

ON ONE MIXED PROBLEM OF THE PLANE THEORY
OF ELASTICITY WITH A PARTIALLY UNKNOWN
BOUNDARY

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ABSTRACT. An elastic square-like body weakened by a hole with an unknown equidurable boundary, is considered. Neighborhoods of the square vertices, cut by unknown equidurable arcs, are symmetric with respect to diagonals of the square. Tangential stresses on the whole boundary are equal to zero. Normal displacements on the linear parts of the boundary are constant, while on unknown parts of the boundary they are equal to zero. Using the methods of the theory of analytic functions, the exact solution of the problem, as well as unknown parts of the boundary are found.

რეზიუმე. განხილულია უცნობი თანაბრადმტკიცე საზღვრიანი ხვრელით შესუსტებული დრეკადი კვადრატი, რომლის წვეროების მიდამო ამოჭრილია უცნობი თანაბრადმტკიცე ტოლი სიდიდის რკალეებით, რომლებიც სიმეტრიულია კვადრატის დიაგონალების მიმართ. საზღვრის წრფივ ნაწილებზე მოქმედებს სწორფუძიანი გლუვი ხისტი შტამპები. საზღვრის უცნობი ნაწილი თავისუფალია გარე ძაბვებისაგან.

ანალიზურ ფუნქციათა თეორიის მეთოდების გამოყენებით მოძებნილია ამოცანის ზუსტი ამონახსნი და საზღვრის უცნობი ნაწილები.

Let an elastic isotropic body on a plane $z = x + iy$ occupy a doubly-connected square region which is weakened by an unknown hole. Suppose that the vertices of the square lie on the coordinate axes and its neighbourhoods are cut by the equal unknown arcs, symmetric with respect to the coordinate axes. Length of the side of the square we denote by $2a$.

Let to every segment of the boundary of the body be applied absolutely smooth rigid stamps with rectilinear bases which move towards the normal under the action of concentrated normally contracting force p applied to the midpoint of the stamp. Unknown parts of the boundary of the body are

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free from external forces. On the whole boundary of the body the tangential stress $\tau_{ns} = 0$.

We formulate the problem as follows: Find a stressed state of the body and unknown parts of the boundary under the condition that the tangential normal stress on them takes the constant value $\sigma_s = k = \text{const}$.

Since the problem is cyclically symmetric, therefore on the segments A_4A_5, A_6A_1 the normal displacements and tangential stresses $V_n = \tau_{ns} = 0$.

To study the above-formulated problem it is sufficient to consider a curvilinear hexagon which we denote by D_k . Since $A_1A_2 \parallel A_4A_5$ and $A_3A_4 \parallel A_6A_1$, the value of the principal vector of stresses on the segments A_4A_5, A_6A_1 will be equal to $-\frac{p}{2}$. Introduce the notation: $\Gamma_1 = A_1A_2$, $\Gamma_2 = A_3A_4$, $\Gamma_3 = A_4A_5$, $\Gamma_4 = A_6A_1$; $\Gamma = \bigcup_{j=1}^4 \Gamma_j$.

On the basis of Kolosov-Miskhelishvili's formulas [1], the above-posed problem is reduced to finding two analytic functions $\varphi(z)$ and $\psi(z)$ in the domain D by the boundary conditions

$$\text{Re} [e^{-i\alpha(t)} (\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)})] = 2\mu v_n(t), \quad t \in \Gamma, \quad (1)$$

$$\text{Re} [e^{-i\alpha(t)} (\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})] = C(t), \quad t \in \Gamma, \quad (2)$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = B(t), \quad t \in A_2A_3 \cup A_5A_6 \quad (3)$$

$$\text{Re} \varphi'(t) = \frac{k}{4}, \quad t \in A_2A_3 \cup A_5A_6 \quad (4)$$

where $\varkappa$ and μ are elastic constants, $\alpha(t)$ is size of the angle made by the outer normal n and the OX -axis,

$$C(t) = \text{Re} \int_{A_1}^t (i\sigma_n \exp(\alpha(t_0) - \alpha(t))) ds_0, \quad t \in \Gamma, \quad (5)$$

