

MAXIMAL SUBGROUPS OF COMPLETE SEMI-GROUPS OF BINARY RELATIONS

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ABSTRACT. On the basis of the notation of normal representations of binary relations we describe elements of maximal subgroups of the given semi-group and the semi-groups themselves.

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Constructions of maximal subgroups of the semigroup B_X are well known (see [1]). In this paper, using normal representations of binary relations, we describe maximal subgroups of the semigroup $B_X(D)$ which are defined by complete X -semilattices of unions D .

Let X and $X \times X$ be respectively an arbitrary nonempty set and the Cartesian product of the set X by itself. Recall that we call a binary relation on the set X a subset of the Cartesian product $X \times X$. If α and β are binary relations on the set X and $x, y, z \in X$, then the conditions $(x, y) \in \alpha$ is written in the form $x\alpha y$, while its negation in the form $(x, y) \notin \alpha$. The notation $x\alpha y\beta z$ means that the conditions $x\alpha y$ and $y\beta z$ hold simultaneously. Furthermore, let Y be a nonempty subset from X . Then by α^{-1} , $\emptyset$, $Y\alpha$ and αY we respectively denote the following sets: $\alpha^{-1} = \{(x, y) \mid y\alpha x\}$, an empty binary relation or an empty set, $Y\alpha = \{x \in X \mid \text{for some } y \in Y\}$ and $\alpha Y = Y\alpha^{-1}$. Note that instead of $\{y\}\alpha$ and $\alpha\{y\}$ we always write $y\alpha$ and αy .

We call the product of binary relations α and β the binary relation $\delta = \alpha \circ \beta$ on X by which $x\delta z$ ($x, z \in X$) is true if and only if there exists $y \in X$ such that $x\alpha y\beta z$. The binary operation $(\circ)$ is associative so that the set B_X of all binary relations on the set X is a semigroup with respect to the operation $(\circ)$.

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Let D be some nonempty set of subsets from X , which is closed with respect to the set-theoretic union of elements from D . In that case the set D is called the complete X -semilattice of unions. Furthermore, to each mapping $f : X \rightarrow D$ we assign a binary relation α_f on the set X which satisfies the condition

$$\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x)).$$

The set of all such α_f ($f : X \rightarrow D$), which is denoted by $B_X(D)$, is a subgroup of the semigroup B_X which is a complete semigroup of binary relations defined by a complete X -semilattice of unions D . It is obvious that $B_X(D) = B_X$ if D is the set of all subsets of the set X , i.e., when $D = 2^X$. Moreover, for any $\alpha \in B_X(D)$ and a complete X -semilattice of unions D the sets $V(D; \alpha) = \{Y \mid Y \in D\}$ and $X^* = 2^X \setminus \{\emptyset\}$ are complete X -semilattices of unions.

Definition 1. Let $\alpha \in B_X(D)$ and φ be the one-to-one mapping of the subset D'_1 of the semilattice D on the semilattice $V(X^*; \alpha)$. A representation of the binary relation α written in the form

$$\alpha = \bigcup_{T \in D_1} (Y_T^\alpha \times \varphi(T)) \quad (1)$$

is said to be normal if for Y_T^α and $\varphi(T)$ of this representation the following conditions hold:

- a) $D_1 \subseteq D'_1$;
- b) $\emptyset \neq Y_T^\alpha \subseteq X$ for any $T \in D_1$;
- c) $Y_T^\alpha = Y_{T'}^\alpha$ if $\varphi(T) = \varphi(T')$;
- d) $Y_T^\alpha \cap Y_{T'}^\alpha = \emptyset$ if $T, T' \in D_1$ and $T \neq T'$;
- e) $\bigcup_{T \in D_1} Y_T^\alpha = X$.

It is obvious that by virtue of the conditions b), d) and e) from Definition 1 the system of sets $\{Y_T^\alpha\}_{T \in D_1}$ is a partition of the set X . The condition c) makes it impossible for union (1) to contain elements of the form $Y_T^\alpha \times \varphi(T)$ and $Y_{T'}^\alpha \times \varphi(T)$ for various Y_T^α and $Y_{T'}^\alpha$. The condition a) restricts the choice of indexes for the family $\{Y_T^\alpha\}_{T \in D_1}$ to the subset of the semilattice D , while the mapping φ emphasizes the connection between the elements Y_T^α and $\varphi(T)$.

Note that when φ is the identical mapping of the semilattice $V(X^*; \alpha)$, the normal representation of the relation α takes the form

$$\alpha = \bigcup_{T \in D_1} (Y_T^\alpha \times T). \quad (2)$$

Lamma 1. *If a normal representation of the binary relation α of the semigroup $B_X(D)$ has form (1), then $Y_T^\alpha = \{y \in X \mid y\alpha = \varphi(T)\}$ for any T from D_1 .*

Proof. Let $T \in D_1$ and $y \in Y_T^\alpha$. Then by virtue of d) and e) from Definition 1 we have $y \notin X \setminus Y_T^\alpha$. Therefore

$$y\alpha = y(Y_T^\alpha \times \varphi(T)) = y(\{y\} \times \varphi(T)) = \varphi(T),$$

i.e., $y\alpha = \varphi(T)$ for any $y \in Y_T^\alpha$.

The lemma is proved. $\square$

The next lemma shows that the normal representation of the binary relation α of the semigroup $B_X(D)$ depends only on a choice of α and does not depend on choices of the set of indices and the mapping φ .

Lamma 2. *Let $\alpha \in B_X(D)$ and*

$$\alpha = \bigcup_{T \in D_1} (Y_T^\alpha \times \varphi(T)) = \bigcup_{Z \in D_2} (X_Z^\alpha \times \psi(Z))$$

be the normal representations of the binary relation α . If $D'_1 = \{\varphi(T) \mid T \in D_1\}$ and $D'_2 = \{\psi(Z) \mid Z \in D_2\}$, then $D'_1 = D'_2$ and $Y_T^\alpha = X_Z^\alpha$ if $\varphi(T) = \psi(Z)$.

Proof. Let $\varphi(T') \in D'_1$. Then by the condition b) from Definition 1 we have $Y_{T'}^\alpha \neq \emptyset$, i.e., $y' \in Y_{T'}^\alpha$ for some $y' \in X$. Hence by Lemma 1 we obtain $y'\alpha = \varphi(T')$.

On the other hand, by virtue of the equality

$$\bigcup_{Z \in D_2} X_Z^\alpha = X$$

there exist $X_{Z'}^\alpha \subseteq X$ and $\psi(Z') \in D'_2$ such that $y' \in X_{Z'}^\alpha$ and $y'\alpha = \psi(Z')$ in view of Lemma 1.

Thus $\varphi(T') = y'\alpha = \psi(Z')$ and we find that $D'_1 \subseteq D'_2$.

Analogously, one can prove the inverse inclusion $D'_2 \subseteq D'_1$. Now, taking into account the inclusions $D'_1 \subseteq D'_2 \subseteq D'_1$, we obtain $D'_1 = D'_2$.

