

ABSOLUTE CONVERGENCE OF THE FOURIER SERIES AND THE FOURIER–HAAR COEFFICIENTS

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ABSTRACT. In the present paper we study the questions on the absolute convergence of general orthonormal Fourier series for functions with finite variation.

We have also obtained the results regarding the Fourier-Haar coefficients and proved that a series of absolute values of the Fourier-Haar coefficients and the variation of the same coefficients have, in some sense, the same properties.

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1. INTRODUCTION

Let $f : [0, 1] \rightarrow R$. As usual, By V we denote a class of functions with finite variations on $[0, 1]$ and by $V_0^1(f)$ the variation of a function f on $[0, 1]$.

If

$$W_0^1(f) = \sup_{\Pi} \left(\sum_{k=1}^{n-1} |f(x_{k+1}) - 2f(x_k) + f(x_{k-1}))| \right) < +\infty,$$

then $f \in W$, where $\Pi = (a = x_0 < x_1 < \dots < x_n = b)$ is an arbitrary partitioning of the segment $[0, 1]$.

Let $0 < p < 2$ be an arbitrary number. The symbol V_p denotes a class of all sequences (a_n) for which

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n|^p < +\infty.$$

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If, however,

$$\sum_{n=1}^{\infty} |a_n|^p < +\infty.$$

then in this case as conventionally $(a_n) \in \ell_p$.

By $c_n(f)$ we denote the Fourier coefficient of the function f with respect to the orthonormalized on $[0, 1]$ system (φ_n) , i.e.

$$c_n(f) = \int_0^1 f(x) \varphi_n(x) dx, \quad n = 1, 2, \dots$$

Suppose that for $p \in (0, 2)$ the relation

$$\sum_{n=1}^{\infty} |c_n(f)|^p < \infty.$$

takes place. Then $f \in A(p, \varphi)$.

The definition of the Haar system is well-known (see [1], pp.54–55).

STATEMENT OF THE BASIC PROBLEMS

The questions of the absolute convergence of general Fourier series have been investigated by various authors. Among these questions there are such which do not require additional conditions on a system of functions to obtain some definite result. A large amount of work in this direction have been done by S. V. Stechkin [2], A. M. Olevskii [3], S. V. Bochkarev [4], B. S. Mitjagin [5], etc. In some cases, especially when we study the functions with differential properties, we have to impose certain conditions on the given system. The rather that otherwise, as we will see below, the statement of the problem is devoid sense. Among such kind of fact the results obtained by L. Bosanquet and H. Kestelman [7], J. R. Mc Laughlin [8] are worth while mentioning.

Here we present S. B. Olevski's theorem [3] which, in particular, shows that differential properties of the function do not affect the order of divergence of its Fourier coefficients with respect to general orthogonal systems.

Theorem A. *Let $(c_n) \in \ell_2$ be an arbitrary sequence and $f \in L^2(0, 1)$ be an arbitrary function. It is assumed that the equality*

$$\sum_{n=1}^{\infty} c_n^2 = \int_0^1 f^2(x) dx.$$

is fulfilled.

Then there exists a complete in $L^2(0,1)$ orthonormalized system (φ_n) such that

$$\int_0^1 f(x)\varphi_n(x) dx = c_n \quad (n = 1, 2, \dots).$$

Let now $f_0 \in V$ be a function and $(c_n) \in \ell_2$ an arbitrary sequence satisfying the condition $(c_n) \notin \ell_p$ for some $p \in (0, 2)$.

Assume

$$\sum_{n=1}^{\infty} c_n^2 = \int_0^1 f^2(x) dx.$$

Then by Theorem A, there exists an orthonormalized on $[0, 1]$ system (φ_n) such that

$$\int_0^1 f_0(x)\varphi_n(x) dx = c_n \quad (n = 1, 2, \dots).$$

Consequently,

$$c_n = c_n(f_0) \quad n = 1, 2, \dots$$

whence, because

$$(c_n) \notin \ell_p,$$

we conclude that

$$f_0 \notin A(p, \varphi).$$

Thus the series

$$\sum_{n=1}^{\infty} |c_n(f)|^p \quad (p \in (0, 2))$$

converges for $f \in V$ only under certain conditions imposed on the system (φ_n) .

Clearly, the conditions must be efficient, i.e., easily verifiable at least for classical orthogonal systems.

In the present paper the necessary and sufficient conditions are found for the given on $[0, 1]$ orthonormalized system (φ_n) for the Fourier series of the function $f \in V$ to satisfy the condition

$$\sum_{n=1}^{\infty} |c_n(f)|^p < \infty,$$

for $p \in (0, 2)$.

The efficiency of the obtained condition is illustrated by examples.

Properties of the Haar system and the Fourier coefficients for that system are well studied. Fundamental results concerning these problems have been obtained by P. L. Ul'janov [10]. Here we note some of them (see [9] and [10]).

Theorem B. *If $f \in V$, then*

$$\sum_{m=1}^{\infty} |c_m(f)|^{\gamma} < \infty,$$

for $\gamma > \frac{2}{3}$. If, however, $\gamma = \frac{2}{3}$, then the series diverges for some function $f_0 \in V$.

Theorem C. *If $f \in V$, then*

$$\sum_{m=1}^{\infty} m^{\gamma} |c_m(f)| < \infty,$$

for all $\gamma < \frac{1}{2}$. For $\gamma = \frac{1}{2}$ the convergence does not take place for some $f_0 \in V$.

The result of M. Khabazi [13] who generalized Theorem B is worth mentioning. It consists in the follows

Theorem D. *If $f \in V_p^{(2)}$, ($p \geq 1$), then*

$$\sum_{m=1}^{\infty} |c_m(f)|^{\gamma} < \infty$$

for $\gamma > \frac{2p}{2+p}$.