$$B(t) = i \int_{A_1}^t \sigma_n(t_0) ds_0, \quad t \in \Gamma.$$

Taking into account that $\sigma_n(t) = 0$, $t \in A_2A_3 \cup A_5A_6$,

$$\alpha(t) = \alpha_j = \frac{\pi}{4} + \frac{\pi(j-1)}{2}, \quad t \in \Gamma_j \quad (j = 1, 2, 3, 4) \quad (6)$$

we obtain

$$C(t) = \begin{cases} -\frac{p}{2}, & t \in \Gamma_2 \cup \Gamma_3 \\ 0, & t \in \Gamma_1 \cup \Gamma_4 \end{cases} \quad (7)$$

$$B = \begin{cases} -\frac{p}{2}ie^{\frac{\pi}{4}i}, & t \in A_2A_3 \\ \frac{p}{2}e^{\frac{\pi}{4}i}, & t \in A_5A_6. \end{cases} \quad (8)$$

Combining (1) and (2) and then differentiating with respect to s , as well as taking into account that $\alpha(t)$, $C(t)$ and $v_n(t)$ are piecewise constant functions on Γ , we get

$$\operatorname{Im} \varphi'(t) = 0, \quad t \in \Gamma. \quad (9)$$

In the sequel, the function $\varphi(z)$ will be assumed to be continuous in a closed domain D , and $\varphi'(z)$ and $\psi(z)$ are continuously extendable on the boundary of the body S , except possibly of the points A_1, A_2, A_5, A_6 in the neighbourhood of which the estimate

$$|\varphi'(z)|, |\psi(z)| < |z - A_j|^{-\frac{1}{2}}, \quad j = 1, 2, 5, 6 \quad (10)$$

is quite admissible.

Equalities (4) and (9) is, in fact, the Keldysh-Sedov problem for the domain D (see [2]).

$$\begin{aligned} \operatorname{Re} \varphi'(z) &= \frac{k}{4}, \quad t \in A_2A_3 \cup A_5A_6 \\ \operatorname{Im} \varphi'(t) &= 0, \quad t \in \Gamma. \end{aligned} \quad (11)$$

The problem (11) has the unique solution [3]

$$\varphi(z) = \frac{k}{4}z + (1+i)e^{\frac{\pi}{4}i}. \quad (12)$$

If we substitute the values $\varphi(t)$, $C(t)$, $B(t)$ into the boundary conditions (2) and (3), then we can find that

$$\operatorname{Re} e^{-i\alpha_j} \left(\frac{k}{2}t + \overline{\psi(t)} \right) = -\frac{p}{4}, \quad t \in \Gamma_j, \quad j = 1, 2, 3, 4 \quad (13)$$

$$e^{-\frac{\pi}{4}i} \left(\frac{k}{2}t + \overline{\psi(t)} \right) = \begin{cases} -\frac{p}{4}(1+i), & t \in A_2A_3 \\ \frac{p}{4}(1+i), & t \in A_5A_6 \end{cases} \quad (14)$$

If $t \in \Gamma_j$, then $(t - A_j) = i|t - A_j|e^{i\alpha_j}$, whence

$$\operatorname{Re}(e^{i\alpha_j}t) = \operatorname{Re}(e^{i\alpha_j}A_j),$$

that is,

$$\operatorname{Re} e^{i\alpha_j}t = a, \quad t \in \Gamma_j, \quad j = 1, 3, 4. \quad (15)$$

Taking into account equality (6), we rewrite the conditions (13)-(15) in the form

$$\operatorname{Re} \left[\frac{k}{2} e^{-\frac{i\pi}{4}} t + e^{\frac{i\pi}{4}} \psi(t) \right] = \begin{cases} -\frac{p}{4}, & t \in \Gamma_1 \\ \frac{p}{4}, & t \in \Gamma_3 \end{cases} \quad (16)$$

$$\operatorname{Im} \left[\frac{k}{2} e^{-\frac{i\pi}{4}} t - e^{\frac{i\pi}{4}} \psi(t) \right] = \begin{cases} -\frac{p}{4}, & t \in \Gamma_2 \\ \frac{p}{4}, & t \in \Gamma_4 \end{cases} \quad (17)$$

$$\operatorname{Re}(e^{-\frac{\pi}{4}i} t) = \begin{cases} a, & t \in \Gamma_1 \\ -a, & t \in \Gamma_3 \end{cases} \quad (18)$$

$$\operatorname{Im}(e^{-\frac{\pi}{4}i} t) = \begin{cases} a, & t \in \Gamma_2 \\ -a, & t \in \Gamma_4 \end{cases} \quad (19)$$

