then*

$$\|H\|_{M(w,v)} = 2(1-\alpha) \int_0^\infty \frac{1}{y^\alpha} \left[\log\left(1 + \frac{1}{y}\right)\right] dy.$$

Proof. We have

$$S(1) = \int_0^\infty \frac{1}{y^\alpha} c(1, y)\, dy = \int_0^\infty \frac{1}{y^\alpha} \left[\log\left(1 + \frac{1}{y}\right)\right] dy.$$

Applying Corollary 3.1, we have the statement. □

In particular, if $v(x) = w(x) = 1/x^\alpha$, we have the following statement.

Proposition 3.3. *Let* $w(x) = 1/x^\alpha$, *where* $0 < \alpha < 1$ *and* $b(x, y) = 1/(x + y)$. *Then*

$$\|H\|_{M(w)} = (1-\alpha) \int_0^\infty \frac{1}{y^\alpha} \left[\log\left(1 + \frac{1}{y}\right)\right] dy.$$

Proposition 3.4. *Let C be the Copson operator, and let $0<\alpha<1$ and $w(x)=1/x^\alpha$ and $V(x)=\frac{1}{1-\alpha}\frac{x^{1-\alpha}(1+x)}{2x+1}$. Then*

$$\|C\|_{M(w,v)}=\frac{2}{\alpha}\,.$$

Proof. We have

$$c(1,y)=\begin{cases}1/y & \text{for} \quad y>1,\\ 1 & \text{for} \quad y\le 1,\end{cases}$$

so that

$$S(1)=\int_0^\infty \frac{1}{y^\alpha}\,c(1,y)\,dy=\frac{1}{\alpha(1-\alpha)}\,. \qquad \square$$

Applying Corollary 3.2 for Copson operator we have

Proposition 3.5. *Let C be the Copson operator, and let $w(x)=1/x^\alpha$ where $0<\alpha<1$. Then*

$$\|C\|_{M(w)}=\frac{1}{\alpha}\,.$$

Proposition 3.6. *Let C be the Copson operator from $M(w)$ into itself. If $w(x)=\frac{1}{x}$, then*

$$L_{M(w)}(C)=1.$$

Proof. Suppose that $f\in M(w)$ is a decreasing non-negative function,

$$\int_0^l C\,f(x)\,dx \ge \int_0^l \Big(\int_0^l b(x,y)\,f(y)\,dy\Big)dx =$$
$$=\int_0^l f(y)\Big(\int_0^l b(x,y)dx\Big)dy \int_0^l f(y)\,dy.$$

Hence

$$\int_0^l C\,f(x)\,dx \ge \int_0^l f(y)\,dy, \quad (\forall\, l>0),$$

and so, we have

$$\|Cf\|_{M(w)}\ge\|f\|_{M(w)}.$$

Therefore $L_{M(w)}(C)\ge 1$. If $f=w$, then $\|f\|_{M(w)}=1$ and $Cf(x)=f(x)$ for each $x>0$; hence $\|Cf\|_{M(w)}=\|f\|_{M(w)}$. This completes the proof of the statement. $\square$

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