Moreover, by virtue of Lemma 1 the equalities

$$Y_T^\alpha = \{y \in X \mid y\alpha = \varphi(T)\} = X_Z^\alpha$$

hold, if $\varphi(T) = \psi(Z)$.

The lemma is proved. $\square$

Lamma 3. *For any binary relation α of the semigroup $B_X(D)$ there exists its normal representation.*

Proof. Let $\alpha \in B_X(D)$, $T \in V(X^*; \alpha)$,

$$Y_T^\alpha = \{y \in X \mid y\alpha = T\} \text{ and } \tilde{D}(\alpha) = \{T \in V(X^*; \alpha) \mid Y_T^\alpha \neq \emptyset\}.$$

Then the binary relation α is uniquely represented as

$$\alpha = \bigcup_{T \in \tilde{D}(\alpha)} (Y_T^\alpha \times T). \quad (3)$$

For the elements of the latter representation, all conditions from Definition 1 hold (it is assumed that $D_1 = \tilde{D}(\alpha)$ and φ is the identical mapping of the set $V(X^*; \alpha)$), i.e., this representation of the relation α is normal. $\square$

Example 1. Let $X = \{1, 2, 3, 4\}$, $D = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $\alpha = \{(2, 1), (3, 1), (4, 2)\}$. Then $V(X^*; \alpha) = D$, $Y_\emptyset^\alpha = \{1\}$, $Y_{\{1\}}^\alpha = \{2, 3\}$, $Y_{\{2\}}^\alpha = \{4\}$, $Y_{\{1, 2\}}^\alpha = \emptyset$ and $\tilde{D}(\alpha) = \{\emptyset, \{1\}, \{2\}\}$, and the normal representation of the binary relation α is written as

$$\begin{aligned} \alpha &= (Y_\emptyset^\alpha \times \emptyset) \bigcup (Y_{\{1\}}^\alpha \times \{1\}) \bigcup (Y_{\{2\}}^\alpha \times \{2\}) = \\ &= (\{1\} \times \emptyset) \bigcup (\{2, 3\} \times \{1\}) \bigcup (\{4\} \times \{2\}). \end{aligned}$$

Note that the above results immediately imply that a normal representation of the binary relation α of the semigroup $B_X(D)$ is always defined uniquely.

Lemma 4. *If a normal representation of the binary relation α of the semigroup $B_X(D)$ has form (3), then $\tilde{D}(\alpha)$ is a unique generating set of the semilattice $V(X^*; \alpha)$, satisfying the condition $Y_T^\alpha \neq \emptyset$ for any $T \in \tilde{D}(\alpha)$.*

Proof. Let $Z \in V(X^*; \alpha)$. Then $Z = Y\alpha$ for some $Y \in X^*$. Let us prove that the sets

$$D' = \{y\alpha \mid y \in Y\} \text{ and } D'' = \{T \in \tilde{D}(\alpha) \mid Y_T^\alpha \cap Y \neq \emptyset\}$$

are equal. Indeed, if $y\alpha \in D'$ for some $y' \in Y$, then by virtue of the equality

$$\bigcup_{T \in \tilde{D}(\alpha)} Y_T^\alpha = X$$

there exists an element $T' \in \tilde{D}(\alpha)$ such that $y' \in Y_{T'}^\alpha$ and $y'\alpha = T'$ in view of Lemma 1. Moreover, $Y_{T'}^\alpha \cap Y \neq \emptyset$ since $y' \in Y_{T'}^\alpha \cap Y$, i.e., $y'\alpha = T' \in D''$. Therefore $D' \subseteq D''$.

Conversely, if $T'' \in D''$, then $Y_{T''}^\alpha \cap Y \neq \emptyset$, i.e., $y'' \in Y_{T''}^\alpha$ for some $y'' \in Y$ and $y''\alpha = T''$ in view of Lemma 1. From the conditions $T'' = y''\alpha$ and $y'' \in Y$ it follows that $T'' = y''\alpha \in D'$. Therefore

$$D'' \subseteq D'.$$

Now, taking into account the inclusions $D' \subseteq D'' \subseteq D'$, we obtain $D' = D''$. Then the following equalities

$$Z = Y\alpha = \bigcup_{y \in Y} y\alpha = \bigcup D' = \bigcup D''$$

hold, where $D'' \subseteq \tilde{D}(\alpha)$. Thus any element of the semilattice $V(X^*; \alpha)$ is a union of some elements of the set $\tilde{D}(\alpha)$, i.e., $\tilde{D}(\alpha)$ is generating set of the semilattice $V(X^*; \alpha)$.

Furthermore, let $\tilde{D}(\alpha)'$ be another generating set of the semilattice $V(X^*; \alpha)$, satisfying the condition $Y_{T'}^\alpha \neq \emptyset$ for all $T' \in \tilde{D}(\alpha)'$. Then, by virtue of the condition $Y_{T'}^\alpha \neq \emptyset$, for T' from $\tilde{D}(\alpha)'$ we have $y\alpha = T'$ for some $y \in Y_{T'}^\alpha$. On the other hand, $y \in X = \bigcup_{T \in \tilde{D}(\alpha)} Y_T^\alpha$. Therefore there exists

$T(y) \in \tilde{D}(\alpha)$ such that $y \in Y_{T(y)}^\alpha$. Hence by virtue of Lemma 1 we obtain $y\alpha = T(y)$, i.e., $T' = y\alpha = T(y) \in \tilde{D}(\alpha)$. Therefore $\tilde{D}(\alpha)' \subseteq \tilde{D}(\alpha)$.

Analogously, one can prove the inverse inclusion $\tilde{D}(\alpha) \subseteq \tilde{D}(\alpha)'$.

Now, taking into account the inclusions $\tilde{D}(\alpha)' \subseteq \tilde{D}(\alpha) \subseteq D(\alpha)'$, we conclude that $\tilde{D}(\alpha) = D(\alpha)'$.

The lemma is proved. $\square$

Lemma 5. Let $\delta, \beta \in B_X(D)$.

$$\delta = \bigcup_{T \in \tilde{D}(\delta)} (Y_T^\delta \times T)$$

is a normal representation of the binary relation δ , $\alpha = \delta \circ \beta$, $\tilde{D}(\alpha) = \{T\beta \mid T \in \tilde{D}(\delta)\}$, $Z \in \tilde{D}(\alpha)$ and

$$Y_Z^\alpha = \bigcup \{Y_T^\delta \mid T \in \tilde{D}(\delta), T\beta = Z\}.$$

Then

$$\alpha = \bigcup_{Z \in \tilde{D}(\alpha)} (Y_Z^\alpha \times Z)$$

is a normal representation of the binary relation α of the semigroup $B_X(D)$.