The condition (14) can be represented as

$$\operatorname{Re}(e^{-\frac{\pi}{4}i} t + e^{\frac{\pi}{4}i} \psi(t)) = \begin{cases} -\frac{p}{4}, & t \in A_2 A_3 \\ \frac{p}{4}, & t \in A_5 A_6 \end{cases} \quad (20)$$

$$\operatorname{Im}(e^{-\frac{\pi}{4}i} t - e^{\frac{\pi}{4}i} \psi(t)) = \begin{cases} -\frac{p}{4}, & t \in A_2 A_3 \\ \frac{p}{4}, & t \in A_5 A_6 \end{cases} \quad (21)$$

Combining equality (19), multiplied by k , and equality (17) with the changed sign, we obtain

$$\operatorname{Im} \left[\frac{k}{2} e^{-\frac{\pi}{4}i} t + e^{\frac{\pi}{4}i} \psi(t) \right] = \begin{cases} \frac{p}{4} + ka, & t \in \Gamma_2 \\ -(\frac{p}{4} + ka), & t \in \Gamma_4 \end{cases} \quad (22)$$

Analogously, from equalities (16) and (18) we get

$$\operatorname{Re} \left[\frac{k}{2} e^{-\frac{\pi}{4}i} \psi(t) \right] = \begin{cases} \frac{p}{4} + ka, & t \in \Gamma_2 \\ -(\frac{p}{4} + ka), & t \in \Gamma_3. \end{cases} \quad (23)$$

Introduce the notation

$$\phi(z) = e^{-\frac{i\pi}{4}} z + e^{\frac{i\pi}{4}} \psi(z), \quad \Psi(z) = e^{-\frac{i\pi}{4}} z - e^{\frac{i\pi}{4}} \psi(z). \quad (24)$$

under which the boundary conditions (16),(20), (22) and (17), (21), (23) can be written as

$$\operatorname{Re} \phi(t) = \begin{cases} -\frac{p}{4}, & t \in \Gamma_1 \cup A_2A_3 \\ \frac{p}{4}, & t \in \Gamma_3 \cup A_5A_6 \end{cases} \quad (25)$$

$$\operatorname{Im} \phi(t) = \begin{cases} \frac{p}{4} + ka, & t \in \Gamma_2 \cup A_2A_3 \\ -(\frac{p}{4} + ka), & t \in \Gamma_4 \end{cases} \quad (26)$$

$$\operatorname{Im} \Psi(t) = \begin{cases} -\frac{p}{4}, & t \in \Gamma_2 \cup A_2A_3 \\ \frac{p}{4}, & t \in \Gamma_4 \cup A_5A_6 \end{cases} \quad (27)$$

$$\operatorname{Re} \Psi(t) = \begin{cases} \frac{p}{4} + ka, & t \in \Gamma_1 \\ -(\frac{p}{4} + ka), & t \in \Gamma_3 \end{cases} \quad (28)$$

Thus we have come to the Keldysh-Sedov problem [2]. Let the function $z = \omega(\zeta)$, $\zeta = \xi + i\eta$ map conformally the domain D onto the upper half-plane $\operatorname{Im} \zeta > 0$, $\zeta = \xi + i\eta$. Denote by a_j the image of the point A_j , $j = 1, \dots, 6$ and assume that $a_6 = 1$, $a_5 = -1$, while a midpoint of the arc A_2A_3 transforms into $\zeta = \infty$. Since the domain D is symmetric with respect to the Oy -axis, we have $-a_2 = a_3$, $-a_1 = a_4$.

