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SOME PROPERTIES OF FUNCTIONS FROM THE INTERSECTION OF BESOV-MORREY TYPE SPACES WITH DOMINANT MIXED DERIVATIVES

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Abstract. In the present paper we study differential properties of functions belonging to the intersection of Besov-Morrey type spaces with dominant mixed derivatives.

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Sobolev and Nikolskii's spaces with dominant mixed derivatives (with the difference) $S_p^l W$ and $S_p^l H$ have been introduced and studied by S.M. Nikolskii [1]. By different methods, A.D. Djabrailov [2] and T.I. Amanov [3] have studied the Besov space with dominant mixed derivatives $S_p^l B$. The Sobolev-Liuville space with dominant mixed derivatives $S_p^l L$ were investigated by P.I. Lizorkin and S.M. Nikolskii [4].

Later on, V.P. Il'yin [5] introduced and studied the Sobolev-Morrey space $W_{p,a,æ}^l(G)$. The Nikolskii-Morrey spaces $H_{p,\lambda}^l$ were introduced and studied by I. Ross [6]. By Y.V. Netrusov [7] were introduced the Besov-Morrey space $B_{p,\theta,a,æ}^l(G)$. The spaces of Besov-Morrey type $B_{p,\theta,a,æ,\tau}^l(G)$ and of Lizorkin-Tribel-Morrey type $F_{p,\theta,a,æ,\tau}^l(G)$ were introduced by V.S. Guliev and studied in [8,9]. The space of Besov-Morrey type with dominant mixed derivatives $S_{p,\theta,a,æ,\tau}^l B(G)$ was introduced and studied in [10].

In this article, from the point of view of the imbedding theory we study some properties of functions $f(x)$ belonging to the intersection of Besov-Morrey type spaces with dominant mixed derivatives $S_{p_\mu,\theta_\mu,a,æ,\tau}^{l^\mu} B(G)$ $(\mu = 1, 2, \dots, N)$, where the domain $G \subset R^n$, $p_\mu \in [1,\infty)^n$; $\theta_\mu, \tau \in [1,\infty]$; $a \in [0,1]^n$; $æ, l^\mu \in (0,\infty)^n$, $\mu = 1, 2, \dots, N$.

Let $e_n = \{1, 2, \dots, n\}$, $e \subseteq e_n$, $m = (m_1, m_2, \cdots, m_n)$, m_j be natural, $k = (k_1, k_2, \cdots, k_n)$, k_j be integers for $j \in e_n$; $k^e = (k_1^e, k_2^e, \cdots, k_n^e)$, $k_j^e = k_j$

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for $j\in e$; $k_j^e=0$ for $j\in e_n\backslash e=e'$; $[t_j]_1=\min\{1,t_j\}$, for $j\in e_n$; and let h_0, $t_0\in(0,\infty)^n$ be fixed positive vectors and

$$\int\limits_{a^e}^{b^e} f(x)dx^e=\left(\prod_{j\in e}\int\limits_{a_j}^{b_j} dx_j\right)f(x),$$

i.e., the integration takes place only with respect to the variable x_j, whose indices belong to the set e.

We say that the open set $G\subset R^n$ satisfies the condition (A_1), if for any $x\in G$ and $T\in(0,\infty)^n$ there exists the vector-function

$$\rho(t,x)=(\rho_1(t_1,x),\rho_2(t_2,x),\ldots,\rho_n(t_n,x)),0\le t_j\le T_j,j\in e_n,$$

with the following properties:

1) for all $j\in e_n$, the functions $\rho_j(t_j,x)$ are absolutely continuous with respect to t_j on $[0,T_j]$, and $\left|\rho_j'(t_j,x)\right|\le 1$ for almost all $t_j\in[0,T_j]$, where $\rho_j'(t_j,x)=\frac{\partial}{\partial t_j}\rho_j(t_j,x)$;

2) $\rho_j(0,x)=0$ for all $j\in e_n$,

$$x+V(x,\omega)=x+\bigcup_{\substack{0\le t_j\le T_j,\\ j\in e_n}}[\rho(t,x)+t\omega I]\subset G,$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$, $\omega_j\in(0,1]$ for $j\in e_n$, $I=[-1,1]^n$, $t\omega I=\{(t_1\omega_1y_1,t_2\omega_2y_2,\ldots,t_n\omega_ny_n):y\in I\}$. If $t_1=t^{\lambda_1},\ldots,t_n=t^{\lambda_n}$, $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$, $\rho(t,x)=\rho(t^\lambda,x)$, $\omega^\lambda=(\omega^{\lambda_1},\omega^{\lambda_2},\ldots,\omega^{\lambda_n})$, $\omega\in(0,1]$, then $V(\lambda,x,\omega)=\bigcup\limits_{0\le t\le T}\left[\rho(t^\lambda,x)+t^\lambda\omega^\lambda I\right]$ is a flexible λ-horn introduced by O.V. Besov [11].

