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PROBLEM ON THE SMOOTHNESS OF SOLUTIONS OF ONE CLASS OF HYPOELLIPTIC EQUATIONS

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Abstract. In this paper we study the smoothness of solutions of some hypoelliptic equations.

რეზიუმე. ნაშრომში შესწავლილია ჰიპოელიფსურ ოპერატორთა ზოგიერთი კლასის ამონახსნთა სიგლუვე.

In this paper we apply the theorems obtained in [1] to study the local smoothness of solutions of some class of hypoelliptic equations, i.e., we prove theorems claiming that the solution belongs to the Hölder class inside the domain and holds for zero Dirichlet boundary condition up to the boundary.

Let us consider the equation of the type

$$\sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^\mu, \delta_j^\mu \le l_j^\mu, \\ j\in e\subseteq e_n}} D^{\alpha^{\mu^e}} \Big(a_{\alpha_e^\mu \delta_e^\mu} D^{\delta^{\mu^e}} u(x) \Big) = \sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^\mu \le l_j^\mu, \\ j\in e\subseteq e_n}} D^{\alpha^{\mu^e}} f_{\alpha^{\mu^e}}(x), \tag{1}$$

where $e_n = \{1,2,\dots,n\}$, $e \subseteq e_n$, $\alpha^\mu = (\alpha_1^\mu, \alpha_2^\mu, \dots, \alpha_n^\mu)$, $\delta^\mu = (\delta_1^\mu, \delta_2^\mu, \dots, \delta_n^\mu)$, $l^\mu = (l_1^\mu, l_2^\mu, \dots, l_n^\mu)$, $l_j^\mu > 0$ are integers, $j \in e_n$, $\mu = 1,2,\dots,N$; $\alpha^{\mu^e} = (\alpha_1^{\mu^e}, \alpha_2^{\mu^e}, \dots, \alpha_n^{\mu^e})$, $\alpha_j^{\mu^e} = \alpha_j$ for $j \in e$, $\alpha_j^{\mu^e} = 0$ for $j \in e_n \backslash e = e'$. Let the domain $G \subset R^n$ be bounded in R^n, and suppose that the coefficients $a_{\alpha_e^\mu \delta_e^\mu}(x)$ are the bounded measurable functions in the domain G, $a_{\alpha_e^\mu \delta_e^\mu}(x) \equiv a_{\delta_e^\mu \alpha_e^\mu}(x)$ and $\xi \in R^n$,

$$\sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^\mu, \delta_j^\mu \le l_j^\mu, \\ j\in e\subseteq e_n}} (-1)^{|\alpha^{\mu^e}|} a_{\alpha_e^\mu \delta_e^\mu}(x) \xi_{\alpha^{\mu^e}} \xi_{\delta^{\mu^e}} \ge c_0 \sum_{\mu=1}^{N} \sum_{e\subseteq e_n} |\xi_{l^{\mu^e}}|^2, \qquad c_0 = \text{const} > 0, \tag{2}$$

where $|\alpha^{\mu^e}| = \sum\limits_{j\in e} \alpha_j^\mu$. We assume that $f_{\alpha^{\mu^e}} \in L_2(G)$ for $\alpha_j^\mu < l_j^\mu$, and $f_{\alpha^{\mu^e}} \in L_{2,a,æ}(G)$ for $\alpha_j^\mu = l_j^\mu$, $j \in e_n$, $\mu = 1,2,\dots,N$.

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In the case if $\mu = 1$, the existence and uniqueness of a generalized solution of equation (1) has been studied in [2].

The problem on the local smoothness of solutions of quasielliptic equations

$$\sum_{\substack{\left|\alpha,\frac{1}{l}\right|\le 1,\\ \left|\delta,\frac{1}{l}\right|\le 1}} D^{\alpha}\left(a_{\alpha\delta}(x)\, D^{\delta}u(x)\right) = \sum_{\left|\alpha,\frac{1}{l}\right|\le 1} D^{\alpha} f_{\alpha}(x) \tag{3}$$

with continuous or Hölder-continuous coefficients for higher derivatives is considered in [3]. In [4], the estimates for the solutions of equation (3) were studied under the condition that the coefficients for the higher derivatives are infinitely differentiable.