Denote

$$W_1(\zeta) = \phi(\omega(\zeta_1)), \quad \text{and} \quad W_2(\zeta) = \Psi(\omega(\zeta)). \quad (29)$$

Taking into account notation (29), the boundary conditions (25), (26) and (27), (28) take the form

$$W_1(\xi) + \overline{W_1(\xi)} = \begin{cases} -\frac{p}{2}, & \xi \in (-\infty, -a_2) \cup (a_1, \infty) \\ \frac{p}{2}, & \xi \in (-a_1, 1) \end{cases} \quad (30)$$

$$W_1(\xi) - \overline{W_1(\xi)} = \begin{cases} \frac{i(p+4ak)}{2}, & \xi \in (-a_2, -a_1) \\ -\frac{i(p+4ak)}{2}, & \xi \in (1, a_1) \end{cases}$$

$$W_2(\xi) - \overline{W_2(\xi)} = \begin{cases} -\frac{ip}{2}, & \xi \in (-\infty, -a_1) \cup (a_2, \infty) \\ \frac{ip}{2}, & \xi \in (-1, a_1) \end{cases} \quad (31)$$

$$W_2(\xi) + \overline{W_2(\xi)} = \begin{cases} -\frac{(p+4ak)}{2}, & \xi \in (-a_1, -1) \\ \frac{p+4ak}{2}, & \xi \in (a_1, a_2) \end{cases}$$

Multiplying (31) by u and introducing the notation $Q(\xi) = iW_2(\xi)$, we obtain

$$Q(\xi) + \overline{Q(\xi)} = \begin{cases} \frac{p}{2}, & \xi \in (-\infty, -a_1) \cup (a_2, \infty) \\ -\frac{p}{2}, & \xi \in (-1, a_1) \end{cases} \quad (32)$$

$$Q(\xi) - \overline{Q(\xi)} = \begin{cases} -\frac{i(p+4ak)}{2}, & \xi \in (a_1, a_2) \\ \frac{i(p+4ak)}{2}, & \xi \in (-a_1, -1) \end{cases}$$

The solution of the obtained boundary problems (30), (32) is given by the Keldysh-Sedov formula (see [4], §95), and in our case it has the form

$$W_1(\zeta) = \frac{X_1(\zeta)}{4\pi i} \left(-\int_{-\infty}^{-a_2} \frac{pd\xi}{|X_1(\xi)|(\xi - \zeta)} - \int_{-a_2}^{-a_1} \frac{(p + 4ak)d\xi}{|X_1(\xi)|(\xi - \zeta)} + \int_{-a_1}^1 \frac{pd\xi}{|X_1(\xi)|(\xi - \zeta)} + \right. \\ \left. + \int_1^{a_1} \frac{(p + 4ak)d\xi}{|X_1(\xi)|(\xi - \zeta)} - \int_{a_1}^{\infty} \frac{pd\xi}{|X_1(\xi)|(\xi - \zeta)} + C \right) \quad (33)$$

$$Q(\zeta) = \frac{X_1(\zeta)}{4\pi i} \left(-\int_{-\infty}^{-a_1} \frac{pd\xi}{|X_2(\xi)|(\xi - \zeta)} - \int_{-a_1}^{-1} \frac{(p + 4ak)d\xi}{|X_2(\xi)|(\xi - \zeta)} - \int_{-1}^{a_1} \frac{pd\xi}{|X_2(\xi)|(\xi - \zeta)} + \right. \\ \left. + \int_{a_1}^{a_2} \frac{(p + 4ak)d\xi}{|X_2(\xi)|(\xi - \zeta)} + \int_{a_2}^{\infty} \frac{pd\xi}{|X_2(\xi)|(\xi - \zeta)} + C \right), \quad (34)$$

where

$$X_1(\zeta) = \sqrt{\frac{(\zeta + a_2)(\zeta - 1)}{(\zeta + a_1)(\zeta - a_1)}}, \quad \text{Im } \zeta > 0, \quad (35)$$

$$X_2(\zeta) = \sqrt{\frac{(\zeta - a_2)(\zeta + 1)}{(\zeta + a_1)(\zeta - a_1)}}, \quad \text{Im } \zeta > 0, \quad (36)$$

$$X_1(\infty) = x_2(\infty) = 1$$

It can be easily seen that

$$X_1(\xi) = \begin{cases} |X_1(\xi)|, & \xi \in (-\infty, -a_2) \cup (-a_1, 1) \cup (a_1, \infty) \\ -|X_1(\xi)|, & \xi \in (-a_2, -a_1) \cup (1, a_1) \end{cases} \quad (37)$$

$$X_2(\xi) = \begin{cases} |X_2(\xi)|, & \xi \in (-\infty, -a_1) \cup (-1; a_1) \cup (a_2, \infty) \\ |X_2(\xi)|, & \xi \in (-a_1, -1) \cup (a_1, a_2) \end{cases} \quad (38)$$