Definition. By the space of Becov-Morrey type with the dominant mixed derivatives $S^l_{p,\theta,a,æ,\tau}B(G_h)$ is meant the Banach space of locally summable on G functions f with the finite norm ($m_j>l_j-k_j>0$, $j\in e_n$) :

$$\|f\|_{S^l_{p,\theta,a,æ,\tau}B(G_h)}=\sum_{e\subseteq e_n}\left\{\int\limits_{0^e}^{h_0^e}\left[\frac{\left\|\Delta^{m^e}(h,G_h)D^{k^e}f\right\|_{p,a,æ,\tau}}{\prod\limits_{j\in e}h_j^{l_j-k_j}}\right]^\theta\frac{dh^e}{\prod\limits_{j\in e}h_j}\right\}^{\frac{1}{\theta}},\quad(1)$$

where

$$\left\|\Delta^{m^e}(h,G_h)f\right\|_{p,a,æ,\tau}=\left\|\Delta^{m^e}(h)f\right\|_{p,a,æ,\tau;G_h},$$

$$\Delta^{m^e}(h)f=\left(\prod_{j\in e}\Delta_j^{m_j}(h_j)\right)f(x),\ G_h=\{x:x+hI\subset G\},$$

$$\|f\|_{p,a,æ,\tau;G_h}=$$

$$= \sup_{x\in G} \left\{ \int_0^{t_{01}} \cdots \int_0^{t_{0n}} \left[\prod_{j\in e_n} [t_j]_1^{-\frac{æ_j a_j}{p_j}} \|f\|_{p,(G_h)_{t^{æ}}(x)} \right]^{\tau} \prod_{j\in e_n} \frac{dt_j}{t_j} \right\}^{\frac{1}{\tau}}, \ 1\le\tau<\infty, \quad (2)$$

$$\|f\|_{p,a,æ,\infty;G_h} = \|f\|_{p,a,æ;G_h} = \sup_{x\in G, t>0} \left(\prod_{j\in e_n} [t_j]_1^{-\frac{æ_j a_j}{p_j}} \|f\|_{p,(G_h)_{t^{æ}}(x)} \right),$$

$$(G_h)_{t^{æ}}(x) = \{y : y + x + hI + t^{æ} I \subset G\}.$$

If we take $\Delta^{m^e}(h, G_h)f$ instead of $\Delta^{m^e}(h, G)f$ in the norm (1) we get the norm of the space $S^l_{p,\theta,a,æ,\tau}B(G)$. When $\theta = \infty$ the spaces $S^l_{p,\infty,a,æ,\tau}B(G_h) = S^l_{p,a,æ,\tau}H(G_h)$ will be called the Nicolski-Morrey type spaces with dominant mixed derivatives, and if $\tau = \infty$, the spaces $S^l_{p,\theta,a,æ,\infty}B(G_h) = S^l_{p,\theta,a,æ}B(G_h)$ will be is called the Besov-Morrey spaces with dominant mixed derivatives.

Here we point out some properties of the spaces $S^l_{p,\theta,a,æ,\tau}B(G_h)$.

1) The following embedding holds for arbitrary $æ_j > 0$ and $0 \le a_j \le 1$, for $j \in e_n$:

$$S^l_{p,\theta,a,æ,\tau}B(G_h) \to S^l_{p,\theta,a,æ}B(G_h) \to S^l_{p,\theta}B(G_h).$$

2) For all real $c > 0$ the expression

$$\|f\|_{S^l_{p,\theta,a,cæ,\tau}B(G_h)} \approx \|f\|_{S^l_{p,\theta,a,æ,\tau}B(G_h)} \quad (3)$$

is valid.

3) For every $æ_j > 0$, where $j \in e_n$, the relation

$$\|f\|_{S^l_{p,\theta,0,æ,\infty}B(G_h)} = \|f\|_{S^l_{p,\theta}B(G_h)} \quad (4)$$

is valid.

Let $\beta_\mu \ge 0$ $(\mu = 1, 2, \dots, N)$, $\sum_{\mu=1}^N \beta_\mu = 1$, $\frac{1}{p} = \sum_{\mu=1}^N \frac{\beta_\mu}{p_\mu}$, $\frac{1}{\theta} = \sum_{\mu=1}^N \frac{\beta_\mu}{\theta_\mu}$, $l = \sum_{\mu=1}^N l^\mu \beta_\mu$, $\varepsilon_j = \sum_{\mu=1}^N l_j^\mu \beta_\mu - \nu_j - (1 - æ_j a_j)\left(\frac{1}{p_j} - \frac{1}{q_j}\right)$, $\varepsilon_j^0 = \sum_{\mu=1}^N l_j^\mu \beta_\mu - \nu_j - (1 - æ_j a_j)\frac{1}{p_j}$.

Let $\xi_e \in C_0^\infty(R^{|e|})$, $M_e \in C_0^\infty(R^n \times R^n)$ be uniformly finite with respect to z from an arbitrary compact, i.e., $S(M_e) = supp\ (M_e) \subset I_1 = \{x : |x_j| < 1, j \in e_n\}$, and let $0 < T_j \le 1$, $j \in e_n$. Suppose

$$V = \bigcup_{\substack{0\le t_j \le T_j,\\ j\in e_n}} \left\{ y : \left(\frac{y}{t^e + T^{e'}} \right) \in S(M_e) \right\}.$$

Clearly, $V \subset I_T = \{x : |x_j| < T_j, j \in e_n\}$. By U we denote an open set contained in the domain G, and henceforth it will be always assumed that $U + V \subset G$. Further, let $G_{T^æ}(U) = \bigcup_{x\in U} G_{T^æ}(x)$; note that if $0 < æ_j \le 1$,

$0 < T_j \le 1$, $j \in e_n$, then $I_T \subset I_{T^{æ}}$, since $U+V \subset G$ and $U+V \subset G_{T^{æ}}(U) = Q$.