In [5-7], the theorem on belonging of a solution of equation (3) as well as of the solution of equation of the type

$$\sum_{\mu=1}^{N} \sum_{\substack{\left|\alpha^{\mu},\frac{1}{l^{\mu}}\right|\le 1,\\ \left|\delta^{\mu},\frac{1}{l^{\mu}}\right|\le 1}} D^{\alpha^{\mu}}\left(a_{\alpha^{\mu}\delta^{\mu}}(x)\, D^{\delta^{\mu}}u(x)\right) = \sum_{\mu=1}^{N} \sum_{\left|\alpha^{\mu},\frac{1}{l^{\mu}}\right|\le 1} D^{\alpha^{\mu}} f_{\alpha^{\mu}}$$

to the Hölder class inside the domain and for zero Dirichlet boundary conditions up to bounds.

In this paper, as well as in [5-7], the Hölder continuity of a solution is studied without any conditions of smoothness on the functions $a_{\alpha\delta}(x)$.

However, it should be noted that in this paper just as in [6], unlike [5] we have:

1) the Holder "exponent" is greater than that in [5];

2) $f_{\alpha^{\mu e}}$ for $\alpha_j^{\mu} = l_j^{\mu}$, $j \in e_n$, $\mu = 1, 2, \ldots, N$ belongs to the wider class, i.e., $f_{\alpha^{\mu e}} \in L_{2,a,\chi}(G)$;

3) moreover, $\nu \neq 0$, and the amount of vectors ν increases.

A generalized solution of equation (1) in G is a function $u(x) \in \bigcap_{\mu=1}^{N} S_2^{l^{\mu}} W(G)$, such that

$$\sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^{\mu},\delta_j^{\mu}\le l_j^{\mu},\\ j\in e\subseteq e_n}} (-1)^{\left|\alpha^{\mu^e}\right|} \int_G a_{\alpha_e^{\mu}\delta_e^{\mu}}(x)\, D^{\delta^{\mu^e}} u(x) D^{\alpha^{\mu^e}} v(x)dx =$$
$$= \sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^{\mu}\le l_j^{\mu},\\ j\in e\subseteq e_n}} (-1)^{\left|\alpha^{\mu^e}\right|} \int_G f_{\alpha^{\mu e}} D^{\alpha^{\mu^e}} v(x)dx, \tag{4}$$

for every function $v(x) \in \bigcap_{\mu=1}^{N} \overset{\circ}{S}{}_2^{l^{\mu}} W(G)$.

The norm in the space $\bigcap\limits_{\mu=1}^{N} S_2^{l^\mu} W(G)$ is defined as follows:

$$\sum_{\mu=1}^{N} \|u\|_{S_2^{l^\mu} W(G)} \,,$$

the space $S_2^{l^\mu} W(G)$ is the supplement $C^\infty(G)$ in the norm

$$\|u\|_{S_2^{l^\mu} W(G)} = \sum_{e \subseteq e_n} \left\| D^{l^{\mu^e}} u \right\|_{L_2(G)} ,$$

and the space $\overset{\circ}{S}{}_2^{l^\mu} W(G)$ is the supplement $C_0^\infty(G)$ in the norm $S_2^{l^\mu} W(G)$.

Let $\beta_\mu \geq 0$, $\mu = 1, 2, \ldots, N$, $\sum\limits_{\mu=1}^{N} \beta_\mu = 1$, $l = \sum\limits_{\mu=1}^{N} \beta_\mu l^\mu$, $b = (b_1, b_2, \ldots, b_n)$, and $d = (d_1, d_2, \ldots, d_n)$ be the fixed vector, and let $0 < d_j < 1$, $b_j \leq d_j$ for $j \in e_n$.