Here $V_p^{(2)}$ is the class of real measurable functions on $[0, 1]$ for which the sum

$$\sum_{k=0}^{n-1} |\Delta^2(x_k, x_{k+1})|^p, \quad (p > 0)$$

where

$$\Delta^2(\alpha, \beta) = f(\alpha) + f(\beta) - 2f\left(\frac{\alpha + \beta}{2}\right),$$

remains bounded for all possible partitionings of the segment $[0, 1]$ by parts:

$$0 < x_0 < x_1 < \dots < x_n = 1.$$

In the present work the theorems which generalize Theorems A and B to the difference of the Fourier-Haar coefficients of functions of the class W are proved. More precisely, we study the convergence of the series

$$\sum_{m=1}^{\infty} |c_m(f) - c_{m+1}(f)|^{\gamma}, \quad \text{and} \quad \sum_{m=1}^{\infty} |c_m(f) - c_{m+1}(f)| m^{\beta},$$

where $c_m(f)$ are the Fourier-Haar coefficients of functions from the class W .

We also prove that there exist the functions f_1 and f_2 from the class V , such that not only the series

$$\sum_{n=1}^{\infty} |c_n(f_1)|^{\frac{2}{3}} \quad \text{and} \quad \sum_{n=1}^{\infty} |c_n(f_2)| n^{\frac{1}{2}},$$

diverge, but, strange as it may seem, also the series

$$\sum_{n=1}^{\infty} |c_n(f_1) - c_{n+1}(f_1)|^{\frac{2}{3}} \quad \text{and} \quad \sum_{n=1}^{\infty} |c_n(f_2) - c_{n+1}(f_2)| n^{\frac{1}{2}}.$$

ABSOLUTE CONVERGENCE

Let (φ_n) be the orthonormalized on $[0, 1]$ system of functions with properties $1 \in A(p, \varphi)$ and $(a_k) \in \ell_q$ for some $q > 1$.

Introduce the notation

$$K_n = \max_{x \in [0, 1]} \left| \int_0^x p_n(t) dt \right|,$$

where

$$p_n(t) = \sum_{k=1}^n a_k \varphi_k(t).$$

Theorem 1. *For the condition $f \in V$ to imply the inclusion $f \in A(p, \varphi)$ for some $p \in (1, 2)$, it is necessary and sufficient that the inequality*

$$K_n < M.$$

take place for any sequence $(a_k) \in \ell_q$ for $\frac{1}{p} + \frac{1}{q} = 1$ and $M > 0$ does not depend on n .

Proof. The sufficiency. As is known, for the finite function $f \in L_2$ and for $\Phi \in L_2$ (see [11]) the equality

$$\begin{aligned} \int_0^1 f(x) \Phi(x) dx &= \sum_{k=1}^{n-1} \left(f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n}\right) \right) \int_0^{\frac{k}{n}} \Phi(x) dx + \\ &+ \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(f(x) - f\left(\frac{k}{n}\right) \right) \Phi(x) dx + f(1) \int_0^1 \Phi(x) dx. \end{aligned} \quad (1)$$

holds.

Let $f \in V$ be some function, $(a_k) \in \ell_q$ any sequence ($\frac{1}{p} + \frac{1}{q} = 1$), and $c_n(f)$ the Fourier coefficients of the function f . Then the equality

$$\sum_{k=1}^n c_k(f) a_k = \int_0^1 f(x) \sum_{k=1}^n a_k \varphi_k(x) dx = \int_0^1 f(x) p_n(x) dx. \quad (2)$$

is valid. $\square$

Substituting $\Phi(x) = p_n(x)$ into equality (1), we obtain (see [2]):

$$\begin{aligned} \sum_{k=1}^n c_k(f) a_k &= \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{\frac{i}{n}} p_n(x) dx + \\ &+ \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(f(x) - f\left(\frac{i}{n}\right) \right) p_n(x) dx + f(1) \int_0^1 p_n(x) dx. \end{aligned} \quad (3)$$

By the condition of Theorem 1, we have

$$\begin{aligned} &\left| \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{\frac{i}{n}} p_n(x) dx \right| \leq \\ &\leq \max_{1 \leq i \leq n} \left| \int_0^{\frac{i}{n}} p_n(x) dx \right| \sum_{i=1}^{n-1} \left| \left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right| \leq \\ &\leq V_0^1(f) \cdot K_n < V_0^1(f) \cdot M. \end{aligned} \quad (4)$$

Applying then Hölder's inequality, we arrive at

$$\begin{aligned} &\left| \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(f(x) - f\left(\frac{i}{n}\right) \right) p_n(x) dx \right| \leq \\ &\leq \sum_{i=1}^n \sup_{x \in [\frac{i-1}{n}, \frac{i}{n}]} \left| f(x) - f\left(\frac{i}{n}\right) \right| \int_{\frac{i-1}{n}}^{\frac{i}{n}} |p_n(x)| dx \leq \\ &\leq V_0^1(f) \max_{1 \leq i \leq n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left| \sum_{k=1}^n a_k \varphi_k(x) \right| dx \leq \\ &\leq V_0^1(f) \frac{1}{\sqrt{n}} \left(\int_0^1 \left(\sum_{k=1}^n a_k \varphi_k(x) \right)^2 dx \right)^{1/2} \leq \end{aligned}$$

$$\leq V_0^1(f) \cdot \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n a_k^2 \right)^{1/2} < B, \quad (5)$$

where $B > 0$ does not depend on n .