Lemma 1. *Let* $1 \le p_{\mu j} \le q_{\mu j} \le r_{\mu j} \le \infty$, $\mu = 1,2,\dots,N$, $0 < æ_j \le 1$, $0 < t_j \le T_j \le 1$, $j \in e_n$; $\eta = (\eta_1,\eta_2,\dots,\eta_n)$, $0 < \eta_j \le T_j$, $\nu = (\nu_1,\nu_2,\dots,\nu_n)$, $\nu_j \ge 0$*-are integers,* $j \in e_n$, $\Delta^{m^e}(t)f \in L_{p_\mu,a,æ,\tau}(G)$,

$$B_\eta^e(x) = \prod_{j\in e'} T_j^{-1-\nu_j} \int_{0^e}^{\eta^e} \prod_{j\in e} t_j^{-3-\nu_j} K_e(x, t^e + T^{e'}) dt^e, \tag{5}$$

$$B_{\eta,T}^e(x) = \prod_{j\in e'} T_j^{-1-\nu_j} \int_{\eta^e}^{T^e} \prod_{j\in e} t_j^{-3-\nu_j} K_e(x, t^e + T^{e'}) dt^e, \tag{6}$$

$$K_e(x, t^e + T^{e'}) = \int_{R^n} \int_{-\infty^e}^{\infty^e} M_e^{(\nu)} \left(\frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}} \right) \times$$
$$\times \zeta_e \left(\frac{u}{t}, \frac{\rho(t,x)}{2t}, \frac{1}{2}\rho'(t,x) \right) \Delta^{m^e}(\delta u) f(x+y+u^e) du^e dy.$$

Then the following inequalities hold:

$$\sup_{\overline{x}\in U} \left\| B_\eta^e \right\|_{q,U_{\gamma^{æ}}(\overline{x})} \le C_1 \prod_{\mu=1}^N \left\{ \left\| \prod_{j\in e} t_j^{-l_j^\mu} \Delta^{m^e}(t,Q) f \right\|_{p_\mu,a,æ,\tau} \right\}^{\beta_\mu} \times$$
$$\times \prod_{j\in e'} T_j^{-\nu_j - (1-æ_j a_j)\left(\frac{1}{p_j} - \frac{1}{q_j}\right)} \prod_{j\in e_n} [\gamma_j]_1^{\frac{æ_j a_j}{p_j}} \prod_{j\in e} \eta_j^{\varepsilon_j} (\varepsilon_j > 0), \tag{7}$$

$$\sup_{\overline{x}\in U} \left\| B_{\eta,T}^e \right\|_{q,U_{\gamma^{æ}}(\overline{x})} \le C_2 \prod_{\mu=1}^N \left\{ \left\| \prod_{j\in e} t_j^{-l_j^\mu} \Delta^{m^e}(t,Q) f \right\|_{p_\mu,a,æ,\tau} \right\}^{\beta_\mu} \times$$
$$\times \prod_{j\in e'} T_j^{-\nu_j - (1-æ_j a_j)\left(\frac{1}{p_j} - \frac{1}{q_j}\right)} \prod_{j\in e_n} [\gamma_j]_1^{\frac{æ_j a_j}{p_j}} \begin{cases} \prod_{j\in e} T_j^{\varepsilon_j}, & \varepsilon_j > 0, \\ \prod_{j\in e} \ln \frac{T_j}{\eta_j}, & \varepsilon_j = 0, \\ \prod_{j\in e} \eta_j^{\varepsilon_j}, & \varepsilon_j < 0, \end{cases} \tag{8}$$

where C_1 *and* C_2 *are the constants independent of* f,γ, η, T.

Proof. Assume first that $p_{\mu 1} = p_{\mu 2} = \dots = p_{\mu n} = p_\mu$, $q_{\mu 1} = q_{\mu 2} = \dots = q_{\mu n} = q_\mu$, and $r_{\mu 1} = r_{\mu 2} = \dots = r_{\mu n} = r_\mu$ ($\mu = 1,2,\dots,N$). For the given $\overline{x} \in U$, applying the generalized Minkovskii inequality, we obtain

$$\left\| B_\eta^e \right\|_{q,U_{\gamma^{æ}}(\overline{x})} \le$$

$$\leq C_1 \prod_{j\in e'} T_j^{-1-\nu_j} \int\limits_{0^e}^{\eta^e} \prod_{j\in e} t_j^{-3-\nu_j} \left\| K_e(\cdot, t^e + T^{e'}) \right\|_{q, U_{\gamma æ}(\overline{x})} dt^e, \qquad (9)$$