Theorem 1. *Let* $\sum\limits_{\mu=1}^{N} l_j^\mu \beta_\mu - \nu_j > 0$ *for* $j \in e_n$, *then any generalized solution of equation* (1) *from* $\bigcap\limits_{\mu=1}^{N} S_2^{l^\mu} W(G)$ *belongs to the space* $C_{\nu+\sigma^{1,0}}(G^d)$, $(\overline{G}_d \subset G)$, *where* $C_{\nu+\sigma^{1,0}}$ *is Hölder's space.*

Proof. Consider first the case when all $a_{\alpha_e^\mu \delta_e^\mu}(x) \equiv 0$, except for ones for which $\alpha_j^\mu = \delta_j^\mu = l_j^\mu$, and all $f_{\alpha^{\mu^e}} \equiv 0$, $\mu = 1, 2, \ldots, N$. Let $x_0 \in G$ and $\Pi(x_0)$ be parallelepiped in R^n

$$\Pi_b(x_0) = \big\{ x : |x_j - x_{j0}| < b_j ,\ j \in e_n \big\}$$

and G^d be a subdomain of the domain G such that

$$G^d = \big\{ y : |y_j - x_j| > d_j ,\quad x \in \partial G, \quad j \in e_n \big\}.$$

The validity of the theorem we prove in G^d. From the variational principle it follows that

$$\int\limits_{\Pi_b(x_0)} \sum_{\substack{\alpha_j^\mu, \delta_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} \big(-1 \big)^{|\alpha^{\mu^e}|} a_{\alpha_e^\mu \delta_e^\mu}(x) \times$$
$$\times D^{\delta^{\mu^e}} \big[\theta(x) \big(u(x) - p(x) \big) \big] D^{\alpha^{\mu^e}} \big[\theta(x) \big(u(x) - p(x) \big) \big] \, dx \geq$$
$$\geq \int\limits_{\Pi_b(x_0)} \sum_{\substack{\alpha_j^\mu, \delta_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} \big(-1 \big)^{|\alpha^{\mu^e}|} a_{\alpha_e^\mu \delta_e^\mu}(x) D^{\delta^{\mu^e}} \big(u(x) - p(x) \big) \times$$
$$\times D^{\alpha^{\mu^e}} \big(u(x) - p(x) \big) \, dx = A_\mu \Big(\big(u(x) - p(x) \big), \Pi_b(x_0) \Big) \qquad (5)$$

for any $\theta(x) \in C^{\infty}(\Pi_b(x_0))$, such that $\theta(x) \equiv 1$ in the neighbourhood of $\partial\Pi_b(x_0)$, and for any polynomial $p(x) = \sum\limits_{\substack{\alpha_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} C_{\alpha^\mu} x^{\alpha^\mu}$ and for an arbitrary solution $u(x)$ of equation (1).

Let

$$\theta(x) = 1 - \prod_{j \in e_n} \omega_j \Big(\frac{x_j - x_{j,0}}{b_j} \Big),$$

where $\omega_j(t) \in C^{\infty}(R)$, such that $\omega_j(t_j) = 1$ for $|t_j| < \dfrac{1}{2}$, $\omega_j(t_j) = 0$ for $|t_j| \geq \dfrac{1}{2}$ it follows that $0 \leq \omega_j(t_j) \leq 1$, for $j \in e_n$. It is clear that $\theta(x) \equiv 0$ in $\Pi_{\frac{b}{2}}(x_0)$, $\theta(x) \equiv 1$ in the neighborhood of $\partial\Pi_b(x_0)$, we taken the coefficients $p(x)$ such that

$$\int\limits_{\Pi_b(x_0) \setminus \Pi_{\frac{b}{2}}(x_0)} \big[u(x) - p(x) \big] x^{\alpha^\mu} dx = 0.$$

By inequality (7) in [1], with the help of (5) we obtain

$$\begin{aligned} A_\mu\big(u(x) - p(x), \Pi_b(x_0)\big) \leq A_\mu\big(u(x) - p(x), \Pi_b(x_0) \setminus \Pi_{\frac{b}{2}}(x_0)\big) + \\ + C \int\limits_{\Pi_b(x_0) \setminus \Pi_{\frac{b}{2}}(x_0)} \sum_{\substack{\alpha^\mu < l_j^\mu, \\ \alpha_j^\mu + s_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} \prod_{j \in e} b_j^{-2s_j} \left[D^{\alpha^{\mu^e}} \big(u(x) - p(x) \big) \right]^2 dx + \\ + C_1 A_\mu\big(u(x) - p(x), \Pi_b(x_0) \setminus \Pi_{\frac{b}{2}}(x_0)\big) \leq \\ \leq q A_\mu(u(x) - p(x), \Pi_b(x_0) \setminus \Pi_{\frac{b}{2}}(x_0)\big). \end{aligned} \tag{6}$$

