

COUNTEREXAMPLE WITHOUT CASES

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ABSTRACT. The standard examples of functions without a one-sided limit are defined piecewise, with at least two pieces. We give an example of a function defined everywhere by a single formula. This example is an infinite sum of simple rational functions.

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Does there exist a function which does not have a right hand limit at $x = 0$? The standard example is

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

This example is satisfactory from almost every modern point of view, but I distinctly remember that when I first took calculus, whenever a counterexample was called for, to my slight dissatisfaction the machinery of cases was always brought in. Let me try to make this vague complaint into a precise question. What I want is an everywhere defined function given by a single formula. It would be nice if the formula could be an analytic function. But at each point of discontinuity, such a function either has a pole or an essential singularity and hence is not defined. So I am forced to relax the rule. I will allow the formula to be an infinite sum of analytic functions. It follows that, sadly, any example must be inaccessible to beginning calculus students. Notice that such a function would have been an admissible example in, say, the year 1800, while $f(x)$ would not have been considered a function at that time. So the goal is to create an everywhere convergent infinite series of elementary functions that does not have a right hand limit at $x = 0$.

If $\{s_n(x)\}$ is *any* sequence of analytic functions tending pointwise to $f(x)$, then $-s_0(x) + \sum_{k=1}^{\infty} (s_k(x) - s_{k-1}(x)) = f(x)$. This provides a general

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solution, but fails to satisfy an additional criterion, which is harder to make precise: that the example be as simple as possible. I will give an example which is simple enough to be accessible to students who have completed a year of calculus. If the reader can think of a simpler example, I would certainly enjoy seeing it.

My example is an infinite sum of spikes

$$g(x) = \sum_{n=1}^{\infty} S_{a_n, M_n}(x),$$

where a spike, defined by

$$S_{a, M}(x) = \frac{1}{1 + M(x - a)^2},$$

is a positive function whose maximal height 1 is attained at $x = a$ and which is very sharply peaked if $M > 0$ is large. The $\{a_n\}$ are decreasing monotonically to zero and the M_n are picked to be so large that the $S_n(x) = S_{a_n, M_n}(x)$ satisfy

$$\sum_{n=1}^{\infty} \max \{S_n(b_n) S_n(b_{n-1})\} < \frac{1}{2},$$

where

$$b_n = \frac{a_{n+1} + a_n}{2}, \quad n \geq 1, \quad b_0 = a_1 + 1.$$

Then $g(x)$ is large at the peaks $\{a_k\}$,

$$\limsup_{x \rightarrow 0^+} g(x) \geq \limsup_{k \rightarrow \infty} g(a_k) \geq \lim_{k \rightarrow \infty} S_k(a_k) = 1,$$

while $g(x)$ is small in the valleys $\{b_k\}$,

$$\begin{aligned} \liminf_{x \rightarrow 0^+} g(x) &\leq \liminf_{k \rightarrow \infty} \sum_{n=1}^{\infty} S_n(b_k) \leq \\ &\leq \liminf_{k \rightarrow \infty} \sum_{n=1}^{\infty} \max \{S_n(b_n), S_n(b_{n-1})\} < \frac{1}{2}. \end{aligned}$$

To see that $S_n(b_k) \leq \max\{S_n(b_n), S_n(b_{n-1})\}$, note that $S_n(b_k) \leq S_n(b_n)$ if $k \geq n$, while $S_n(b_k) \leq S_n(b_{n-1})$ if $k < n$, since $S_n \nearrow$ on $(-\infty, a_n)$ and $S_n \searrow$ on (a_n, ∞) . Observe that the proof that $g(x)$ converges finitely at every x is similar to the calculation that g is small in the valleys.

In particular, if we set

$$a_n = \frac{1}{n}, \quad n = 1, 2, \dots,$$

and

$$M_n = 64n^6;$$

then for $n \geq 1$,

$$\begin{aligned} S_n(b_n) &= \frac{1}{1 + 64n^6 \left[\frac{1}{2} \left(\frac{1}{n+1} - \frac{1}{n} \right) \right]^2} = \\ &= \frac{1}{1 + \frac{16n^4}{(n+1)^2}} < \frac{(n+1)^2}{16n^4} \leq \frac{1}{2n(n+1)}, \end{aligned}$$

and for $n \geq 2$,

$$\begin{aligned} S_n(b_{n-1}) &= \frac{1}{1 + 64n^6 \left[\frac{1}{2} \left(\frac{1}{n} - \frac{1}{n-1} \right) \right]^2} = \\ &= \frac{1}{1 + \frac{16n^4}{(n-1)^2}} < \frac{(n-1)^2}{16n^4} < \frac{1}{16n(n+1)}, \end{aligned}$$

while for $n = 1$,

$$S_1(b_0) = \frac{1}{1 + 64(2-1)^2} \leq \frac{1}{(32)(1)(2)}.$$

Thus

$$\begin{aligned} \sum_{n=1}^{\infty} \max \{S_n(b_n) S_n(b_{n-1})\} &< \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{2}, \end{aligned}$$

and

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{1 + 64n^6 \left(x - \frac{1}{n} \right)^2} = \sum_{n=1}^{\infty} \frac{1}{1 + 64n^4 (nx - 1)^2}$$

does not have a limit as $x \rightarrow 0^+$.

Remark 1. A faster example is the Fourier series of $f(x)$ restricted to $[-\pi, \pi)$.

However, the coefficients are not easily expressed in closed form and a little theory is needed to demonstrate convergence.

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