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M. OKROPIRIDZE

Abstract. If the 2π periodic and summable on the interval $(-\pi,\pi)$ function $f(x)$ has a finite, symmetric derivative at the point x_0 and $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$, then

$$\lim_{(r,t)\to(1,0)} \sum_{k=1}^{\infty} r^k \sin kt (b_k \cos kx_0 - a_k \sin kx_0) = 0.$$

1. The 2π-periodic and summable on the interval $(-\pi,\pi)$ function $f(x)$ corresponds to Poisson's integral for the unit disk

$$U_f(r,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(t-x)\,dt, \quad P_r(u) = \frac{1-r^2}{1-2r\cos u + r^2}. \tag{1}$$

P. Fatou has obtained fundamental results on the boundary values of the integral (1) and its derivative. His theorem on the radial limit for the derivative of the integral (1) sounds as follows:

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Theorem (**P. Fatou** [1], [2], **p.**99) *If $f(x)$ has at the point x_0 a finite symmetric derivative*

$$f^{(1)}(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2h}, \tag{2}$$

then the derivative $\frac{\partial}{\partial x} U_f(r,x)$ of the integral (1) *has the radial limit at the point* $(1, x_0)$ *and this limit is equal to the number $f^{(1)}(x_0)$:*

$$\lim_{r\to 1} \Big(\frac{\partial U_f(r,x)}{\partial x}\Big)_{x=x_0} = f^{(1)}(x_0). \tag{3}$$

The proof presented here differs from the already known ones and allows us to get new facts (see Corollaries 1–3 below).
Proof of Equality (3). Note that the existence of the symmetric derivative $f^{(1)}(x_0)$ does not, generally speaking, require of the function f to be necessarily defined at the point x_0.

We introduce the function $\phi(t)$ in the form

$$\phi(t) = \begin{cases} \frac{1}{2}\big[f(x_0+t) - f(x_0-t)\big] & \text{for } t \neq 0 \\ 0 & \text{for } t = 0. \end{cases} \tag{4}$$

The finiteness of the limit (2) and the equality (4) imply that

$$\lim_{t\to 0} \phi(t) = 0 = \phi(0). \tag{5}$$

Hence the function $\phi(t)$ is continuous at the point $t = 0$ and $\phi(0) = 0$.

According to Schwarz's theorem, the Poisson integral $U_\phi(r,x)$, corresponding to the function $\phi(t)$, has at the point $(1,0)$ the zero limit ([3]; [4], p.147-8).

Consequently,

$$\lim_{(r,x)\to(1,0)} U_\phi(r,x) = 0. \tag{6}$$

For the unilateral derivatives we have at the point $t = 0$ the following equalities:

$$\phi'_+ = \lim_{t\to 0+} \frac{\phi(t) - \phi(0)}{t} = \lim_{t\to 0+} \frac{\phi(t)}{t} = f^{(1)}(x_0) \tag{7}$$

and

$$\phi'_- = \lim_{t\to 0-} \frac{\phi(t) - \phi(0)}{t} = \lim_{t\to 0-} \frac{\phi(t)}{t} = f^{(1)}(x_0). \tag{8}$$

Thus the function $\phi(t)$ has at the point $t = 0$ equal between themselves unilateral derivatives. Hence at the point $t = 0$ there exists the finite ordinary derivative $\phi'(0)$ equal to $f^{(1)}(x_0)$. Therefore the derivative $\frac{\partial}{\partial x} U_\phi(r,x)$

has at the point $(1,0)$ the angular limit equal to $\phi'(0)=f^{(1)}(x_0)$, by P. Fatou's theorem ([2], p.100). This means that for every constant $c>0$ the equality

$$\lim_{\substack{(r,x)\to(1,0)\\ |x|<c(1-r)}} \frac{\partial}{\partial x} U_\phi(r,x) = f^{(1)}(x_0) \tag{9}$$

is fulfilled. □

Now let us find the connection between the integrals $U_\phi(r,x)$ and $U_f(r,x)$. We have

$$2\pi U_\phi(r,x) = \frac{1}{2}\int_{-\pi}^{\pi} \big[f(x_0+t)-f(x_0-t)\big] P_r(t-x)\,dt =$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} f(\tau)P_r(\tau-x-x_0)\,d\tau - \frac{1}{2}\int_{-\pi}^{\pi} f(\tau)P_r(\tau+x-x_0)\,d\tau.$$