Finally, since $(a_k) \in \ell_q$ and $1 \in A(p, \varphi)$, we have

$$\begin{aligned} \left| \int_0^1 p_n(x) dx \right| &= \left| \int_0^1 \sum_{k=1}^n a_k \varphi_k(x) dx \right| = \left| \sum_{k=1}^n a_k \int_0^1 \varphi_k(x) dx \right| \leq \\ &\leq \left(\sum_{k=1}^n |a_k|^q \right)^{1/q} \left(\sum_{k=1}^n \left| \int_0^1 \varphi_k(x) dx \right|^p \right)^{1/p} < C, \end{aligned} \quad (6)$$

where $C > 0$ does not depend on n .

Taking into account (4), (5) and (6), from (3) we get

$$\sum_{k=1}^n |c_k(f) a_k| < V_0^1(f) M + B + |f(1)| \cdot C < D. \quad (7)$$

where $D > 0$ does not depend on n .

Because of the fact that (7) is fulfilled for any $(a_k) \in \ell_q$, the series

$$\sum_{k=1}^{\infty} |c_k(f)|^p$$

converges, i.e., $f \in A(p, \varphi)$.

The necessity. Suppose that for some sequence of numbers $(a_k) \in \ell_q$ ($q > 1$) the equality

$$\overline{\lim}_{n \rightarrow \infty} K_n = +\infty.$$

is fulfilled.

Let for some $x_n \in [\frac{i_n}{n}, \frac{i_n+1}{n}]$ there be given the equality

$$K_n = \left| \int_0^{x_n} p_n(t) dt \right|.$$

Then

$$\begin{aligned} \left| \int_0^{x_n} p_n(t) dt \right| &\leq \left| \int_0^{\frac{i_n}{n}} p_n(t) dt \right| + \left| \int_{\frac{i_n}{n}}^{x_n} p_n(t) dt \right| \leq \\ &\leq \left| \int_0^{\frac{i_n}{n}} p_n(t) dx \right| + \left| \frac{i_n}{n} - x_n \right|^{\frac{1}{2}} \left(\int_0^1 p_n^2(t) dt \right)^{\frac{1}{2}} \leq \end{aligned}$$

$$\leq \left| \int_0^{\frac{i_n}{n}} p_n(t) dt \right| + \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n a_k^2 \right)^{1/2} \leq \left| \int_0^{\frac{i_n}{n}} p_n(t) dt \right| + C, \quad (8)$$

where $C > 0$ does not depend on n .

Consequently (see (8)),

$$\overline{\lim}_{n \rightarrow \infty} \left| \int_0^{\frac{i_n}{n}} p_n(t) dt \right| = +\infty.$$

Let us consider the continuous on $[0, 1]$ function

$$f_n(x) = \begin{cases} 0 & \text{for } x \in [0, \frac{i_n}{n}], \\ 1 & \text{for } x \in [\frac{i_n+1}{n}, 1], \\ \text{linear on} & [\frac{i_n}{n}, \frac{i_n+1}{n}]. \end{cases}$$

Obviously,

$$f_n\left(\frac{i}{n}\right) - f_n\left(\frac{i+1}{n}\right) = \begin{cases} 0 & \text{for } i \neq i_n \\ -1 & \text{for } i = i_n \end{cases} \quad (9)$$

On the other hand, (9) implies

$$\begin{aligned} \left| \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(f_n(x) - f_n\left(\frac{i}{n}\right) \right) p_n(x) dx \right| &\leq \int_{\frac{i_n}{n}}^{\frac{i_n+1}{n}} |p_n(t)| dt \leq \\ &\leq \frac{1}{\sqrt{n}} \left(\int_0^1 p_n^2(t) dt \right)^{1/2} = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n a_k^2 \right)^{1/2} = O(1). \end{aligned} \quad (10)$$

Next, Putting $f = f_n$ in equality (3) and taking into account conditions (6), (8), (9) and (10), we obtain

$$\sum_{k=1}^n c_k(f_n) a_k = - \int_0^{\frac{i_n}{n}} p_n(x) dx + O(1).$$

Hence, with regard for (2), we have

$$\int_0^1 f_n(x) p_n(x) dx = - \int_0^{\frac{i_n}{n}} p_n(x) dx + O(1).$$

which, according to the condition of Theorem 1, results in

$$\overline{\lim}_{n \rightarrow \infty} \left| \int_0^1 f_n(x) p_n(x) dx \right| = \overline{\lim}_{n \rightarrow \infty} \left| \int_0^{\frac{i_n}{n}} p_n(x) dx \right| = +\infty.$$

If now $\|f\|_A$ denotes the norm of the function f in the space A of absolutely continuous functions, then by definition of the norm we obtain

$$\|f_n\|_A = \|f_n\|_c + \int_0^1 |f'_n(x)| dx = 1 + \int_{\frac{i_n}{n}}^{\frac{i_n+1}{n}} n dx = 2.$$

Finally, by the Banach-Steinhaus theorem there exists the function $f_0 \in A$ with the property

$$\overline{\lim}_{n \rightarrow \infty} \left| \int_0^1 f_0(x) p_n(x) dx \right| = +\infty.$$

This implies (see (2)) that

$$\sum_{k=1}^{\infty} C_k(f_0) a_k = +\infty,$$

for some sequence $(a_k) \in \ell_q$.

Consequently,

$$(c_n(f_0)) \notin \ell_p, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Thus $f_0 \notin A(p, \varphi)$.