Proof. Let Z be an element of the set $V(X^*; \alpha)$ such that $Z = T'\beta$ for some $T' \in \tilde{D}(\alpha)$,

$$Y_Z^\alpha = \{y \in X \mid y\alpha = Z\} \text{ and } Y_Z' = \bigcup \{Y_Z^\alpha \mid T \in \tilde{D}(\alpha) \text{ and } T\beta = Z\}.$$

Let us prove that $Y_Z^\alpha = Y_Z'$. Indeed, let $y_1 \in Y_Z^\alpha$ for some $y_1 \in X$. Then by the definition of the set Y_Z^α and the binary relation α we have $Z = y_1\alpha = y_1(\delta \circ \beta) = (y_1\delta)\beta$. Since, by assumption, $\bigcup_{T \in \tilde{D}(\alpha)} Y_T^\delta = X$ we have $y_1 \in Y_{T_1}^\delta$

for some $T_1 \in \tilde{D}(\alpha)$ and $y_1\delta = T_1$ in view of Lemma 1. But the conditions

$y_1 \in Y_{T_1}^\delta$, $T_1 \in \tilde{D}(\delta)$, $Z = (y_1\delta)\beta = T_1\beta$ imply that $y_1 \in Y_Z'$. Therefore $Y_Z^\alpha \subseteq Y_Z'$.

Conversely, let $y_2 \in Y_Z'$. Then $y_2 \in Y_{T_2}^\delta$ for some $T_2 \in V(X^*; \delta)$, satisfying the conditions $T_2 \in \tilde{D}(\delta)$ and $T_2\beta = Z$. By virtue of Lemma 1, from the condition $y_2 \in Y_{T_2}^\delta$ we obtain $y_2\delta = T_2$. Therefore $y_2\alpha = y_2(\delta \circ \beta) = (y_2\delta)\beta = T_2\beta = Z$, i.e., $y_2 \in Y_Z^\alpha$ by the definition of the set Y_Z^α . Thus $Y_Z' \subseteq Y_Z^\alpha$.

Now, taking into account the inclusions $Y_Z^\alpha \subseteq Y_Z' \subseteq Y_Z^\alpha$, we obtain

$$Y_Z^\alpha = Y'. \quad (1)$$

Further, let $\tilde{D}(\alpha) = \{Z \in V(X^*; \alpha) \mid Y_Z^\alpha \neq \emptyset\}$ and $D' = \{T\beta \mid T \in \tilde{D}(\delta)\}$. If $Z \in \tilde{D}(\alpha)$, then $Y_Z^\alpha \neq \emptyset$ and therefore $x'\alpha = Z$ for some $x' \in X$ satisfying the condition $x' \in Y_Z^\alpha$. By assumption, $\bigcup_{T \in \tilde{D}(\delta)} Y_T^\delta = X$, i.e., for

an element $x' \in X$ there exists $T' \in \tilde{D}(\delta)$ such that $x' \in Y_{T'}^\delta$. Hence by Lemma 1 we have $x'\delta = T'$ and $Z = x'\alpha = x'(\delta \circ \beta) = (x'\delta)\beta = T'\beta$, i.e., $Z \in D'$. Therefore $\tilde{D}(\alpha) \subseteq D'$.

Conversely, if $Z' \in D'$, then, by the definition of a normal representation of the relation δ , we have $Z' = T_1\beta$ for some $T_1 \in \tilde{D}(\delta)$ and $Y_{T_1}^\delta \neq \emptyset$. Hence we obtain $Y_{Z'}^\alpha = Y_{Z'}' = Y_{T_1\beta}' \supseteq Y_{T_1}^\delta \neq \emptyset$, i.e., $Z' \in \tilde{D}(\alpha)$. Therefore $D' \subseteq \tilde{D}(\alpha)$.

Now, taking into account the valid inclusions $\tilde{D}(\alpha) \subseteq D' \subseteq \tilde{D}(\alpha)$, we have

$$\tilde{D}(\alpha) = D'. \quad (2)$$

Moreover, by virtue of equalities (1) and (2) we have

$$\bigcup_{Z \in \tilde{D}(\alpha)} Y_Z^\alpha = \bigcup_{T \in \tilde{D}(\delta)} Y_{T\beta}^\alpha \supseteq \bigcup_{T \in \tilde{D}(\delta)} Y_T^\delta = X,$$

i.e.,

$$\bigcup_{Z \in \tilde{D}(\alpha)} Y_Z^\alpha = X. \quad (3)$$

According to Lemma 3, equalities (1), (2) and (3) imply that a representation of the binary relation α of the form

$$\alpha = \bigcup_{Z \in \tilde{D}(\alpha)} (Y_Z^\alpha \times Z)$$

is normal.

The lemma is proved. $\square$

Corollary 1. Let $\delta, \beta \in B_X(D)$,

$$\delta = \bigcup_{T \in \tilde{D}(\alpha)} (Y_T^\delta \times T)$$

be a normal representation of the binary relation δ and $\alpha = \delta \circ \beta$. If $T\beta \neq T'\beta$ for any and different T and T' from the set $\tilde{D}(\delta)$, then a representation of the relation α in the form

$$\alpha = \bigcup_{T \in \tilde{D}(\delta)} (Y_T^\delta \times T\beta)$$

is always normal.

Proof. In proving this corollary, we use the notation of Lemma 5. Further, by assumption, $T\beta \neq T'\beta$ for any and different T and T' from the set $\tilde{D}(\delta)$. Hence it follows that any element Z of the set $\tilde{D}(\alpha)$ is uniquely represented as $Z = T\beta$, where $T \in \tilde{D}(\delta)$. Therefore the following equalities are valid:

$$Y_Z^\alpha = Y_{T\beta}^\alpha = \bigcup \{Y_{T'}^\delta \mid T' \in \tilde{D}(\delta), T'\beta = T\beta = Z\} = Y_T^\delta,$$

$$\tilde{D}(\alpha) = \{T\beta \mid T \in \tilde{D}(\delta)\}$$

and

$$\alpha = \bigcup_{Z \in \tilde{D}(\alpha)} (Y_Z^\alpha \times Z) = \bigcup_{T \in \tilde{D}(\delta)} (Y_T^\delta \times T\beta). \quad (4)$$

By assumption, the mapping φ_1 , which to every element T of the set $\tilde{D}(\delta)$ assigns an element $T\beta$ of the set $\tilde{D}(\alpha)$, is one-to-one. Moreover, this and the condition $\tilde{D}(\alpha) \subseteq V(X^*; \alpha)$ immediately imply

$$|V(X^*; \alpha) \setminus \tilde{D}(\delta)| \geq |V(X^*; \alpha) \setminus \tilde{D}(\alpha)|.$$

Therefore in the set $V(X^*; \alpha) \setminus \tilde{D}(\delta)$ there exists a subset $\overline{D}$ equivalent to the set $V(X^*; \alpha) \setminus \tilde{D}(\alpha)$. Hence there exists the one-to-one mapping φ_2 of the set $\overline{D}$ on the set $V(X^*; \alpha) \setminus \tilde{D}(\alpha)$. Next we define the mapping φ of the set $D'_1 = \overline{D} \cup \tilde{D}(\delta)$ in the set $V(X^*; \alpha)$ as follows:

$$\varphi(T) = \begin{cases} \varphi_1(T) & \text{if } T \in \tilde{D}(\delta), \\ \varphi_2(T) & \text{if } T \in \overline{D}. \end{cases}$$

It is obvious that φ is the one-to-one mapping of the subset D'_1 of the semilattice D on the semilattice $V(X^*; \alpha)$ for which the binary relation α is represented in the form

$$\alpha = \bigcup_{T \in \tilde{D}(\delta)} (Y_T^\alpha \times \varphi(T)).$$

Moreover, all conditions of Definition 1 hold for the mapping φ and elements of this representation, which means that the latter is normal.