Consequently,

$$2U_\phi(r,x) = U_f(r,x_0+x) - U_f(r,x_0-x). \tag{10}$$

Equalities (6) and (10) yield

$$\lim_{(r,x)\to(1,0)} \big[U_f(r,x_0+x)-U_f(r,x_0-x)\big] = 0. \tag{11}$$

Next, from equality (10) we arrive at the following relations:

$$2\frac{\partial}{\partial x}U_\phi(r,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{\partial}{\partial x}P_r(t-x-x_0) - \frac{\partial}{\partial x}P_r(t+x-x_0)\Big]\,dt =$$

$$= \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{2r\sin(t-x-x_0)}{(1-2r\cos(t-x-x_0)+r^2)^2} +$$

$$+\frac{2r\sin(t+x-x_0)}{(1-2r\cos(t+x-x_0)+r^2)^2}\Big]\,dt. \tag{12}$$

Taking into account equality (9), it follows from (12) that

$$\lim_{\substack{(r,x)\to(1,0)\\ |x|<c(1-r)}} \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{r\sin(t+x-x_0)}{(1-2r\cos(t+x-x_0)+r^2)^2} +$$

$$+\frac{2r\sin(t-x-x_0)}{(1-2r\cos(t-x-x_0)+r^2)^2}\Big]\,dt = f^{(1)}(x_0), \quad c>0. \tag{13}$$

If now we put in equality (12) the particular value $x = 0$, then we shall get the equality

$$\Big(\frac{\partial}{\partial x} U_\phi(r,x)\Big)_{x=0} = \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{2r\sin(t-x_0)}{(1-2r\cos(t-x_0)+r^2)^2}\, dt \tag{14}$$

which by virtue of (13) implies that

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(t-x_0)\cdot Q_r(t-x_0)\, dt = f^{(1)}(x_0), \tag{15}$$

where Q_r is the conjugate Poisson kernel

$$Q_r(u) = \frac{2r\sin u}{1-2r\cos u + r^2}.$$

On the other hand, the equality

$$\Big(\frac{\partial}{\partial x} P_r(t-x)\Big)_{x=0} = P_r(t-x_0)\cdot Q_r(t-x_0) \tag{16}$$

is valid and therefore equality (15) can be written as

$$\lim_{r\to 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\partial}{\partial x} P_r(t-x)\, dt\right)_{x=x_0} = \\ = \lim_{r\to 1} \Big(\frac{\partial}{\partial x} U_f(r,x)\Big)_{x=x_0} = f^{(1)}(x_0). \tag{17}$$

Thus we have established equality (3).

2. The main point of equality (11) can be stated as follows:

Corollary 1. *If the 2π-periodic function $f \in L(-\pi,\pi)$ has at some point x_0 a finite, symmetric derivative $f^{(1)}(x_0)$, then the corresponding to it Poisson integral $U_f(r,x)$ has the property*

$$\lim_{(r,x)\to(1,0)} \big[U_f(r,x_0+x) - U_f(r,x_0-x)\big] = 0. \tag{18}$$

3. Equality (15) admits the following statement:

Corollary 2. *Let the function $f \in L(-\pi,\pi)$, $f(x+2\pi) = f(x)$ have at some point x_0 a finite, symmetric derivative $f^{(1)}(x_0)$. Then the equality*

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t-x_0)\cdot Q_r(t-x_0)\, dt = f^{(1)}(x_0) \tag{19}$$

holds.

4. If the Fourier series has for the 2π-periodic function $f \in L(-\pi, \pi)$ the form

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big),$$

then

$$U_f(r, x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \big(a_n \cos nx + b_n \sin nx\big),$$

and equality (3) is written as

$$\lim_{r \to 1} \sum_{n=1}^{\infty} n \cdot r^n \big(b_n \cos nx_0 - a_n \sin nx_0\big) = f^{(1)}(x_0). \tag{20}$$

Since

$$U_f(r, x_0 + x) – U_f(r, x_0 - x) = 2 \sum_{n=1}^{\infty} r^n \sin nx \big(b_n \cos nx_0 - a_n \sin nx_0\big), \tag{21}$$

Corollary 1 can be rephrased in the form of

Corollary 3. *If the 2π-periodic function $f \in L(-\pi, \pi)$ has at some point x_0 a finite, symmetric derivative $f^{(1)}(x_0)$, then the equality*

$$\lim_{(r,t) \to (1,0)} \sum_{n=1}^{\infty} r^n \sin nt \big(b_n \cos nx_0 - a_n \sin nx_0\big) = 0 \tag{22}$$

holds.

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Author's address:
Secondary School No. 51
2, Melikishvili St., Tbilisi, 380057
Georgia