Theorem 2. For any number $p \in (1, 2)$ there exists the orthonormalized on $[0, 1]$ system of functions (φ_n) satisfying the following conditions:

- 1) $\overline{\lim}_{n \rightarrow \infty} k_n = +\infty$ for some sequence $(a_k) \in \ell_q$, $(\frac{1}{p} + \frac{1}{q} = 1)$;
- 2) $\overline{\lim}_{n \rightarrow \infty} k_n < +\infty$ for any number $\varepsilon > 0$ and for any sequence $(b_k) \in \ell_{q_1}$,

where $\frac{1}{p+\varepsilon} + \frac{1}{q_1} = 1$;

- 3) $\int_0^1 \varphi_n(x) dx = 0$, $n = 1, 2, \dots$

Proof. Denote

$$m_p = \sum_{k=1}^{\infty} k^{-\frac{2}{p}}.$$

Let E_k be an infinite set of non-intersecting intervals, where

$$\bigcup_{k=1}^{\infty} E_k = \left[0, \frac{1}{2}\right] \quad \text{and} \quad |E_k| = \frac{1}{2m_p} k^{-\frac{2}{p}}.$$

By F_k we denote the interval which is symmetric with respect to the point $\frac{1}{2}$ of the set E_k . Obviously,

$$\bigcup_{k=1}^{\infty} F_k = \left(\frac{1}{2}, 1\right) \quad \text{and} \quad |F_k| = \frac{1}{2m_p} k^{-\frac{2}{p}}.$$

Let

$$\varphi_k(x) = \begin{cases} \frac{1}{\sqrt{2}} k^{\frac{1}{p}} \sqrt{m_p} & \text{for } x \in E_k \\ -\frac{1}{\sqrt{2}} k^{\frac{1}{p}} \sqrt{m_p} & \text{for } x \in F_k \\ 0 & \text{for } x \in [0, 1] \setminus (E_k \cup F_k). \end{cases}$$

It is evident that the system (φ_n) is orthonormalized on $[0, 1]$ and

$$\int_0^1 \varphi_n(x) dx = 0 \quad n = 1, 2, \dots$$

Suppose

$$a_k = \frac{1}{k^{\frac{1}{q}} \ell_n k}.$$

Clearly

$$(a_k) \in \ell_q \quad \text{for } q > 1.$$

We have

$$\begin{aligned} k_n &= \max_{x \in [0, 1]} \left| \int_0^x \sum_{k=1}^n a_k \varphi_k(t) dt \right| = \left| \int_0^{\frac{1}{2}} \sum_{k=1}^n a_k \varphi_k(t) dt \right| = \\ &= \frac{\sqrt{m_p}}{\sqrt{2}} \sum_{k=1}^n \frac{1}{k^{\frac{1}{q}} \ell_n k} \int_{E_k} k^{\frac{1}{p}} dt = \frac{1}{2\sqrt{2m_p}} \sum_{k=1}^n \frac{k^{-\frac{1}{p}}}{k^{\frac{1}{q}} \ell_n k} = \\ &= \frac{1}{2\sqrt{2m_p}} \sum_{k=1}^n \frac{1}{k \ell_n k}. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} k_n = +\infty.$$

Suppose that $q_1 = \frac{p+\varepsilon}{p+\varepsilon-1}$. Then

$$\begin{aligned} k_n &= \max_{x \in [0, 1]} \left| \int_0^1 \sum_{k=1}^n b_k \varphi_k(t) dt \right| \leq \sum_{k=1}^n b_k \cdot k^{-\frac{1}{p}} \leq \\ &\leq \left(\sum_{k=1}^n b_k^{q_1} \right)^{\frac{1}{q_1}} \cdot \left(\sum_{k=1}^n k^{-\frac{p+\varepsilon}{p}} \right)^{\frac{1}{p+\varepsilon}} < c, \end{aligned}$$

where c does not depend on n . □

Remark. It follows from Theorem 2 that for any $p \in (1, 2)$ there exists the orthonormalized system (φ_n) for which the condition $f \in V$ does not imply that $f \in A(p, \varphi)$, although it implies that $f \in A(p + \varepsilon, \varphi)$ for any number $\varepsilon > 0$.

Theorem 3. *Of any orthonormalized on $[0, 1]$ system (φ_n) we can distinguish a subsystem $\psi_k \equiv \varphi_{n_k}$ for which*

$$\overline{\lim}_{n \rightarrow \infty} k_n < M < +\infty$$

for any sequence $(a_k) \in \ell_q$, where $q > 1$.

Proof. Let

$$N_i(k) = \sum_{s=1}^{\infty} \left(\int_0^{i \cdot 2^{-k}} \varphi_s(x) dx \right)^2,$$

where $i = 1, 2, \dots, 2^k$.

Since $N_i(k) < \infty$ for $i = 1, 2, \dots, 2^k$, for any fixed i we can choose a number $s_i(k)$ such that

$$\sum_{s=s_i(k)}^{\infty} \left(\int_0^{i \cdot 2^{-k}} \varphi_s(x) dx \right)^2 < 2^{-3k},$$

If we introduce the notation

$$s(k) = \max_{1 \leq i \leq 2^k} s_i(k),$$

then

$$\sum_{s=s(k)}^{\infty} \left(\int_0^{i \cdot 2^{-k}} \varphi_s(x) dx \right)^2 < 2^{-3k},$$

for any $i = 1, 2, \dots, 2^k$.

From the latter inequality we have

$$\left| \int_0^{i \cdot 2^{-k}} \varphi_{s_m}(x) dx \right| < 2^{-\frac{3}{2}k},$$

for $s \geq s(k)$ and for any $i = 1, 2, \dots, 2^k$.

Let $(s_m)_{m=2^k}^{2^{k+1}-1}$ be the sequence of natural numbers satisfying the conditions

a) $s_m < s_{m+1}$ and

b) $s_m > s(k)$ for any $m = 2^k, 2^k + 1, \dots, 2^{k+1} - 1$. Then

$$\left| \int_0^{i \cdot 2^{-k}} \varphi_{s_m}(x) dx \right| < 2^{-\frac{3}{2}k}, \quad (11)$$

for $i = 1, 2, \dots, 2^k$.