Estimate the norm $\left\| K_e(\cdot, t^e + T^{e'}) \right\|_{q, U_{\gamma æ}(\overline{x})}$. Applying the Hölder inequality with the exponents

$$\alpha_\mu = \frac{q_\mu}{\beta_\mu q}, \quad \mu = 1, 2, \ldots, N, \quad \left(\sum_{\mu=1}^N \frac{1}{\alpha_\mu} = q \sum_{\mu=1}^N \frac{\beta_\mu}{q_\mu} = 1 \right),$$

for $\left| K_e(x, t^e + T^{e'}) \right|$ we obtain

$$\left\| K_e(\cdot, t^e + T^{e'}) \right\|_{q, U_{\gamma æ}(\overline{x})} \leq C_2 \prod_{\mu=1}^N \left\{ \left\| K_e(\cdot, t^e + T^{e'}) \right\|_{q_\mu, U_{\gamma æ}(\overline{x})} \right\}^{\beta_\mu}. \qquad (10)$$

By virtue of the Hölder's inequality, with regard for $q_\mu \leq r_\mu$, $\mu = 1, 2, \ldots, N$ we have

$$\left\| K_e(\cdot, t^e + T^{e'}) \right\|_{q_\mu, U_{\gamma æ}(\overline{x})} \leq \prod_{j\in e_n} \gamma_j^{\left(\frac{1}{q_\mu} - \frac{1}{r_\mu}\right)} \left\| K_e(\cdot, t^e + T^{e'}) \right\|_{r_\mu, U_{\gamma æ}(\overline{x})}. \qquad (11)$$

Let χ be the characteristic function of the set $S(M_e)$. Note that $1 \leq p_\mu \leq r_\mu \leq \infty$, $s_\mu \leq r_\mu$ $\left(\frac{1}{s_\mu} = 1 - \frac{1}{p_\mu} + \frac{1}{r_\mu}\right)$, and if we represent the integrand in the expression $K_e(x, t^e + T^{e'})$ in the form

$$\left| \int\limits_{-\infty^e}^{\infty^e} M_e^{(\nu)} \zeta_e \Delta^{m^e}(\delta u) f du^e \right| = \left(\left| \int\limits_{-\infty^e}^{\infty^e} \zeta_e \Delta^{m^e}(\delta u\Big) f du^e \right|^{p_\mu} \left| M_e^{(\nu)} \right|^{s_\mu} \right)^{\frac{1}{r_\mu}} \times$$

$$\times \left(\left| \int\limits_{-\infty^e}^{\infty^e} \zeta_e \Delta^{m^e}(\delta u) f du^e \right|^{p_\mu} \chi\Big(\frac{y}{t^e + T^{e'}}\Big) \right)^{\frac{1}{p_\mu} - \frac{1}{r_\mu}} \left(\left| M_e^{(\nu)} \right|^{s_\mu} \right)^{\frac{1}{s_\mu} - \frac{1}{r_\mu}}$$

and apply Hölder's inequality for $\left| K_e(x, t^e + T^{e'}) \right|$ (by virtue of $\frac{1}{r_\mu} + \left(\frac{1}{p_\mu} - \frac{1}{r_\mu}\right) + \left(\frac{1}{s_\mu} - \frac{1}{r_\mu}\right) = 1$), we obtain

$$\left\| K_e(\cdot, t^e + T^{e'}) \right\|_{r_\mu, U_{\gamma æ}(\overline{x})} \leq \sup_{x\in U_{\gamma æ}(\overline{x})} \Bigg(\int\limits_{R^n} \Bigg| \int\limits_{-\infty^e}^{\infty^e} \zeta_e\Big(\frac{u}{t}, \frac{\rho(t,x)}{2t}, \frac{1}{2}\rho'(t,x)\Big) \times$$

$$\times \Delta^{m^e}(\delta u) f(x + y + u^e) du^e \Bigg|^{p_\mu} \chi\Big(\frac{y}{t^e + T^{e'}}\Big) dy \Bigg)^{\frac{1}{p_\mu} - \frac{1}{r_\mu}} \times$$

$$\times \sup_{y\in V}\Big(\int\limits_{U_{\gamma æ}(\overline{x})}\Big|\int\limits_{-\infty^e}^{\infty^e}\zeta_e\Big(\frac{u}{t},\frac{\rho(t,x)}{2t},\frac{1}{2}\rho'(t,x)\Big)\times$$

$$\times\Delta^{m^e}(\delta u)f(x+y+u^e)du^e\Big|^{p_\mu}dx\Big)^{\frac{1}{r_\mu}}\times$$

$$\times\Big(\int\limits_{R^n}\Big|M_e^{(\nu)}\Big(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}}\Big)\Big|^{s_\mu}dy\Big)^{\frac{1}{s_\mu}}. \tag{12}$$