The corollary is proved. $\square$

Lamma 6. Let D and $G_X(D; \varepsilon)$ be respectively a complete X -semilattice of unions and a maximal subgroup of the semigroup $B_X(D)$, whose unit is an idempotent binary relation ε from $B_X(D)$. Then for the element δ of the group $G_X(D; \varepsilon)$ the following statements are true:

- a) $V(D; \delta) = V(X^*; \delta) = V(X^*; \varepsilon) = V(D; \varepsilon)$ if $\emptyset \notin D$;
- b) $V(D; \delta) = V(X^*; \delta) \cup \{\emptyset\} = V(X^*; \varepsilon) \cup \{\emptyset\} = V(D; \varepsilon)$ if $\emptyset \in D$.

Proof. Assume that $\delta \in G_X(D; \varepsilon)$. Then $\delta \circ \varepsilon = \delta$ and $\delta' \circ \delta = \delta \circ \delta' = \varepsilon$ for some $\delta' \in G_X(D; \varepsilon)$. By Theorem 1 from [3], the conditions $\delta' \circ \delta = \varepsilon$ and $\delta \circ \varepsilon = \delta$ respectively imply

$$V(X^*; \varepsilon) \subseteq V(D; \delta) \text{ and } V(X^*; \delta) \subseteq V(D; \varepsilon). \quad (1)$$

Moreover,

$$V(D; \delta) \subseteq V(X^*; \delta) \bigcup \{\emptyset\} \quad (2)$$

since $D \subseteq X^* \cup \{\emptyset\}$. Further, by virtue of Corollary 2 of Subsection 6.1 in [4] we obtain

$$V(D; \varepsilon) = \begin{cases} V(X^*; \varepsilon) & \text{if } \emptyset \notin D, \\ V(X^*; \varepsilon) \cup \{\emptyset\} & \text{if } \emptyset \in D. \end{cases} \quad (3)$$

For the semilattice D we will consider the following cases:

- 1) $\emptyset \notin D$. Then by conditions (1), (2) and (3) we have

$$V(D; \varepsilon) = V(X^*; \varepsilon) \subseteq V(D; \delta) \subseteq V(X^*; \delta) \subseteq V(D; \varepsilon),$$

i.e.,

$$V(D; \delta) = V(X^*; \delta) = V(X^*; \varepsilon) = V(D; \varepsilon).$$

The statement a) is proved.

- 2) $\emptyset \in D$. Then, taking into account the conditions (1), (2) and (3), we obtain

$$V(D; \varepsilon) = V(X^*; \varepsilon) \cup \{\emptyset\} \subseteq V(D; \delta) \subseteq V(X^*; \delta) \cup \{\emptyset\} \subseteq V(D; \varepsilon),$$

i.e.,

$$V(D; \delta) = V(X^*; \delta) \cup \{\emptyset\} = V(X^*; \varepsilon) \cup \{\emptyset\} = V(D; \varepsilon).$$

The lemma is proved. $\square$

Theorem 1. Let

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times T)$$

be a normal representation of the binary relation ε of the semigroup $B_X(D)$, $\emptyset \neq Z \in V(X^*; \varepsilon)$ and $\tilde{D}_\varepsilon(Z) = \{T \in \tilde{D}(\varepsilon) \mid Y_T^\varepsilon \cap Z \neq \emptyset\}$. The relation ε is an idempotent element of this semigroup if and only if $Z = \bigcup \tilde{D}_\varepsilon(Z)$ for any nonempty Z from $V(X^*; \varepsilon)$.

Proof. Assume that $\varepsilon \circ \varepsilon = \varepsilon$. Then for any Z from $V(X^*; \varepsilon)$ we have $Z = Y\varepsilon$ for some $Y \in X^*$ and $Z\varepsilon = (Y\varepsilon)\varepsilon = Y(\varepsilon \circ \varepsilon) = Y\varepsilon = Z$, i.e., $Z\varepsilon = Z$ for any $Z \in V(X^*; \varepsilon)$.

Let now $Z' = \cup \tilde{D}_\varepsilon(Z)$ and $t \in Z = Z\varepsilon$. Then $t \in t'\varepsilon$ for some $t' \in Z$. By assumption,

$$t' \in X = \bigcup_{T \in \tilde{D}(\varepsilon)} Y_T^\varepsilon$$

and therefore there exists an element $T(t') \in \tilde{D}(\varepsilon)$ such that $t' \in Y_{T(t')}^\varepsilon$ and $t' \in Y_{T(t')}^\varepsilon \cap Z$. Hence we obtain $t'\varepsilon = T(t')$ and $Y_{T(t')}^\varepsilon \cap Z \neq \emptyset$. Therefore $t \in \cup \tilde{D}_\varepsilon(Z) = Z'$, i.e., the inclusion $Z \subseteq Z'$ is valid.

Conversely, let $t_1 \in Z'$, i.e., there exists $T(t_1) \in \tilde{D}(\varepsilon)$ such that $t_1 \in T(t_1)$ and $Y_{T(t_1)}^\varepsilon \cap Z \neq \emptyset$. By the condition $Y_{T(t_1)}^\varepsilon \cap Z \neq \emptyset$ we obtain $t_1'\varepsilon = T(t_1)$ for some $t_1' \in Z$. Note that $t_1 \in t_1'\varepsilon$ since $t_1 \in T(t_1)$. By assumption, $Z = Z\varepsilon$ for any $Z \in V(X^*; \varepsilon)$. Therefore

$$t_1 \in t_1'\varepsilon \subseteq \bigcup_{t \in Z} t\varepsilon = Z\varepsilon = Z.$$

Thus we have $Z' \in Z$.

Now, taking into account the inclusions $Z \subseteq Z' \subseteq Z$, we obtain $Z = Z'$.

The necessity of the conditions of this theorem is proved.

Further, assume that ε is a binary relation of the semigroup $B_X(D)$ such that $Z = \cup \tilde{D}_\varepsilon(Z)$ for any nonempty Z from $V(X^*; \varepsilon)$ and prove that it is idempotent. Indeed, let $x\varepsilon y$ for some $x, y \in X$. Therefore $y \in x\varepsilon$ and $x\varepsilon \in V(X^*; \varepsilon)$. Hence by virtue of the equality

$$x\varepsilon = \bigcup \tilde{D}_\varepsilon(x\varepsilon) = \bigcup \{T \in \tilde{D}(\varepsilon) \mid Y_T^\varepsilon \cap x\varepsilon \neq \emptyset\}$$

and $y \in x\varepsilon$ there exists $T(y) \in \tilde{D}(\varepsilon)$ such that $y \in T(y)$ and $Y_{T(y)}^\varepsilon \cap x\varepsilon \neq \emptyset$. From the latter inequality we have $z\varepsilon = T(y)$ for some $z \in x\varepsilon$, i.e., $y \in z\varepsilon$ and $z \in x\varepsilon$, i.e., $x\varepsilon z\varepsilon y$ and $x(\varepsilon \circ \varepsilon)y$. Thus the inclusion $\varepsilon \subseteq \varepsilon \circ \varepsilon$ is valid.