Since $A_\mu(u(x) - p(x), G) = A_\mu(u(x), G)$, therefore

$$A_\mu\big(u(x), \Pi_{\frac{b}{2}}(x_0)\big) \leq \Big(1 - \frac{1}{q}\Big) A_\mu\big(u(x), \Pi_b(x_0)\big),$$

and hence by induction we obtain

$$A_\mu\Big(u(x), \Pi_{\frac{b}{2^k}}(x_0)\Big) \leq \left(1 - \frac{1}{q}\right)^k A_\mu\left(u(x), \Pi_b\left(x_0\right)\right).$$

Let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$, $0 < \eta_j < \dfrac{b_j}{2^k}$, $j \in e_n$. Then it follows that $\Pi_\eta(x_0) \subset \Pi_{\frac{b}{2^k}}(x_0)$. Further, $k \ln 2 < \prod\limits_{j \in e_n} \dfrac{b_j}{\eta_j}$, we take $k = \left[\frac{\ln \prod\limits_{j \in e_n} \frac{b_j}{\eta_j}}{\ln 2} \right]$, $\lambda =$

$1-\dfrac{1}{q}$. Then

$$A_\mu\left(u(x),\Pi_\eta(x_0)\right)\leq\lambda^k A_\mu\left(u(x),G\right)<\lambda^{\frac{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}{\ln 2}-1}A_\mu\left(u(x),G\right)=$$

$$=e^{\frac{\ln\lambda}{\ln 2}\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}-\ln\lambda}A_\mu\left(u(x),G\right)=e^{\left(\frac{\ln\lambda}{\ln 2}-\frac{\ln\lambda}{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}\right)\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}A_\mu\left(u(x),G\right)=$$

$$=\left(e^{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}\right)^{\left(\frac{\ln\lambda}{\ln 2}-\frac{\ln\lambda}{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}\right)}A_\mu\left(u(x),G\right)=$$

$$=\left(\ln\prod_{j\in e_n}\frac{b_j}{\eta_j}\right)^{\left(\frac{\ln\lambda}{\ln 2}-\frac{\ln\lambda}{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}\right)}A_\mu\left(u(x),G\right)=$$

$$=\prod_{j\in e_n}\left(\frac{\eta_j}{b_j}\right)^{\left|\frac{\ln\lambda}{\ln 2}-\frac{\ln\lambda}{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}\right|}A_\mu\left(u(x),G\right)\leq$$

$$\leq\prod_{j\in e_n}\left(\frac{\eta_j}{b_j}\right)^{\left|\frac{\ln\lambda}{\ln 2}\right|-\left|\frac{\ln\lambda}{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}\right|}A_\mu\left(u(x),G\right),$$

for any $x_0\in G^d$. Denote $\xi_j=\left|\frac{\ln\lambda}{\ln 2}\right|=\chi_j a_j$, $\zeta_j=\left|\dfrac{\ln\lambda}{\ln\prod\limits_{j\in e_n}\frac{b_j}{\eta_j}}\right|$. It becomes evident that $0<\xi_i,\,\zeta_j<1$ for $j\in e_n$ and

$$A_\mu\left(u(x),\Pi_\eta\left(x_0\right)\right)\leq\prod_{j\in e_n}\left(\frac{\eta_j}{b_j}\right)^{\xi_j-\zeta_j}A_\mu\left(u(x),G\right). \tag{7}$$