Since $U+V\subset Q$, $Q_{t^e+T^{e'}}(x)\subset Q_{t^{æ^e}+T^{æ^{e'}}}(x)$ for arbitrary $0<æ_j\le 1$, $0<t_j\le T_j\le 1$, $j\in e_n$, and $x\in U$, we have

$$\int\limits_{R^n}\Big|\int\limits_{-\infty^e}^{\infty^e}\zeta_e\Big(\frac{u}{t},\frac{\rho(t,x)}{2t},\frac{1}{2}\rho'(t,x)\Big)\Delta^{m^e}(\delta u)f(x+y+u^e)du^e\Big|^{p_\mu}\times$$

$$\times\chi\Big(\frac{y}{t^e+T^{e'}}\Big)dy\le\int\limits_{Q_{t^{æ^e}+T^{æ^{e'}}}(x)}\Big|\int\limits_{-\infty^e}^{\infty^e}\zeta_e\Big(\frac{u}{t},\frac{\rho(t,x)}{2t},\frac{1}{2}\rho'(t,x)\Big)\times$$

$$\times\Delta^{m^e}(\delta u)f(x+y+u^e)du^e\Big|^{p_\mu}dy\le$$

$$\le\prod_{j\in e}t_j^{p_\mu l_j^\mu+p_\mu}\Big\|\prod_{j\in e}t_j^{-l_j^\mu}\Delta^{m^e}(t,Q)f\Big\|^{p_\mu}_{p_\mu,a,æ}\prod_{j\in e'}T_j^{æ_j a_j}\prod_{j\in e}t_j^{æ_j a_j}, \tag{13}$$

where $y\in V$

$$\int\limits_{U_{\gamma æ}(\overline{x})}\Big|\int\limits_{-\infty^e}^{\infty^e}\zeta_e\Big(\frac{u}{t},\frac{\rho(t,x)}{2t},\frac{1}{2}\rho'(t,x)\Big)\Delta^{m^e}(\delta u)f(x+y+u^e)du^e\Big|^{p_\mu}dx\le$$

$$\le\int\limits_{Q_{t^{æ^e}+T^{æ^{e'}}}(\overline{x}+y)}\Big|\int\limits_{-\infty^e}^{\infty^e}\zeta_e\Big(\frac{u}{t},\frac{\rho(t,x)}{2t},\frac{1}{2}\rho'(t,x)\Big)\Delta^{m^e}(\delta u)f(x+y+u^e)du^e\Big|^{p_\mu}dx\le$$

$$\le\prod_{j\in e}t_j^{p_\mu l_j^\mu+p_\mu}\Big\|\prod_{j\in e}t_j^{-l_j^\mu}\Delta^{m^e}(t,Q)f\Big\|^{p_\mu}_{p_\mu,a,æ}\prod_{j\in e_n}[\gamma_j]_1^{æ_j a_j}, \tag{14}$$

$$\int\limits_{R^n}\Big|M_e^{(\nu)}\Big(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}}\Big)\Big|^{s_\mu}dy=\prod_{j\in e'}T_j\prod_{j\in e}t_j\,\Big\|M_e^{(\nu)}\Big\|^{s_\mu}_{s_\mu}. \tag{15}$$

It follows from (10)-(15) that

$$\left\|K_e(\cdot,t^e+T^{e'})\right\|_{q,U_{\gamma^{æ}}(\overline{x})}\le C_3\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t_j^{-l_j^{\mu}}\Delta^{m^e}(t,Q)f\right\|_{p_\mu,a,æ}\right\}^{\beta_\mu}\times$$
$$\times\prod_{j\in e'}T_j^{-\nu_j-(1-æ_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)}\prod_{j\in e_n}[\gamma_j]_1^{\frac{æ_j a_j}{p}}\times$$
$$\times\prod_{j\in e_n}\gamma_j^{\left(\frac{1}{q}-\frac{1}{r}\right)}\prod_{j\in e}t_j^{l_j+2-(1-æ_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)}. \tag{16}$$

Taking into account the inequality $\|\cdot\|_{p,a,æ}\le c\,\|\cdot\|_{p,a,æ,\tau}$ and using (16) with respect to every variable, we obtain the following inequality:

$$\left\|K_e(\cdot,t^e+T^{e'})\right\|_{q,U_{\gamma^{æ}}(\overline{x})}\le C_4\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t_j^{-l_j^{\mu}}\Delta^{m^e}(t,Q)f\right\|_{p_\mu,a,æ,\tau}\right\}^{\beta_\mu}\times$$
$$\times\prod_{j\in e'}T_j^{-\nu_j-(1-æ_j a_j)\left(\frac{1}{p_j}-\frac{1}{r_j}\right)}\prod_{j\in e_n}[\gamma_j]_1^{\frac{æ_j a_j}{p_j}}\times$$
$$\times\prod_{j\in e_n}\gamma_j^{\left(\frac{1}{q_j}-\frac{1}{r_j}\right)}\prod_{j\in e}t_j^{l_j+2-(1-æ_j a_j)\left(\frac{1}{p_j}-\frac{1}{r_j}\right)} \tag{17}$$

for $p_\mu=(p_{\mu1},p_{\mu2},\dots,p_{\mu n})$, $q_\mu=(q_{\mu1},q_{\mu2},\dots,q_{\mu n})$ and $r_\mu=(r_{\mu1},r_{\mu2},\dots,r_{\mu n})$, $(\mu=1,2,\dots,N)$ Substituting (17) in (9) for $r_j=q_j$,$j\in e_n$, we arrive at (7). In a similar way we can prove inequality (8). □