On the other hand, let $x'\varepsilon z'\varepsilon y'$ for some $x', z', y' \in X$, i.e., $z' \in x'\varepsilon$ and $y' \in z'\varepsilon$. By assumption,

$$z' \in X = \bigcup_{T \in \tilde{D}(\varepsilon)} Y_T^\varepsilon$$

and therefore $z' \in Y_{T(z')}^\varepsilon$ for some $T(z') \in \tilde{D}(\varepsilon)$. Therefore $z'\varepsilon = T(z')$, $y' \in T(z')$ and $z' \in Y_{T(z')}^\varepsilon \cap x'\varepsilon$. From the conditions $y' \in T(z')$ and $Y_{T(z')}^\varepsilon \cap x'\varepsilon \neq \emptyset$ and by the definition of the set $\cup \tilde{D}_\varepsilon(x'\varepsilon)$ we have $y'\varepsilon \cup \tilde{D}_\varepsilon(x'\varepsilon)$. By assumption, $\cup \tilde{D}_\varepsilon(x'\varepsilon) = x'\varepsilon$ since $x'\varepsilon \in V(X^*; \varepsilon)$. Therefore $y' \in x'\varepsilon$ and $x'\varepsilon y'$. Hence the inclusion $\varepsilon \circ \varepsilon \subseteq \varepsilon$ is valid.

Now, taking into account the inclusions, we have $\varepsilon \subseteq \varepsilon \circ \varepsilon \subseteq \varepsilon$, which means that the binary relation ε is idempotent.

The sufficiency of the conditions of this theorem is proved.

The theorem is proved. $\square$

Definition 2. The bijection φ of the complete X -semilattice of unions D onto itself is called the complete automorphism of this semilattice if it preserves all exact upper faces of its any nonempty subsets, i.e.,

$$\varphi\left(\bigcup D'\right) = \bigcup_{T \in D'} \varphi(T)$$

for any nonempty subset D' of the semilattice D .

Lamma 7. Let

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\alpha \times T)$$

be a normal representation of the binary relation ε of the semigroup $B_X(D)$, $\emptyset \neq Z \in V(D; \varepsilon)$, $\tilde{D}_\varepsilon(Z) = \{T \in \tilde{D}(\varepsilon) \mid Y_T^\varepsilon \cap Z \neq \emptyset\}$ and φ be a bijection of the semilattice $V(D; \varepsilon)$ onto itself such that

$$\varphi(Z) = \bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi(T)$$

for any nonempty Z from $V(D; \varepsilon)$. Then the equalities

$$\bigcup_{Z \in D'} \left(\bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi(T) \right) = \bigcup_{T' \in \tilde{D}_\varepsilon(\cup D')} \varphi(T')$$

hold for any nonempty subset D' of the semilattice $V(D; \varepsilon)$.

Proof. Assume that

$$\overline{Z} = \bigcup_{Z \in D'} \left(\bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi(T) \right) \text{ and } \overline{Z}' = \bigcup_{T' \in \tilde{D}_\varepsilon(\cup D')} \varphi(T').$$

We will show that $\overline{Z} = \overline{Z}'$. Indeed, if $x_1 \in \overline{Z}$, then there exists $Z_1 \in D'$ such that $T_1 \in \tilde{D}_\varepsilon(Z_1)$ and $x_1 \in \varphi(T_1)$. From the conditions $T_1 \in \tilde{D}_\varepsilon(Z_1)$ it follows that $T_1 \in \tilde{D}(\varepsilon)$ and $Y_{T_1}^\varepsilon \cap Z_1 \neq \emptyset$. Moreover, $Z_1 \subseteq \cup D'$ since $Z_1 \in D'$. Hence $Y_{T_1}^\varepsilon \cap (\cup D') \supseteq Y_{T_1}^\varepsilon \cap Z_1 \neq \emptyset$. Therefore $T_1 \in \tilde{D}(\varepsilon)$ and $Y_{T_1}^\varepsilon \cap (\cup D') \neq \emptyset$. Hence, by the definition of the set $\tilde{D}_\varepsilon(\cup D')$, we have $T_1 \in \tilde{D}_\varepsilon(\cup D')$. Now, using the condition $x_1 \in \varphi(T_1)$ and the definition of the set $\overline{Z}'$, we obtain $x_1 \in \overline{Z}'$. Thus the inclusion $\overline{Z} \subseteq \overline{Z}'$ is valid.

Now assume that $y' \in \overline{Z}'$. Then $y' \in \varphi(T'_1)$ for some $T'_1 \in \tilde{D}_\varepsilon(\cup D')$. The conditions $T'_1 \in \tilde{D}_\varepsilon(\cup D')$ imply that $T'_1 \in \tilde{D}(\varepsilon)$ and $Y_{T'_1}^\varepsilon \cap (\cup D') \neq \emptyset$. From the latter inequality we obtain $y'_1 \in Y_{T'_1}^\varepsilon$ and $y'_1 \in \cup D'$. At that, according

to the conditions $y'_1 \in \cup D'$ there exists $Z' \in D'$ such that $y'_1 \in Z'$ and therefore $Y_{T'_1}^\varepsilon \cap Z' \neq \emptyset$. Thus $y' \in \varphi(T'_1)$, $T'_1 \in \tilde{D}(\varepsilon)$ and $Y_{T'_1}^\varepsilon \cap Z' \neq \emptyset$, i.e.,

$$y' \in \bigcup_{T \in \tilde{D}_\varepsilon(Z')} \varphi(T).$$

Hence we obtain $y' \in \overline{Z}$. Therefore $\overline{Z}' \subseteq \overline{Z}$.

Now, by the inclusions $\overline{Z} \subseteq \overline{Z}' \subseteq \overline{Z}$ we have $\overline{Z} = \overline{Z}'$.

The lemma is proved. $\square$

Theorem 2. *Let*

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\alpha \times T)$$

be a normal representation of the idempotent binary relation ε of the semi-group $B_X(D)$. The bijection φ of the semilattice $V(D; \varepsilon)$ onto itself is a complete automorphism of this semilattice if and only if $\varphi(\emptyset) = \emptyset$ if $\emptyset \in V(D; \varepsilon)$, and the conditions

$$\varphi(Z) = \bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi(T) \quad (1)$$

hold for any nonempty Z from $V(D; \varepsilon)$ and the subset $\tilde{D}_\varepsilon(Z) = \{T \in \tilde{D}(\varepsilon) | Y_T^\varepsilon \cap Z \neq \emptyset\}$ of the set $\tilde{D}(\varepsilon)$.

Proof. Assume first that the bijection φ of the semilattice $V(D; \varepsilon)$ onto itself is a complete automorphism of this semilattice. Then by Theorem 1 and the definition of a complete automorphism of complete X -semilattices of unions (see Definition 2) we have

$$\varphi(Z) = \varphi\left(\bigcup \tilde{D}_\varepsilon(Z)\right) = \varphi\left(\bigcup_{T \in \tilde{D}_\varepsilon(Z)} T\right) = \bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi(T).$$

for any nonempty Z from $V(D; \varepsilon)$.