$$|X_1(\xi)| = |X_2(-\xi)| \quad (39)$$

Since the functions $X_1(\xi)$ and $X_2(\xi)$ have singularity of order $\frac{1}{2}$ at the points $\xi = \pm a$, therefore for $W_1(\xi)$ and $Q(\zeta)$ to be bounded in the vicinity of the

points $-a_1, a_1$, it is necessary and sufficient that the conditions

$$\left(\int_{-\infty}^{-a_2} \frac{d\xi}{|X_1(\xi)|(\xi \pm a_1)} + \int_{-a_2}^{-a_1} \frac{(p+4ak)d\xi}{p|X_1(\xi)|(\xi \pm a_1)} - \int_{-a_1}^1 \frac{d\xi}{|X_1(\xi)|(\xi \pm a_1)} + \right. \\ \left. + \int_1^{a_1} \frac{(p+4ak)d\xi}{p|X_1(\xi)|(\xi \pm a_1)} + \int_{a_1}^{\infty} \frac{d\xi}{|X_1(\xi)|(\xi \pm a_1)} - C \right) = 0 \quad (40)$$

$$\left(\int_{-\infty}^{-a_1} \frac{d\xi}{|X(\xi)|(\xi \pm a_1)} - \int_{-a_1}^{-1} \frac{(p+4ak)d\xi}{p|X_2(\xi)|(\xi \pm a_1)} - \int_{-1}^{a_1} \frac{d\xi}{|X_2(\xi)|(\xi \pm a_1)} + \right. \\ \left. + \int_{a_1}^{a_2} \frac{(p+4ak)d\xi}{p|X_2(\xi)|(\xi \pm a_1)} + \int_{a_2}^{\infty} \frac{d\xi}{|X_2(\xi)|(\xi \pm a_1)} + C \right) = 0 \quad (41)$$

be fulfilled.

If in the condition (41) we replace ξ by $-\xi$ and take into account the condition (39), we obtain the condition which coincides with the condition (40) which is, in fact, the system of two equations containing unknown parameters a_1, a_2, C, k . Omitting C from that system, we obtain

$$\frac{p+4ak}{p} \left(\int_{-a_2}^{-a_1} \frac{d\xi}{|X_1(\xi)|(\xi^2 - a_1^2)} + \int_1^{a_1} \frac{d\xi}{|X_1(\xi)|\xi_1(a_1^2 - \xi^2)} \right) = \\ = - \int_{-\infty}^{-a_2} \frac{d\xi}{|X_1(\xi)|(\xi^2 - a_1^2)} + \int_{-a_1}^1 \frac{d\xi}{|x_1(\xi)|(a_1^2 - \xi^2)} + \int_{a_1}^{\infty} \frac{d\xi}{|X_1(\xi)|(\xi^2 - a_1^2)}. \quad (42)$$

Equation (42) contains a_1, a_2, k as unknowns. If we fix k , then with respect to the unknowns a_1 and a_2 we obtain a nonlinear equation whose solution is connected with great difficulties. Therefore we can act as follows: we endow the unknowns a_1 and a_2 with the values from the interval $1 < a_1 < a_2 < +\infty$ and find the unknown k . Introducing the notation

$$A = \int_{-a_2}^{-a_1} \frac{d\xi}{|X_1(\xi)|(\xi^2 - a_1^2)} + \int_1^{a_1} \frac{d\xi}{|X_1(\xi)|(a_1^2 - \xi^2)}, \quad (43)$$

$$B = - \left(\int_{-\infty}^{-a_2} \frac{d\xi}{|X_1(\xi)|(X_1(\xi) - a_1^2)} + \int_{-a_1}^1 \frac{d\xi}{|X_1(\xi)|(a_1^2 - \xi^2)} + \int_{a_1}^{\infty} \frac{d\xi}{|X_1(\xi)|(\xi^2 - a_1^2)} \right), \quad (44)$$

equation (42) takes the form

$$p + 4ak = -\frac{Bp}{A}.$$

We can easily show that $A > 0$ and $B > 0$, and hence $p + 4ak < 0$. By the condition of the problem, $k < 0$, but we have found that not only $k < 0$, but also $p + 4ak < 0$.

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