Lemma 2. *Let $1\le p_{\mu j}\le q_{\mu j}<\infty$, $\mu=1,2,\dots,N$; $0<æ_j\le 1$, $0<t_j\le T_j\le 1$, $j\in e_n$; $1\le\tau_1\le\tau_2\le\infty$ and $\varepsilon_j>0$. Then for every function $B_T^e(x)$ defined by* (5) *the estimate*

$$\|B_T^e\|_{q,b,æ,\tau_2;U}\le C\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t_j^{-l_j^{\mu}}\Delta^{m^e}(t,Q)f\right\|_{p_\mu,a,æ,\tau}\right\}^{\beta_\mu} \tag{18}$$

is valid, where C is the constant, independent of f, and $b=(b_1,b_2,\dots,b_n)$, b_j is an arbitrary number satisfying the inequalities

$0\le b_j\le 1$, if $\varepsilon_{j,0}>0$ for $j\in e$;
$0\le b_j<1$, if $\varepsilon_{j,0}=0$ for $j\in e$; $0\le b_j\le a_j$ for $j\in e_n\backslash e=e'$;
$0\le b_j<1+\frac{\varepsilon_{j,0}q_j(1-a_j)}{1-æ_j a_j}$, if $\varepsilon_{j,0}<0$ for $j\in e$.

Theorem 1. *Assume that an open set G satisfies the condition (A_1), $1\le p_{\mu j}\le q_{\mu j}\le\infty$, $j\in e_n$, $1\le\theta_\mu\le\infty$, $\mu=1,2,\dots,N$; $\overline{æ}=cæ$, $\frac{1}{c}=\max\limits_{j\in e_n}l_jæ_j$, $\nu=(\nu_1,\nu_2,\dots,\nu_n)$, $\nu_j\ge 0$ be integers , $j\in e_n$, $1\le\tau_1\le\tau_2\le\infty$, $f\in\bigcap\limits_{\mu=1}^{N}S^{l^{\mu}}_{p_\mu,\theta_\mu,a,æ,\tau}B(G_h)$, and let $\varepsilon_j>0$ for $j\in e_n$. Then there exists the* *generalized derivative* $D^\nu f$ *in the domain* G *and the following inequalities are valid:*

$$\|D^\nu f\|_{q,G} \le C_1 A(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,æ,\tau}B(G_h)} \right\}^{\beta_\mu}, \tag{19}$$

$$\|D^\nu f\|_{q,b,æ,\tau_2;G} \le C_2 \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,æ,\tau}B(G_h)} \right\}^{\beta_\mu}, \tag{20}$$

$$p_{\mu j} \le q_{\mu j} < \infty, \quad j \in e_n,$$

where

$$A(T) = \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}}, \quad s_{e,j} = \begin{cases} \varepsilon_j, & j \in e \\ -\nu_j - (1 - æ_j a_j)\left(\frac{1}{p_j} - \frac{1}{q_j}\right), & j \in e'. \end{cases}$$

In particular, if $\varepsilon_{j,0} > 0$, $j \in e_n$, *then* $D^\nu f$ *is continuous in* G *and*

$$\sup_{x\in G} |D^\nu f| \le C_1 A(T^0) \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,æ,\tau}B(G_h)} \right\}^{\beta_\mu}, \tag{21}$$

where $A(T^0) = \sum\limits_{e\subseteq e_n} \prod\limits_{j\in e_n} T_j^{s^0_{e,j}}$, $s^0_{e,j} = \begin{cases} \varepsilon_{j,0}, & j \in e \\ -\nu_j - (1 - æ_j a_j)\frac{1}{p_j}, & j \in e', \end{cases}$,
$T_j \in (0, \min(1, t_{0,j})]$, $j \in e_n$, $C_1 C_2$ *are the constants independent of* f, *and* C_1 *is independent of* T *as well.*

Proof. First of all, it should be noted that since $\overline{æ} = cæ$, $c > 0$, using Property 2), we can assume that $f \in \bigcap\limits_{\mu=1}^{N} S^{l^\mu}_{p_\mu,\theta_\mu,a,cæ,\tau}B(G_h)$, and in inequalities (19)-(21) and in the expression for ε_j we can replace $æ_j$, $j \in e_n$, by $\overline{æ}_j$, $j \in e_n$. Here we will prove these very inequalities (the greater $æ_j$, the greater ε_j, $j \in e_n$). Let $f \in \bigcap\limits_{\mu=1}^{N} S^{l^\mu}_{p_\mu,\theta_\mu,a,\overline{æ},\tau}B(G_h)$, then $f \in S^{l^\mu}_{p_\mu,\theta_\mu,a,\overline{æ},\tau}B(G_h)$ for all $\mu = 1, 2, \dots, N$. The existence of the generalized derivative $D^\nu f$ under the conditions of the theorem follows from Theorem 1 of [10]. Then for almost every point $x \in G$ the following equality holds:

$$D^\nu f(x) = \sum_{e\subseteq e_n} (-1)^{|\nu|} \prod_{j\in e'} T_j^{-1-\nu_j} \times$$

$$\times \int_{0^e}^{T^e} \frac{dt^e}{\prod\limits_{j\in e} t_j^{3+\nu_j}} \int_{R^n} \int_{-\infty^e}^{\infty^e} M_e^{(\nu)}\Big(\frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}}\Big) \times$$

$$\times \zeta_e\Big(\frac{u}{t}, \frac{\rho(t,x)}{2t}, \frac{1}{2}\rho'(t,x)\Big) \Delta^{m^e}(\delta u) f(x + y + u^e) du^e dy, \tag{22}$$