Moreover, if $\emptyset \in V(D; \varepsilon)$, then $\emptyset$ is the least element of the semilattice $V(D; \varepsilon)$ and therefore $\emptyset \cup Z = Z$ for any Z from $V(D; \varepsilon)$. Hence by the definition of the complete isomorphism φ we obtain $\varphi(\emptyset) \cup \varphi(Z) = \varphi(Z)$, i.e., $\varphi(\emptyset) \subseteq \varphi(Z)$ for any $Z \in V(D; \varepsilon)$. The latter condition implies that $\varphi(\emptyset)$ is the least element of the set $\{\varphi(Z) | Z \in V(D; \varepsilon)\} = V(D; \varepsilon)$. Therefore $\varphi(\emptyset) = \emptyset$ since the least element of the set $V(D; \varepsilon)$ is defined uniquely.

The necessity of the conditions of this theorem is proved.

Now assume that equalities of form (1) hold for any nonempty Z from $V(D; \varepsilon)$ and subsets $\tilde{D}_\varepsilon(Z)$ of the set $\tilde{D}(\varepsilon)$. Let us prove that

$$\varphi\left(\bigcup D'\right) = \bigcup_{Z \in D'} \varphi(Z)$$

for any nonempty subset D' of the semilattice $V(D; \varepsilon)$. Indeed, by equality (1), for $\cup D' = Z'$ we have

$$\varphi\left(\bigcup D'\right) = \varphi(Z') = \bigcup_{T' \in \tilde{D}_\varepsilon(Z')} \varphi(T') = \bigcup_{T' \in \tilde{D}_\varepsilon(\cup D')} \varphi(T').$$

But by virtue of Lemma 7 the equality

$$\bigcup_{T' \in \tilde{D}_\varepsilon(\cup D')} \varphi(T') = \bigcup_{Z \in D'} \left(\bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi(T) \right)$$

is true, i.e.,

$$\varphi\left(\bigcup D'\right) = \bigcup_{Z \in D'} \left(\bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi(T) \right) = \bigcup_{Z \in D'} \varphi(Z)$$

since

$$\bigcup_{T' \in \tilde{D}_\varepsilon(Z)} \varphi(T) = \varphi(Z)$$

by virtue of (1).

The sufficiency of the conditions of this theorem is proved.

The theorem is proved. $\square$

Lemma 8. *The set of all complete automorphisms Φ_D of the complete X -semilattice of unions D is a group with respect to operations of multiplication of transforms.*

Proof. Let φ and ψ be arbitrary complete automorphisms of the complete X -semilattice of unions D , and ε and φ^{-1} be respectively the identical mapping of the semilattice D and the inverse mapping of the mapping φ . Then for any nonempty subset D' of the semilattice D we have

$$\begin{aligned} (\varphi\psi)\left(\bigcup D'\right) &= \varphi\left(\psi\left(\bigcup D'\right)\right) = \varphi\left(\psi\left(\bigcup_{T \in D'} T\right)\right) = \varphi\left(\bigcup_{T \in D'} \psi(T)\right) = \\ &= \bigcup_{T \in D'} \varphi(\psi(T)) = \bigcup_{T \in D'} (\varphi\psi)(T), \end{aligned} \quad (1)$$

$$e\left(\bigcup D'\right) = \bigcup D' = \bigcup_{T \in D'} T = \bigcup_{T \in D'} e(T), \quad (2)$$

$$\begin{aligned} \varphi^{-1}\left(\bigcup D'\right) &= \varphi^{-1}\left(\bigcup_{T \in D'} T\right) = \varphi^{-1}\left(\bigcup_{T \in D'} e(T)\right) = \\ &= \varphi^{-1}\left(\bigcup_{T \in D'} (\varphi\varphi^{-1})(T)\right) = \varphi^{-1}\left(\bigcup_{T \in D'} \varphi(\varphi^{-1}(T))\right) = \\ &= \varphi^{-1}\left(\varphi\left(\bigcup_{T \in D'} \varphi^{-1}(T)\right)\right) = (\varphi^{-1}\varphi)\left(\bigcup_{T \in D'} \varphi^{-1}(T)\right) = \end{aligned}$$

$$= e\left(\bigcup_{T \in D'} \varphi^{-1}(T)\right) = \bigcup_{T \in D'} \varphi^{-1}(T). \quad (3)$$

Now, bearing in mind that the product of two bijections is a bijection, from conditions (1), (2) and (3) it immediately follows that the set Φ_D is a group with respect to operations of multiplication of transforms.

The lemma is proved. $\square$

Lemma 9. *If φ and ψ are complete automorphisms of the complete X -semilattice of unions D such that their values on some generating set $\tilde{D}$ of the semilattice D coincide, then $\varphi = \psi$.*

Proof. Let $\varphi(Z) = \psi(Z)$ for any $Z \in \tilde{D}$. Assume that $Z \in D \setminus \tilde{D}$. Then $Z = \bigcup D'$ for some subset D' from the set $\tilde{D}$ and

$$\begin{aligned} \varphi(Z) &= \varphi\left(\bigcup D'\right) = \varphi\left(\bigcup_{T \in D'} T\right) = \bigcup_{T \in D'} \varphi(T) = \bigcup_{T \in D'} \psi(T) = \\ &= \psi\left(\bigcup_{T \in D'} T\right) = \psi\left(\bigcup D'\right) = \psi(Z). \end{aligned}$$

Therefore $\varphi(Z) = \psi(Z)$ for any $Z \in D$, i.e., $\varphi = \psi$.

The lemma is proved. $\square$

Lemma 10. *Let $\varepsilon, \alpha \in B_X(D)$, $\varepsilon \circ \varepsilon = \varepsilon$ and φ_α be some complete automorphism of the complete X -semilattice of unions $V(D; \varepsilon)$. If*

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times T) \quad \text{and} \quad \alpha = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha(T))$$

are normal representations of the binary relations ε and α , then $Z\alpha = \varphi_\alpha(Z)$ for any nonempty element Z from the semilattice $V(D; \varepsilon)$.