Consequently,

$$\int\limits_0^1\cdots\int\limits_0^1\left(\prod_{j\in e_n}\gamma^{-\xi_j}\int\limits_{\Pi_\gamma(x_0)}u^2dx\right)^2\prod_{j\in e_n}\frac{d\gamma_j}{\gamma_j}\leq C\int\limits_0^1\cdots\int\limits_0^1\prod_{j\in e_n}\frac{db_j}{b_j^{1-\frac{1}{2}}}$$

It means that $u(x)\in L_{2,a,\chi,1}\left(G^d\right)\subset L_{2,a,\chi,\tau}\left(G^d\right)$, and also $D^{l^{\mu^e}}\in L_{2,a,\chi,\tau}\left(G^d\right)$, for all $e\subseteq e_n$. Then $u(x)\in S^{l^\mu}_{2,a,\chi,\tau}W\left(G^d\right)$. If we check the conditions of Theorems 1 and 2 in [1], then it turns out that $\varepsilon_j>0$, $\varepsilon_j^0=\sum\limits_{\mu=1}^{N}l_j^\mu\beta_\mu-\nu_j-(1-\chi_j a_j)\,\frac{1}{2}>0$, $j\in e_n$ at $p_{\mu_1}=p_{\mu_2}=\cdots=p_{\mu_n}=\theta_\mu=2$, $l^\mu\in N$, $\mu=1,2,\ldots,N$. Thus by Theorem 1 in [1], $D^\nu u(x)$ is continuous on G^d, and by Theorem 2 in [1], $D^\nu u(x)$ satisfies the Hölder condition, i.e., $u(x) \in C_{\nu+\sigma^{1,0}}\left(G^d\right)$.

Consider now a nonhomogeneous quasi-elliptic equation. Assume that $a_{\alpha_e^\mu \delta_e^\mu}(x) = 0$, except those $a_{\alpha_e^\mu \delta_e^\mu}(x)$, with $\alpha_j^\mu = \delta_j^\mu = l_j^\mu$ for $j \in e_n$, $\mu = 1, 2, \ldots, N$, and the right-hand sides of (1) can now be nonzero; i.e., the equation has the form

$$\sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^\mu, \delta_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} D^{\alpha^{\mu^e}} \left(a_{\alpha_e^\mu \delta_e^\mu} D^{\delta^{\mu^e}} u(x)\right) = \sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^\mu \leq l_j^\mu, \\ j \in e \subseteq e_n}} D^{\alpha^{\mu^e}} f_{\alpha^{\mu^e}}(x), \tag{8}$$

Let $x_0 \in G^d$, and consider the solutions u_{b,x_0} of (8) in $\Pi_b(x_0)$ of the class $\bigcap_{\mu=1}^{N} \overset{\circ}{S}{}_2^{l^\mu} W(G)$. A solution is understood in a distributive sense; i.e., the identity

$$\sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^\mu, \delta_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} (-1)^{|\alpha^{\mu^e}|} \int\limits_{\Pi_b(x_0)} a_{\alpha_e^\mu} D^{\delta^{\mu^e}} u(x) D^{\alpha^{\mu^e}} v(x) dx =$$

$$= \sum_{\mu=1}^{N} \sum_{\substack{\alpha_j^\mu \geq l_j^\mu, \\ j \in e \subseteq e_n}} (-1)^{|\alpha^{\mu^e}|} \int\limits_{\Pi_b(x_0)} f_{\alpha^{\mu^e}} D^{\alpha^{\mu^e}} v(x) dx \tag{9}$$

is valid for every $v \in \bigcap_{\mu=1}^{N} \overset{\circ}{S}{}_2^{l^\mu} W(G)$.

Putting $v(x) \equiv u_{b,x_0}$ in (9), we obtain

$$\int\limits_{\Pi_b(x_0)} \sum_{\substack{\alpha_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} \left(D^{\delta^{\mu^e}} u_{b,x_0}\right)^2 dx \leq \sum_{\substack{\alpha_j^\mu < l_j^\mu, \\ j \in e \subseteq e_n}} \prod_{j \in e_n} b_j^{2s_j} \int\limits_{\Pi_b(x_0)} f_{\alpha^{\mu^e}}^2 dx +$$