the support of the representation (22) is contained in the set $x+V\subset G$ and the parameter of the representation $\delta>0$ is assumed to be sufficiently small. Therefore $\Delta^{m^e}(\delta u, G_{\delta u})f=\Delta^{m^e}(\delta u)f$. On the basis of Minkovski's inequality, we have

$$\|D^\nu f\|_{q,G}\le C_1\sum_{e\subseteq e_n}\|B_T^e\|_{q,G}\,. \tag{23}$$

By means of (7), for $U=G$, $\eta=T$ as $\gamma\to\infty$, we find that

$$\|B_T^e\|_{q,G}\le C_2\prod_{j\in e_n}T_j^{s_{e,j}}\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t_j^{-l_j^\mu}\Delta^{m^e}(t,G_t)f\right\|_{p_\mu,a,\overline{æ},\tau}\right\}^{\beta_\mu}.$$

Then from inequality (23) we have

$$\|D^\nu f\|_{q,G}\le C_3\sum_{e\subseteq e_n}\prod_{j\in e_n}T_j^{s_{e,j}}\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t_j^{-l_j^\mu}\Delta^{m^e}(t,G_t)f\right\|_{p_\mu,a,\overline{æ},\tau}\right\}^{\beta_\mu},$$

whence with regard for $1\le\theta_\mu\le\infty$, $\mu=1,2,\dots,N$ we get inequality (19). Similarly, using (18), we obtain estimate (20).

Assume now that $\varepsilon_{j,0}>0$, $j\in e_n$. Let us show that $D^\nu f$ is continuous in G. On the basis of identity (22) and inequality (18), for $q_j=\infty$, $\varepsilon_j=\varepsilon_{j,0}>0$, $j\in e_n$, we have

$$\|D^\nu f-D^\nu f_T\|_{\infty,G}\le$$
$$\le C_1\sum_{\emptyset\ne e\subseteq e_n}\prod_{j\in e_n}T_j^{s_{e,j}}\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t_j^{-l_j^\mu}\Delta^{m^e}(t,G_t)f\right\|_{p_\mu,a,\overline{æ},\tau}\right\}^{\beta_\mu},$$

whence it follows that the left-hand side of the inequality tends to zero as $T_j\to0$, $j\in e$. Since $D^\nu f_T$ is continuous in G, in this case the convergence of $L_\infty(G)$ coincides with the uniform convergence and, consequently, the limit function $D^\nu f$ is continuous in G. □

Let ξ be an n-dimensional vector.

Theorem 2. *Let the domain G, the vectors p_μ,q_μ, ($\mu=1,2,\dots,N$), $\overline{æ}$, ν and parameters τ, θ_μ, ($\mu=1,2,\dots,N$), satisfy the conditions of Theorem 1. If $\varepsilon_j>0$,$j\in e_n$, then the derivative $D^\nu f$ satisfies the Holder condition in G in the metric of L_q with exponent σ_j^1, or more exactly,*

$$\|\Delta(\xi,G)D^\nu f\|_{q,G}\le C\prod_{\mu=1}^{N}\left\{\|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,æ,\tau}B(G_h)}\right\}^{\beta_\mu}\prod_{j\in e_n}|\xi_j|^{\sigma_j^1}\,, \tag{24}$$

where σ_j^1 is an arbitrary number satisfying the inequalites:
$0\le\sigma_j^1\le1$, *if $\varepsilon_j>1$ for $j\in e$;*
$0\le\sigma_j^1<1$, *if $\varepsilon_j=1$ for $j\in e$; $0\le\sigma_j^1\le1$ for $j\in e'$;*
$0\le\sigma_j^1\le\ \varepsilon_j$, *if $\varepsilon_j<1$ for $j\in e$.*

If $\varepsilon_{j,0}>0$ $(j\in e_n)$, *then*

$$\sup_{x\in G}|\Delta(\xi,G)D^{\nu}f|\leq C\prod_{\mu=1}^{N}\left\{\|f\|_{S^{l^{\mu}}_{p_\mu,\theta_\mu,a,æ,\tau}B(G_h)}\right\}^{\beta_\mu}\prod_{j\in e_n}|\xi_j|^{\sigma_j^{1,0}}, \qquad (25)$$

where $\sigma_j^{1,0}$ *satisfy the same conditions as* σ_j^1, *but with the substitution of* ε_j *by* $\varepsilon_{j,0}$.

Proof. Just as in the proof of Theorem 1, below we can replace the vector æ by $\overline{æ}=cæ$. By Lemma 8.6 [11], there is a domain $G_z\subset G$ ($z=(z_1,z_2,\ldots,z_n)$, $z_j=\alpha_j r(x)$, $\alpha_j>0$, $j\in e_n$, $r(x)=dist(x,\partial G)$, $x\in G$). Suppose that $|\xi_j|<z_j$ $(j\in e_n)$, then for every $x\in G_\zeta$ the segment with endpoints x and $x+\xi$ lies in G. Consequently, (22) is valid for all points of that segment with the same kernels. After several transformations we obtain

$$|\Delta(\xi,G)D^{\nu}f|\leq C_1\sum_{e\subseteq e_n}\prod_{j\in e'}T_j^{-1-\nu_j}\times$$