Thus for any $k = 1, 2, \dots$, we distinguish from (φ_s) the subsystem (φ_{s_m}) which satisfies condition (11), and $s_m \uparrow \infty$ as $m \rightarrow \infty$. If now we distinguish from (φ_{s_m}) the subsystem $(\varphi_{s_{m_j}})$ for which $1 \in A(1, \varphi_{s_{m_j}})$, then for the system $(\varphi_{s_{m_j}}) \equiv (\psi_j)$ we will have (see (11)):

$$\begin{aligned} & \max_{1 \leq i \leq 2^k} \left| \int_0^{i \cdot 2^{-k}} \sum_{m=2^k}^N a_m \psi_m(x) dx \right| = \\ &= \max_{1 \leq i \leq 2^k} \left| \sum_{m=2^k}^N a_m \int_0^{i \cdot 2^{-k}} \psi_m(x) dx \right| \leq \\ &\leq \max_{1 \leq i \leq 2^k} \left(\sum_{m=2^k}^N \left(\int_0^{i \cdot 2^{-k}} \psi_m(x) dx \right)^p \right)^{1/p} \cdot \left(\sum_{m=2^k}^N |a_m|^q \right)^{1/q} \leq \\ &\leq \left(\sum_{m=2^k}^N 2^{-\frac{3p}{2}k} \right)^{1/p} \cdot \left(\sum_{m=2^k}^N |a_m|^q \right)^{1/q} = O(1), \end{aligned} \quad (12)$$

where $(a_m) \in \ell_q$ ($q > 1$) is any sequence, $\frac{1}{p} + \frac{1}{q} = 1$ and $2^k < N < 2^{m+1}$.

Further,

$$\begin{aligned} & \max_{1 \leq i \leq 2^k} \left| \int_0^{i \cdot 2^{-k}} \sum_{m=1}^{2^k-1} a_m \psi_m(x) dx \right| = \\ &= \max_{1 \leq i \leq 2^k} \left| \int_0^{i \cdot 2^{-k}} \sum_{m=0}^{k-1} \sum_{i=2^m}^{2^{m+1}-1} a_{2^m+i} \psi_{2^m+i}(x) dx \right| \leq \\ &\leq \max_{1 \leq i \leq 2^k} \sum_{m=0}^{k-1} \sum_{i=2^m}^{2^{m+1}-1} |a_{2^m+i}| \left| \int_0^{i \cdot 2^{-k}} \psi_{2^m+i}(x) dx \right| \leq \\ &\leq \left(\sum_{m=0}^{k-1} \sum_{i=2^m}^{2^{m+1}-1} |a_{2^m+i}|^q \right)^{1/q} \left(\sum_{m=0}^{k-1} \sum_{i=2^m}^{2^{m+1}-1} 2^{-\frac{3k}{2}p} \right)^{1/p} \leq \\ &\leq \left(\sum_{m=1}^{2^k-1} |a_m|^q \right)^{1/q} \left(\sum_{m=0}^{k-1} 2^{-\frac{3k}{2}p} \cdot 2^m \right)^{1/p} = O(1). \end{aligned} \quad (13)$$

Consequently, by virtue of (12) and (13) we get

$$\max_{1 \leq i \leq n} \left| \int_0^{\frac{i}{n}} \sum_{k=1}^n a_k \psi_k(x) dx \right| = O(1). \quad (14)$$

Finally, for some $x_0 \in [\frac{i_0}{2^k}, \frac{i_0+1}{2^k}]$ we suppose that

$$K_n = \max_{x \in [0,1]} \left| \int_0^x \sum_{m=1}^n a_m \psi_m(t) dt \right| = \left| \int_0^{x_0} \sum_{m=1}^n a_m \psi_m(t) dt \right|.$$

Using then Hölder's inequality and (13), we obtain

$$\begin{aligned} K_n &\leq \int_0^{i_0 \cdot 2^{-k}} \left| \sum_{m=1}^n a_m \psi_m(t) dt \right| + \int_{i_0 \cdot 2^{-k}}^{x_0} \left| \sum_{m=1}^n a_m \psi_m(t) \right| dt \leq \\ &\leq O(1) + 2^{-\frac{k}{2}} \left(\int_0^1 \left(\sum_{m=1}^n a_m \psi_m(t) \right)^2 dt \right)^{1/2} = \\ &= O(1) + 2^{-k/2} \left(\sum_{m=1}^n a_m^2 \right)^{1/2} = \\ &= O(1) + 2^{-\frac{k}{2}} \cdot \sqrt{n} \max_{1 \leq m \leq n} |a_m| = O(1), \end{aligned}$$

where $n < 2^{k+1}$. □

CHECKING OF EFFICIENCY OF THE CONDITION OF THEOREM 1

Let $\varphi_n(x) = \sqrt{2} \cos 2\pi n x$ and $(a_k) \in \ell_q$ ($q > 1$) be any sequence. Using Hölder's inequality for $\frac{1}{p} + \frac{1}{q} = 1$, we find that

$$\begin{aligned} K_n &= \max_{x \in [0,1]} \left| \int_0^x \sum_{k=1}^n a_k \sqrt{2} \cos 2\pi k t dt \right| = \\ &= \max_{x \in [0,1]} \left| \sum_{k=1}^n a_k \sqrt{2} \int_0^x \cos 2\pi k t dt \right| = \frac{\sqrt{2}}{2\pi} \max_{x \in [0,1]} \left| \sum_{k=1}^n a_k \frac{\sin 2\pi k x}{k} \right| \leq \\ &\leq \frac{1}{\sqrt{2}\pi} \left(\sum_{k=1}^n |a_k|^q \right)^{1/q} \left(\sum_{k=1}^n \frac{1}{k^p} \right)^{1/p} = O(1). \end{aligned}$$

The condition $K_n = O(1)$ of Theorem 1 can be easily verified for the Haar and Walsh systems as well.