$$+ \sum_{\substack{\alpha_j^\mu = l_j^\mu, \\ j \in e \subseteq e_n}} \int\limits_{\Pi_b(x_0)} f_{\alpha^{\mu^e}}^2 dx \leq C_1 \prod_{j \in e_n} b_j^{\Delta_j}, \tag{10}$$

where $\Delta_j = \min\{2s_j, æ_j a_j\}$, $j \in e_n$. C_1 and Δ_j are independent of $u(x)$ and x_0. The function $\overline{u}(x) = u(x) - u_{b,x_0}$ is a solution of the homogeneous equation (1) in $\Pi_b(x_0)$, and thus $\overline{u}(x)$ satisfies the inequality

$$A_\mu\left(\overline{u}, \Pi_\eta(x_0)\right) \leq C_2 \prod_{j \in e_n} \left(\frac{\eta_j}{b_j}\right)^{\xi_j - \zeta_j} A_\mu(u, G), \tag{11}$$

for every $\eta_j < b_j$ for $j \in e_n$, if $x_0 \in G^d$. Then from inequalities (10) and (11) we have

$$A_\mu\left(u, \Pi_\eta(x_0)\right) \leq 2A_\mu\left(\overline{u}, \Pi_\eta(x_0)\right) +$$

$$+2A_{\mu}\left(u_{b,x_0},\Pi_{\eta}\left(x_0\right)\right)\leq C_3\prod_{j\in e_n}\left(\frac{\eta_j}{b_j}\right)^{\xi_j-\zeta_j}A_{\mu}\left(u,G\right).$$

Further, we again apply Theorems 1 and 2 from [1] for $p_{\mu_1}=p_{\mu_2}=\cdots=p_{\mu_n}=\theta_{\mu}=2$, $l^{\mu}\in N$, $\mu=1,2,\ldots,N$. In this case we obtain the required result.

Finally, we consider equation (1), whose all coefficients, different from zero, exist for small derivatives of a solution. Then we replace such terms into the right-hand side of the equation and obtain the required result.

Thus the theorem is proved. □

The following theorem on the smoothness of a solution up to the boundary holds when the generalized solution satisfies the Dirichlet boundary condition.

Theorem 2. *Let the domain G be such that there exists $k=\mathrm{const}>0$, such that for any point $x_0\in\partial G$ and the number $r<1$ there exists a parallelepiped $\Pi_{kr}\left(x'\right)$ such that*

$$\Pi_{kr}\left(x'\right)\subset\Pi_r\left(x_0\right)\cap\left(R^n\backslash G\right),$$

and $u(x)$ is a solution of equation (1) *from the space $\bigcap_{\mu=1}^{N}\overset{\circ}{S}{}_2^{l^{\mu}}W\left(G\right)$. If $\sum_{\mu=1}^{N}l_j^{\mu}\beta_{\mu}-\nu_j>0$, $j\in e_n$ then $u(x)$ belong to the space $C_{\nu+\sigma^{1,0}}\left(\overline{G}\right)$.*

Proof. It is sufficiently to restrict ourselves to the case, where all $\alpha_{\alpha_e^{\mu}\delta_e^{\mu}}(x)$, except for ones for which $\alpha_j^{\mu}=\delta_j^{\mu}=l_j^{\mu}, j\in e_n$ are identically equal to zero. Let $x_0\in\partial G$, and $f_{\alpha^{\mu e}}\equiv 0$ in $\Pi_b\left(x_0\right)$, for $e\subseteq e_n$, $\mu=1,2,\ldots,N$ and $u\left(x\right)\equiv 0$ outside of G. From the variational principle it follows that

$$A_{\mu}\left(u\left(x\right),\Pi_b\left(x_0\right)\right)\leq A_{\mu}\left(\theta\left(x\right)u\left(x\right),\Pi_b\left(x_0\right)\right).$$