$$\times\int_{0^e}^{|\xi^e|}\frac{dt^e}{\prod\limits_{j\in e}t_j^{3+\nu_j}}\int_{R^n}\int_{-\infty^e}^{\infty^e}\left|M_e^{\nu}\Big(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}}\Big)\right|\times$$

$$\times\left|\zeta_e\Big(\frac{u}{t},\frac{\rho(t,x)}{2t},\frac{1}{2}\rho'(t,x)\Big)\right|\left|\Delta(\xi,G)\Delta^{m^e}(\delta u)f(x+y+u^e)\right|du^e dy+$$

$$+C_2\sum_{e\subseteq e_n}\prod_{j\in e'}T_j^{-1-\nu_j}\prod_{j\in e}|\xi_j|\int_{|\xi^e|}^{T^e}\frac{dt^e}{\prod\limits_{j\in e}t_j^{4+\nu_j}}\times$$

$$\times\int_{R^n}\int_{-\infty^e}^{\infty^e}\left|M_e^{(\nu+1)}\Big(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}}\Big)\right|\left|\zeta_e\Big(\frac{u}{t},\frac{\rho(t,x)}{2t},\frac{1}{2}\rho'(t,x)\Big)\right|\times$$

$$\times\int_0^1\Delta^{m^e}(\delta u)f(x+y+u^e+\upsilon_1\xi_1+\upsilon_2\xi_2+\cdots\upsilon_n\xi_n)d\upsilon du^e dy=$$

$$=C_1\sum_{e\subseteq e_n}B_1^e(x,\xi)+C_2\sum_{e\subseteq e_n}B_2^e(x,\xi); \qquad (26)$$

where $|\xi^e|=(|\xi_1^e|,|\xi_2^e|,\ldots,|\xi_n^e|)$, $|\xi_j^e|=|\xi_j|$ for $j\in e$, $|\xi_j^e|=0$ for $j\in e_n'$ and $0<T_j\leq\min(1,t_{0,j})$ for $j\in e_n$. We can consider, that $|\xi_j|<T_j$ for $j\in e_n$, cosequently, $|\xi_j|<\min(z_j,T_j)$, $j\in e_n$. if $x\in G\backslash G_z$, then by definition,

$$\Delta(\xi,G)D^{\nu}f(x)=0.$$

By (26) we have

$$\|\Delta(\xi,G)D^{\nu}f\|_{q,G}=\|\Delta(\xi,G)D^{\nu}f\|_{q,G_z}\leq$$

$$\leq C_1 \sum_{e\subseteq e_n} \|B_1^e(\cdot,\xi)\|_{q,G_z} + C_2 \sum_{e\subseteq e_n} \|B_2^e(\cdot,\xi)\|_{q,G_z}. \tag{27}$$

Using inequality (7) for $U = G$, $\eta_j = |\xi_j|$, as $\gamma \to \infty$, we obtain

$$\|B_1^e(\cdot,\xi)\|_{q,G_z} \leq$$

$$\leq C_3 \prod_{j\in e_n} |\xi_j|^{\varepsilon_j} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e} t_j^{-l_j^\mu} \Delta^{m^e}(t,G_t) f \right\|_{p_\mu,a,\overline{æ},\infty} \right\}^{\beta_\mu}, \tag{28}$$

and by inequality (8) for $U = G$, $\eta_j = |\xi_j|$, as $\gamma \to \infty$, we find that

$$\|B_2^e(\cdot,\xi)\|_{q,G_z} \leq C_4 \prod_{j\in e_n'} |\xi_j|^{\varepsilon_j} \prod_{j\in e_n} |\xi_j|^{\varepsilon_j - 1} \times$$

$$\times \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e} t_j^{-l_j^\mu} \Delta^{m^e}(t,G_t) f \right\|_{p_\mu,a,\overline{æ},\infty} \right\}^{\beta_\mu} \leq$$

$$\leq C_5 \prod_{j\in e_n} |\xi_j|^{\sigma_j} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e} t_j^{-l_j^\mu} \Delta^{m^e}(t,G_t) f \right\|_{p_\mu,a,\overline{æ},\infty} \right\}^{\beta_\mu} =$$

$$= C_6 \prod_{j\in e_n} |\xi_j|^{\sigma_j^1} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e} t_j^{-l_j^\mu} \Delta^{m^e}(t,G_t) f \right\|_{p_\mu,a,\overline{æ},\tau} \right\}^{\beta_\mu}, \tag{29}$$

where $\sigma_j^1 > \sigma_j$ for $j \in e_n$. From inequalities (27)-(29), under the condition $1 \leq \theta_\mu \leq \infty$, $\mu = 1, 2, \dots, N$, we obtain inequality (24).

Suppose now that $|\xi_j| \geq \min(z_j, T_j)$, $j \in e_n$. Then

$$\|\Delta(\xi,G) D^\nu f\|_{q,G} \leq 2\, \|D^\nu f\|_{q,G} \leq C(z,T)\, \|D^\nu f\|_{q,G} \prod_{j\in e_n} |\xi_j|^{\sigma_j^1}.$$

In this case, estimating $\|D^\nu f\|_{q,G}$ by means of (19), we again obtain the desired inequality. □

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