VARIATION OF THE FOURIER-HAAR COEFFICIENTS

Lemma. *If $f \in L$ and $c_n^{(k)}(f)$ ($k = 1, 2, \dots, 2^n$) are its Fourier-Haar coefficients, then the equality*

$$\begin{aligned} & c_n^{(k)} - c_n^{(k+1)}(f) = \\ & = 2^{\frac{n}{2}} \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} (f(x - 2^{-n-1}) - 2f(x) + f(x + 2^{-n-1})) dx, \end{aligned} \quad (15)$$

$1 \leq k \leq 2^n - 1$ holds.

Proof. Suppose

- (a) $\Delta c_n^{(k)}(f) \equiv c_n^{(k)}(f) - c_n^{(k+1)}(f)$,
- (b) $\Delta_h f(x) = f(x) - f(x + h)$,
- (c) $\Delta_h^2 f(x) = f(x - h) - 2f(x) + f(x + h)$.

As is known (see [9]),

$$c_n^{(k)}(f) = 2^{\frac{n}{2}} \int_{\frac{2k-2}{2^{n+1}}}^{\frac{2k-1}{2^{n+1}}} (f(x) - f(x + 2^{-n-1})) dx, \quad (16)$$

where $k = 1, 2, \dots, 2^n$.

Consequently,

$$\begin{aligned} \Delta c_n^{(k)}(f) &= c_n^{(k)}(f) - c_n^{(k+1)}(f) = 2^{\frac{n}{2}} \int_{\frac{2k-2}{2^{n+1}}}^{\frac{2k-1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx - \\ &- 2^{\frac{n}{2}} \int_{\frac{2k}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx = 2^{\frac{n}{2}} \left(\int_{\frac{2k-2}{2^{n+1}}}^{\frac{2k-1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx - \right. \\ &- \left. \int_{\frac{2k}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx \right) = 2^{\frac{n}{2}} \left(\int_{\frac{2k-2}{2^{n+1}}}^{\frac{2k-1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx - \right. \\ &- \left. \int_{\frac{2k}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx + \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx - \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx \right) = \end{aligned}$$

$$\begin{aligned}
&= 2^{\frac{n}{2}} \left(\int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f \left(x - \frac{1}{2^{n+1}} \right) dx - \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx + \right. \\
&\quad \left. + \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f(x) dx - \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} f \left(x + \frac{1}{2^{n+1}} \right) dx \right) = \\
&= 2^{\frac{n}{2}} \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \left(\Delta_{\frac{1}{2^{n+1}}} f \left(x - \frac{1}{2^{n+1}} \right) - \Delta_{\frac{1}{2^{n+1}}} f(x) \right) dx + \\
&\quad + 2^{\frac{n}{2}} \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \left(\Delta_{\frac{1}{2^{n+1}}} f(x) - \Delta_{\frac{1}{2^{n+1}}} f \left(x + \frac{1}{2^{n+1}} \right) \right) dx = \\
&= 2^{\frac{n}{2}} \left(\int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 f \left(x - \frac{1}{2^{n+1}} \right) dx + \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 f(x) dx \right) = \\
&= 2^{\frac{n}{2}} \left(\int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 f(x) dx + \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 f(x) dx \right) = 2^{\frac{n}{2}} \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 f(x) dx.
\end{aligned}$$

Thus the lemma is proved completely. $\square$

Theorem 4. a) If $f \in W$ and $\beta > \frac{2}{3}$, then

$$(c_n(f)) \in V_\beta,$$

where $c_n(f)$ are the Fourier-Haar coefficients;

b) there exists the function $f_0 \in V$ such that

$$(c_n(f_0)) \notin V_{\frac{2}{3}},$$

Proof. The validity of item (a) in Theorem 4 follows directly from Theorem D. For $p = 1$ however, because of the fact that the proof of Theorem D is absent in [13] it is advisable to prove item a) of Theorem 4.

Using (15) and Hölder's inequality, we have

$$\sum_{k=1}^{2^n-1} |\Delta c_n^{(k)}|^\beta = 2^{\frac{n\beta}{2}} \sum_{k=1}^{2^n-1} \left| \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 f(x) dx \right|^\beta \leq$$

$$\begin{aligned}
&\leq 2^{\frac{n\beta}{2}} \cdot 2^{n(1-\beta)} \left(\sum_{k=1}^{2^n-1} \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} |\Delta_{\frac{1}{2^{n+1}}}^2 f(x)| dx \right)^\beta \leq \\
&\leq 2^{\frac{n\beta}{2}} \cdot 2^{n(1-\beta)} \cdot 2^{-\beta n} \left(\sum_{k=1}^{2^n-1} M_k^{(n)} \right)^\beta \leq 2^{n(1-\frac{3\beta}{2})} W^\beta(f), \quad (17)
\end{aligned}$$

where

$$M_k^{(n)} = \sup_{x \in [\frac{2k-1}{2^{n+1}}, \frac{2k+1}{2^{n+1}}]} |\Delta_{\frac{1}{2^{n+1}}}^2 f(x)|.$$

Thus if $\beta > \frac{2}{3}$, then $1 - \frac{3\beta}{2} < 0$, and by (17) we get

$$\sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} |\Delta c_n^{(k)}(f)|^\beta < +\infty.$$