Proof. Let $\emptyset \neq Z \in V(D; \varepsilon)$ and

$$Z' = \bigcup \left\{ \varphi_\alpha(T) \mid T \in \tilde{D}(\varepsilon) \text{ and } Y_T^\varepsilon \cap Z \neq \emptyset \right\}.$$

Let us show that $Z\alpha = Z'$. Indeed, let $t \in Z\alpha$ for some $t \in X$. Then $t'\alpha t$ for some $t' \in Z$. By the normal representation of the relation α , the conditions $t'\alpha t$ imply that there exists an element T' from the set $\tilde{D}(\varepsilon)$ such that $t'(Y_{T'}^\varepsilon \times \varphi_\alpha(T'))t$. Therefore the condition $t' \in Y_{T'}^\varepsilon$ and $t \in \varphi_\alpha(T')$ hold. Moreover, $t' \in Y_{T'}^\varepsilon \cap Z$ since $t' \in Z$. Therefore $T' \in \tilde{D}(\varepsilon)$, $t \in \varphi_\alpha(T')$ and $Y_{T'}^\varepsilon \cap Z \neq \emptyset$. Hence, by the definition of the set Z' , we have $t \in Z'$. Thus in this case the inclusion $Z\alpha \subseteq Z'$ is valid. On the other hand, if $x \in Z'$, then $x \in \varphi_\alpha(T_1)$ for some $T_1 \in \tilde{D}(\varepsilon)$ satisfying the condition $Y_{T_1}^\varepsilon \cap Z \neq \emptyset$. If $x' \in Y_{T_1}^\varepsilon \cap Z$, then $x' \in X$ and $x'(Y_{T_1}^\varepsilon \times \varphi_\alpha(T_1))x$. By virtue of the normal representation of the relation α , from the conditions $T_1 \in \tilde{D}(\varepsilon)$ and

$x'(Y_{T_1}^\varepsilon \times \varphi_\alpha(T_1))x$ we have $x'\alpha x$, i.e., $x \in x'\alpha$, where $x' \in X$. Hence we obtain $x \in x'\alpha \subseteq Z\alpha$. Thus the inclusion $Z' \subseteq Z\alpha$ is valid.

Now, taking into account the inclusions $Z\alpha \subseteq Z' \subseteq Z\alpha$, we obtain $Z' = Z\alpha$. Furthermore, if $\tilde{D}_\varepsilon(Z) = \{T \in \tilde{D}(\varepsilon) \mid Y_T^\varepsilon \cap Z \neq \emptyset\}$, then by virtue of Theorem 1 the equality $Z = \cup \tilde{D}_\varepsilon(Z)$ is virtue for any nonempty $Z \in V(D; \varepsilon)$. Hence, since φ_α is a complete automorphism of the semilattice $V(D; \varepsilon)$, we obtain

$$\begin{aligned} Z\alpha &= Z' = \bigcup \left\{ \varphi_\alpha(T) \mid T \in \tilde{D}(\varepsilon) \text{ and } Y_T^\varepsilon \cap Z \neq \emptyset \right\} = \\ &= \bigcup \left\{ \varphi_\alpha(T) \mid T \in \tilde{D}_\varepsilon(Z) \right\} = \bigcup_{T \in \tilde{D}_\varepsilon(Z)} \varphi_\alpha(T) = \\ &= \varphi_\alpha \left(\bigcup_{T \in \tilde{D}_\varepsilon(Z)} T \right) = \varphi_\alpha(Z). \end{aligned}$$

The lemma is proved. $\square$

Lemma 11. *Let $\varepsilon, \alpha, \beta \in B_X(D)$, $\varepsilon \circ \varepsilon = \varepsilon$; φ_α and φ_β be respectively some complete automorphisms of the complete X -semilattice of unions $V(D; \varepsilon)$. If*

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times T), \quad \alpha = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha(T)) \text{ and } \beta = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\beta(T))$$

are normal representations of the binary relations ε, α and β , then

$$\alpha \circ \beta = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times (\varphi_\beta \varphi_\alpha)(T))$$

is always a normal representation of the product $\alpha \circ \beta$ of the binary relation α by the binary relation β .

Proof. Let φ be some complete automorphism of the semilattice $V(D; \varepsilon)$. Then by virtue of Theorem 2 and Lemma 10 we have

$$\begin{aligned} \alpha \circ \beta &= \left(\bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha(T)) \right) \circ \beta = \bigcup_{T \in \tilde{D}(\varepsilon)} \left((Y_T^\varepsilon \times \varphi_\alpha(T)) \circ \beta \right) = \\ &= \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha(T)\beta) = \bigcup_{T \in \tilde{D}(\varepsilon)} \left(Y_T^\varepsilon \times \varphi_\beta(\varphi_\alpha(T)) \right) = \\ &= \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times (\varphi_\beta \varphi_\alpha)(T)). \end{aligned} \tag{1}$$

Taking into account the fact that the product $\varphi_\beta \varphi_\alpha$ of two bijections φ_β and φ_α is again the bijection of the semilattice $V(D; \varepsilon)$, we may conclude that a representation of the binary relation $\alpha \circ \beta$ in form (1) is normal.

The lemma is proved. $\square$

Theorem 3. *Let*

$$\varphi = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times T) \quad (1)$$

be a normal representation of the idempotent binary relation ε of the semi-group $B_X(D)$. Then the binary relation α of the semigroup $B_X(D)$ belongs to the group $G_X(D; \varepsilon)$ if and only if its normal representation has the form

$$\alpha = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha(T)), \quad (2)$$

where φ_α is some complete automorphism of the complete X -semilattice of unions $V(D; \varepsilon)$.

Proof. Assume that $\alpha \in G_X(D; \varepsilon)$ and consider the mapping $\varphi_\alpha : V(D; \alpha) \rightarrow D$ satisfying the condition $\varphi_\alpha(Z) = Z\alpha$ for any $Z \in V(D; \alpha)$. By virtue of Lemma 6 we have $V(X^*; \alpha) \subseteq V(D; \alpha) = V(D; \varepsilon)$, which means that φ_α is the mapping of the semilattice $V(D; \varepsilon)$ in D . Furthermore, for $Z \in V(D; \varepsilon)$ and $Z = \emptyset$ ($\emptyset \in D$) we obtain $\varphi_\alpha(\emptyset) = \emptyset\alpha = \emptyset \in V(D; \varepsilon)$ by the definition of the semilattice $V(D; \varepsilon)$. Therefore it can be assumed that $Z \neq \emptyset$. In that case $Z \in X^*$ and $\varphi_\alpha(Z) = Z\alpha \in V(X^*; \alpha) \subset V(D; \varepsilon)$. Thus φ_α is the mapping of the semilattice $V(D; \varepsilon)$ in $V(D; \varepsilon)$. Let us prove that φ_α is a complete automorphism of the semilattice $V(D; \varepsilon)$. Indeed, let $Z_1\alpha = Z_2\alpha$ for some $Z_1, Z_2 \in V(D; \varepsilon)$. Since, by assumption, $\alpha \in G_X(D; \varepsilon)$, we obtain $\alpha \circ \beta = \beta \circ \alpha = \varepsilon$ for some $\beta \in G_X(D; \varepsilon)$. Moreover, for any $Z \in V(D; \varepsilon)$ we have $Z = Z'\varepsilon$ for some $Z' \in D$ and $Z\varepsilon = (Z'\varepsilon)\varepsilon = Z'(\varepsilon \circ \varepsilon) = Z'\varepsilon = Z$. From this and by virtue of the equality $Z_1\alpha = Z_2\alpha$ we obtain

$$Z_1 = Z_1\varepsilon = Z_1(\alpha \circ \beta) = (Z_1\alpha)\beta = (Z_2\alpha)\beta = Z_2(\alpha \circ \beta) = Z_2\varepsilon = Z_2.$$

Therefore $\varphi_\alpha(Z_1) \neq \varphi_\alpha(Z_2)$ if $Z_1 \neq Z_2$, i.e., the mapping φ_α is one-to-one.