Since $\theta\left(x\right)\equiv 0$ in $\Pi_{\frac{b}{2}}\left(x_0\right)$, therefore just as in Theorem 1, we obtain

$$A_{\mu}\left(u(x),\Pi_{\eta}\left(x_0\right)\right)\leq\prod_{j\in e_n}\left(\frac{\eta_j}{b_j}\right)^{\xi_j-\zeta_j}A_{\mu}\left(u(x),G\right),\qquad(12)$$

for every $\eta_j<b_j$, $j\in e_n$ for all $x_0\in\partial G$, $f_{\alpha^{\mu e}}\equiv 0$ in $\Pi_b\left(x_0\right)$. Estimate now $A_{\mu}\left(u(x),\Pi_{\eta}\left(x_0\right)\right)$ for the given $0<\eta_j<1, j\in e_n$ for all $x_0\in G$ and $f_{\alpha^{\mu e}}\neq 0$, $e\subset e_n$, $\mu=1,2,\ldots,N$. We consider two case:

(a) $x_0\in G^{\sqrt{\eta}}$ and (b) $x_0\notin G^{\sqrt{\eta}}$.

In case (a), for all $\eta_j\leq b_j$ assuming that $b_j=\sqrt{\eta_j}, j\in e$, we have the inequality

$$A_{\mu}\left(u(x),\Pi_{\eta}\left(x_0\right)\right)\leq C_1\prod_{j\in e_n}\left(\frac{\eta_j}{b_j}\right)^{\xi_j-\zeta_j}\left(A_{\mu}\left(u(x),G\right)+1\right).\qquad(13)$$

In case (b) there exists the point $x' \in \partial G$, such that $\Pi_{2\sqrt{\eta}}(x') \supset \Pi_{\sqrt{\eta}}(x_0)$. Let $b_j > 2\sqrt{\eta_j}$, $j \in e_n$. For all b_j, $j \in e_n$ consider the $u_{b,x'}$-solution of equation (1) in $\Pi_b(x') \cap G$ from the space $\bigcap\limits_{\mu=1}^{N} \overset{\circ}{S}{}_2^{l^\mu} W(\Pi_b(x') \cap G)$, for which the inequality

$$A_\mu(u_{b,x'}, \Pi_b(x')) \le C_2 \prod_{j \in e_n} b_j^{\Delta_j} \tag{14}$$

is valid if we assume that $u_{b,x'} = 0$ outside of $\Pi_b(x') \cap G$.

The function $u(x) - u_{b,x'}$ is a solution to the homogeneous equation (1) in $\Pi_b(x')$, where all $a_{\alpha_e^\mu \delta_e^\mu}(x) \equiv 0$, except for ones for which $\alpha_j^\mu = \delta_j^\mu = l_j^\mu$, $j \in e_n$, and $f_{\alpha^{\mu e}} \equiv 0$. From inequalities (12) and (14) we have

$$A_\mu\left(u(x), \Pi_{2\sqrt{\eta}}(x')\right) \le 2A_\mu\left(u - u_{b,x'}, \Pi_{2\sqrt{\eta}}(x')\right) +$$
$$+2A_\mu\left(u_{b,x'}, \Pi_{2\sqrt{\eta}}(x')\right) \le C_3 \Pi_{j\in e_n} \left(\frac{\eta_j}{b_j}\right)^{\xi_j - \zeta_j} A_\mu(u, G).$$

Consequently,

$$A_\mu(u(x), \Pi_\eta(x_0)) \le C_4 \prod_{j \in e_n} \left(\frac{\eta_j}{b_j}\right)^{\xi_j - \zeta_j} A_\mu(u(x), G),$$

and

$$\int_0^1 \cdots \int_0^1 \left(\prod_{j\in e_n} \gamma^{-\xi_j} \int_{\Pi_\gamma(x_0)} u^2(x)dx\right)^2 \prod_{j\in e_n} \frac{d\gamma_j}{\gamma_j} \le C \int_0^1 \cdots \int_0^1 \prod_{j \in e_n} \frac{db_j}{b_j^{1-\frac{1}{2}\zeta_j}}.$$

Further, we again apply Theorems 1 and 2 from [1] for $p_{\mu_1} = p_{\mu_2} = \cdots = p_{\mu_n} = \theta_\mu = 2$, $l^\mu \in N^n$, $\mu = 1, 2, \ldots, N$, and in this case we obtain the required result.

Thus the theorem is proved. □

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