On the other hand, for $1 \leq k \leq 2^n$ from (16) we find that

$$|c_n^{(k)}(f)| \leq M \cdot 2^{-\frac{n}{2}},$$

where $M = \sup_{x \in [0,1]} |f(x)|$. Consequently,

$$|c_n^{(2^n)}(f) - c_{n+1}^{(1)}(f)| \leq 2M \cdot 2^{-\frac{n}{2}},$$

whence

$$\sum_{n=1}^{\infty} |c_n^{(2^n)}(f) - c_{n+1}^{(1)}(f)|^\beta \leq (2M)^\beta \sum_{n=1}^{\infty} 2^{-\frac{n\beta}{2}} < \infty.$$

Finally,

$$\sum_{n=1}^{\infty} |c_{n+1}(f) - c_n(f)|^\beta < \infty,$$

i.e.,

$$(c_n(f)) \in A(\beta, \varphi).$$

b) For some $k = 1, 2, \dots, 2^n$ and $n = 1, 2, \dots$ let

$$g_n^{(k)}(x) = \begin{cases} (x - \frac{2k-2}{2^{n+1}})(\frac{2k-1}{2^{n+1}} - x) & \text{for } x \in (\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}) \equiv \Delta_n^{k+} \\ (\frac{2k-1}{2^{n+1}} - x)(\frac{2k}{2^{n+1}} - x) & \text{for } x \in (\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}) \equiv \Delta_n^{k-} \\ 0 & \text{for } x \in [0, 1] \setminus \Delta_n^{k+} \cup \Delta_n^{k-}. \end{cases} \quad (18)$$

Suppose

$$g_n(x) = 2^n g_n^{(k)}(x)$$

for $x \in \Delta_n^{k+} \cup \Delta_n^{k-}$ and $k = 1, 2, \dots, 2^n$.

Consider the function $(z_m(t))$ Rademacher's function),

$$f_{0,t}(x) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m} \log m} g_m(x) z_m(t). \quad (19)$$

Since $\|g_m\|_c \leq 2^m \cdot 2^{-2m} = 2^{-m}$, series (18) converges uniformly for all $t \in [0, 1]$. Moreover, since

$$\Delta c_n^{(k)}(g_m) = 2^{\frac{n}{2}} \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}} g_m(x) dx$$

and g_m is the periodic function with period $\frac{1}{2^m}$, for $m \geq n$ we have $\Delta_{\frac{1}{2^{n+1}}} g_m(x) = 0$, i.e.,

$$\Delta c_n^{(k)}(g_m) = 0,$$

for $m \geq n$.

In case $m < n$, we have:

1) if $m = n - 1$, then as the function $g_n^{(k)}$ is odd with respect to the point $\frac{2k-1}{2^n}$ on the interval $[\frac{4k-3}{2^{n+1}}, \frac{4k-1}{2^{n+1}}]$, we have

$$\int_{\frac{4k-3}{2^{n+1}}}^{\frac{4k-1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 g_{n-1}^{(k)}(x) dx = \int_{\frac{4k-3}{2^{n+1}}}^{\frac{4k-1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 g_{n-1}(x) dx = 0,$$

where $k = 1, 2, \dots, 2^{n-1}$. Consequently,

$$\int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 g_{n-1}(x) dx = 0,$$

for $k = 1, 2, \dots, 2^n - 1$. That is,

$$\Delta c_n^{(k)}(g_{n-1}) = 0, \quad k = 1, 2, \dots, 2^n - 1.$$

2) Assume that $m < n - 1$. Let $g_m(x_s) = 0$, where $s = 1, 2, \dots, 2^m$. It is evident that for $m < n - 1$ the points x_s serve as a midpoint of some interval $(\frac{2k-1}{2^{n+1}}, \frac{2k+1}{2^{n+1}})$, where $k = 1, 2, \dots, 2^n - 1$ and $s = 1, 2, \dots, 2^m - 1$. Since the function g_m on such intervals is odd with respect to the midpoint x_s , we have

$$\int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} \Delta_{\frac{1}{2^{n+1}}}^2 f(x) dx = 0,$$

where $x_s \in (\frac{2k-1}{2^{n+1}}, \frac{2k+1}{2^{n+1}})$.

Let now N_n be a set of those k for which $x_s \in (\frac{2k-1}{2^{n+1}}, \frac{2k+1}{2^{n+1}})$, where $s = 1, 2, \dots, 2^m - 1$ and $k = 1, 2, \dots, 2^n - 1$. It is clear that power of the set N_n is less than 2^m .

Taking into account definition of the function $g_m(x)$ and also the fact that the function $g_m(x)$ on $(\frac{k-1}{2^m}, \frac{k}{2^m})$ has the form $x^2 + px + q$, we get

$$\Delta_{\frac{1}{2^{n+1}}}^2 g_m(x) = \pm \frac{1}{2} 2^{-2n} \cdot 2^m,$$

whence

$$\Delta C_n^{(k)}(g_m) = \begin{cases} 0 & \text{for } k \in N_n \\ \pm \frac{1}{2} 2^{-2,5n} \cdot 2^m & \text{for } k \notin N_n. \end{cases}$$

By D_n we denote a set of those k for which $\Delta C_n^{(k)}(g_m) \neq 0$, ($k = 1, 2, \dots, 2^n - 1$). It can be easily seen that for $m \geq n - 2$ the power of the set D_n is not less than 2^{n-1} .