Now, if $Z \in V(D; \varepsilon)$ and $Z\beta = Z_1$, then $Z = Z\varepsilon = Z(\beta \circ \alpha) = (Z\beta)\alpha = Z_1\alpha$ and therefore $\varphi_\alpha(Z_1) = Z_1\alpha = Z$. Thus φ_α is the bijection of the semilattice $V(D; \varepsilon)$ onto itself. Moreover, if D' is an arbitrary nonempty subset of $V(D; \varepsilon)$, then

$$\varphi_\alpha\left(\bigcup D'\right) = \left(\bigcup D'\right)\alpha = \left(\bigcup_{T \in D'} T\right)\alpha = \bigcup_{T \in D'} T\alpha = \bigcup_{T \in D'} \varphi_\alpha(T).$$

This and the fact that φ_α is the bijection of $V(D; \varepsilon)$ onto itself allow us to conclude that φ_α is a complete automorphism of the complete X -semilattice of unions $V(D; \varepsilon)$.

Further, by assumption, $\alpha = \varepsilon \circ \alpha$. Since $T_1\alpha \neq T_2\alpha$ for any and different T_1 and T_2 from the set $\tilde{D}(\varepsilon)$ ($\tilde{D}(\varepsilon) \subseteq V(X^*; \varepsilon) \subseteq V(D; \varepsilon)$), a representation of the binary relation α of the form

$$\alpha = \bigcup_{T\alpha \in \tilde{D}(\alpha)} (Y_T^\varepsilon \times T\alpha) = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha(T))$$

is normal by virtue of Corollary 1.

The necessity of the conditions of this Theorem is proved.

Now assume that the set $G_X(D; \varepsilon)$ consists of such binary relations α that are represented in form (2) for some complete automorphism φ_α of the complete X -semilattice of unions $V(D; \varepsilon)$. Let us prove that the set $G_X(D; \varepsilon)$ is a group with respect to the multiplication of binary relations. Indeed, by virtue of Lemma 6 the identical mapping ε of the semilattice $V(D; \varepsilon)$ is a complete automorphism of this semilattice. Moreover, a representation of the relation ε of the form

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} ((Y_T^\varepsilon \times t(T)))$$

is normal. Hence, by the definition of the set $G_X(D; \varepsilon)$, we have $\varepsilon \in G_X(D; \varepsilon)$.

Now, if $\alpha, \beta \in G_X(D; \varepsilon)$ and

$$\alpha = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha(T)), \quad \beta = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\beta(T)),$$

where φ_α and φ_β are complete automorphisms of the semilattice $V(D; \varepsilon)$, then by virtue of Lemma 11 the representation

$$\alpha \circ \beta = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times (\varphi_\beta \varphi_\alpha)(T))$$

is also normal. Moreover, by virtue of Lemma 8 the product $\varphi_\beta \varphi_\alpha$ of two complete automorphisms φ_β and φ_α of the semilattice $V(D; \varepsilon)$ is again a complete automorphism of the semilattice $V(D; \varepsilon)$. Therefore the condition $\alpha \circ \beta \in G_X(D; \varepsilon)$ is valid.

Further, from the assumption that φ_α is a complete automorphism of the semilattice $V(D; \varepsilon)$ it follows, by Lemma 8, that φ_α^{-1} too is a complete automorphism of this semilattice. Therefore the binary relation α' having a representation of the form

$$\alpha' = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\alpha^{-1}(T))$$

also belongs to the set $G_X(D; \varepsilon)$. Moreover,

$$\alpha \circ \alpha' = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times (\varphi_\alpha^{-1} \varphi_\alpha)(T)) = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times e(T)) = \varepsilon.$$

Now, in view of the results obtained above, we see that the set $G_X(D; \varepsilon)$ is a maximal subgroup of the semigroup $B_X(D)$ whose unit is the idempotent relation ε of the semigroup $B_X(D)$.

The theorem is proved. $\square$

Theorem 4. *For any idempotent binary relation ε of the semigroup $B_X(D)$, the group $G_X(D; \varepsilon)$ is anti-isomorphic to the group of all complete automorphisms of the semilattice $V(D; \varepsilon)$.*

Proof. Let ε be an arbitrary idempotent element of the semigroup $B_X(D)$ having a normal representation of the form

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times T). \quad (1)$$

Then by virtue of Theorem 3 any element α of the group $G_X(D; \varepsilon)$ can be represented as

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} ((Y_T^\varepsilon \times \varphi_\alpha(T))), \quad (2)$$

where φ_α is some element of the group Φ_ε of all complete automorphisms of the semilattice $V(D; \varepsilon)$. Let us define the mapping θ of the group $G_X(D; \varepsilon)$ in the group Φ_ε as follows: $\theta(\alpha) = \varphi_\alpha$ for any $\alpha \in G_X(D; \varepsilon)$. First we will show that the mapping is one-to-one. Indeed, if for the binary relation β from $G_X(D; \varepsilon)$ having a normal representation of the form

$$\varepsilon = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times \varphi_\beta(T)), \quad (3)$$

where $\varphi_\beta \in \Phi_\varepsilon$, the conditions $\alpha \neq \beta$ and $\theta(\alpha) = \theta(\beta)$ hold, then $\varphi_\alpha = \varphi_\beta$ and, by virtue of representations (2) and (3) we obtain $\alpha = \beta$. The latter equality however contradicts the assumption that $\alpha \neq \beta$. Therefore $\theta(\alpha) \neq \theta(\beta)$ is virtue for any different α and β from $G_X(D; \varepsilon)$. This and Theorem 3 imply that θ is the one-to-one mapping of the group $G_X(D; \varepsilon)$ onto the group Φ_ε . Moreover, by Lemmas 11 and 8, a representation of the binary relation $\alpha \circ \beta$ of the form

$$\alpha \circ \beta = \bigcup_{T \in \tilde{D}(\varepsilon)} (Y_T^\varepsilon \times (\varphi_\beta \varphi_\alpha)(T))$$

is normal and $\varphi_\beta \varphi_\alpha \in \Phi_\varepsilon$. Hence, by the definition of the mapping θ , we have $\theta(\alpha \circ \beta) = \varphi_\beta \varphi_\alpha$.

Thus the mapping $\theta : G_X(D; \varepsilon) \rightarrow \Phi_\varepsilon$ is an anti-isomorphism of the group $G_X(D; \varepsilon)$ onto the group Φ_ε .

The theorem is proved. $\square$

Corollary 2. *If the complete X -semilattice of unions D is a finite chain, then all maximal subgroups of the semigroup $B_X(D)$ are the unit ones.*

Proof. Let D be a finite chain. Then, for any idempotent binary relation ε the semigroups $B_X(D)$ of the lattice of unions $V(D; \varepsilon)$ are also a finite chain. But in that case the group Φ_ε of all complete automorphisms of the semilattice $V(D; \varepsilon)$ consists of the only identical mapping of $V(D; \varepsilon)$.

Hence, by virtue of Theorem 4, it follows that the group $G_X(D; \varepsilon)$ too is the unit one, i.e., $G_X(D; \varepsilon) = \{\varepsilon\}$.

The corollary is proved. $\square$

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