Using the well-known inequality (see [12], p. 340) for $\beta = \frac{2}{3}$, we find that

$$\begin{aligned} \int_F |\Delta C_n^{(k)}(f_{0,t})|^\beta dt &\geq \int_F \left| \sum_{m=1}^{n-2} \frac{1}{\sqrt{m} \log m} \Delta C_n^{(k)}(g_m) z_n(t) \right|^\beta dt \geq \\ &\geq A(F) \left(\sum_{m=1}^{n-2} \left(\frac{\Delta C_n^{(k)}(g_m)}{\sqrt{m} \log m} \right)^2 \right)^{\beta/2} = A(F) 2^{-\frac{5\beta}{2}n} \left(\sum_{m=1}^{n-2} \frac{2^{2m}}{m \log^2 m} \right)^{\beta/2} \geq \\ &\geq c \cdot 2^{-\frac{5\beta}{2}n} \cdot \frac{2^{\beta n}}{n^{\beta/2} \log^\beta n} = \frac{c \cdot 2^{-n}}{n^{\frac{1}{3}} \cdot \log^{\frac{2}{3}}}, \end{aligned}$$

where $F \subset [0, 1]$ is any set for $|F| > 0$, c and $A(F)$ are independent of n , and the index $k \in D_n$.

Consequently,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} \int_F |\Delta C_n^{(k)}(f_{0,t})|^{\frac{2}{3}} dt \geq c \cdot \sum_{n=1}^{\infty} 2^{n-1} \cdot \frac{2^{-n}}{n^{\frac{1}{3}} \log^{\frac{2}{3}}} = +\infty.$$

which for almost all $t \in [0, 1]$ yields

$$\sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} |\Delta C_n^{(k)}(f_{0,t})|^{\frac{2}{3}} = +\infty. \quad (20)$$

Since

$$\|g'_m(x)\|_c \leq 1,$$

we have

$$\begin{aligned} &\int_0^1 \int_0^1 |f'_{0,t}(x)| dx dt \leq \\ &\leq \int_0^1 \int_0^1 \left| \sum_{m=1}^{\infty} \frac{1}{\sqrt{m} \log m} g'_m(x) z_m(t) \right| dx dt \leq \sum_{m=1}^{\infty} \frac{1}{m \log^2 m} < \infty. \quad \square \end{aligned}$$

This implies that the function $f_{0,t}$ for some set $E \subset [0, 1]$ with $|E| > 0$ for all $t \in E$ has finite variation on $[0, 1]$. Consequently, (19) implies that there exists the function $f_{0,t} \in V$ such that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} |\Delta c_n^{(k)}(f_{0,t})|^{\frac{2}{3}} = +\infty,$$

for some point $t \in [0, 1]$.

Theorem 5. a) If $f \in W$ and $\beta < \frac{1}{2}$, then

$$\sum_{n=1}^{\infty} |c_{n+1}(f) - c_n(f)| \cdot n^{\beta} < \infty.$$

b) There exists the function $f_0 \in V_x$ such that

$$\sum_{n=1}^{\infty} |c_{n+1}(f_0) - c_n(f_0)| \cdot n^{\frac{1}{2}} = +\infty.$$

Proof. (a) Putting $\beta = 1$ in (17), we obtain

$$\sum_{k=1}^{2^n-1} |\Delta c_n^{(k)}(f)| \leq W_0^1(f) 2^{-\frac{n}{2}}.$$

whence

$$\sum_{k=1}^{2^n-1} |\Delta c_n^{(k)}(f)| 2^{\beta n} \leq 2^{\beta n} \cdot 2^{-\frac{n}{2}} W_0^1(f).$$

Therefore for $\beta < \frac{1}{2}$,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} |\Delta c_n^{(k)}(f)| 2^{n\beta} \leq \sum_{n=1}^{\infty} 2^{-n(\frac{1}{2}-\beta)} < +\infty,$$

Next, owing to the fact that

$$|c_n^{(2^n)}(f) - c_{n+1}^{(1)}(f)| \leq M \cdot 2^{-\frac{n}{2}},$$

we arrive at

$$\sum_{n=1}^{\infty} |c_n^{(2^n)}(f) - c_{n+1}^{(1)}(f)| 2^{n\beta} \leq M \sum_{n=1}^{\infty} 2^{-n(\frac{1}{2}-\beta)} < +\infty,$$

Finally we conclude that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} |c_{n+1}(f) - c_n(f)| n^{\beta} < \infty,$$

if $\beta < \frac{1}{2}$.

(b) Consider the function (18). Just as while proving Theorem 4, we have

$$\begin{aligned} \int_E |\Delta c_n^{(k)}(f_{0,t})| dt &\geq A(E) \left(\sum_{m=1}^{n-2} \left(\frac{\Delta c_n^{(k)}(g_m)}{\sqrt{m} \log m} \right)^2 \right)^{1/2} \geq \\ &\geq B \cdot 2^{-\frac{5n}{2}} \left(\sum_{m=1}^{n-2} \frac{2^{2m}}{m \log^2 m} \right)^{1/2} \geq c \cdot 2^{-\frac{5n}{2}} \cdot \frac{2^n}{\sqrt{n} \log n} = c \frac{2^{-\frac{3n}{2}}}{\sqrt{n} \log n}, \end{aligned}$$

where $E \subset [0, 1]$ and $|E| > 0$ is any set. The constants B and c are independent of n .

Thus

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} \int_F |\Delta c_n^{(k)}(f_{0,t})| 2^{\frac{n}{2}} dt \geq \\ &\geq c \cdot \sum_{n=1}^{\infty} 2^{\frac{n}{2}} \cdot 2^{n-1} \cdot \frac{2^{-\frac{3n}{2}}}{\sqrt{n} \log n} = c \cdot \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \log n} = +\infty. \end{aligned}$$

Hence for almost all $t \in [0, 1]$,

$$\sum_{n=1}^{\infty} 2^{\frac{n}{2}} \sum_{k=1}^{2^n-1} |\Delta c_n^{(k)}(f_{0,t})| = +\infty. \quad (21)$$

Since $f_{0,t} \in V$ for all $t \in E$ ($|E| > 0$) (see Theorem 4), equality (21) is fulfilled for some function $f_{0,t} \in V$. $\square$

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