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INVERSION AND CHARACTERIZATION OF RIESZ POTENTIALS IN WEIGHTED SPACES VIA APPROXIMATIVE INVERSE OPERATORS

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Abstract. Within the framework of the method of approximative inverse operators, the inversion of the Riesz potential operator $I^\alpha\varphi$ is constructed in the case of densities φ in weighted spaces $L_p(\rho)$ with the Muckenhoupt-Wheeden $A_{p,q}$-weight functions $\rho(x)$. The range $I^\alpha[L_p(\rho)]$ also is described in terms of the approximative inverse operators.

1. Introduction

Let

$$I^\alpha\varphi = \frac{1}{\gamma_n(\alpha)} \int\limits_{R^n} \frac{\varphi(t)}{|x-t|^{n-\alpha}}, \quad 0 < Re\alpha < n, \tag{1.1}$$

be the well known Riesz potential operator. We consider it in the weighted spaces $L_p(\rho)$, where the weight $\rho(x)$ belongs to the Muckenhoupt-Wheeden class $A_{p,q}$. As is known, the operator inverse to I^α within the framework of L_p-densities φ can be constructed in the form of hypersingular integral

$$D^\alpha f = \lim_{\varepsilon\to 0} \frac{1}{d_{n,l}(\alpha)} \int\limits_{|t|>\varepsilon} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}}\, dt$$

at least for real α. The range $I^\alpha(L_p)$ also can be described in terms of the operator D^α (see [9–13]; [15], Sections 2 and 7; 16, §25–26]). In the papers

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[5–6] (see also the book [15], Ch. 7) there may be found a generalization of these results to weighted spaces $L_p(\rho)$, $\rho(x) \in A_{p,q}$.

We also note, that hypersingular integrals of the form

$$D^\alpha_\Omega f = \int\limits_{R^n} \frac{(\Delta^l_t f)(x)}{|t|^{n+\alpha}} \Omega(t)\, dt$$

are also known to be an effective tool for inverting the generalized Riesz potentials

$$K^\alpha_\theta \varphi = \int\limits_{R^n} \frac{\theta(x-t)}{|x-y|^{n-\alpha}} \varphi(t)\, dt, \quad 0 < Re\alpha < n,$$

in the elliptic case, when the symbol of the operator K^α_θ does not vanish in $R^n \setminus \{0\}$ (we refer to the books [15], Section 8; 16, Chapter 5 and surveying paper [12]).

Recently, an alternative method of inversion, known as the method of approximative inverse operators (AIO), was developed. This method proved to be effective in non-elliptic cases, when the method of hypersingular integrals in its direct form does not work well or does not work at all (see the surveying papers [7]–[8]). Within the framework of AIO's method the inverse operator is constructed as the limit of a sequence of convolutions with integrable kernels.

The AIO's method was also applied for inverting some potentials in the elliptic case. In particular, in the case of operator (1.1), a general approach was developed in [13] (see also [14]). Within the framework of this approach the inversion is constructed in the form

$$T^\alpha f = \lim_{\varepsilon \to 0} T^\alpha_\varepsilon f, \tag{1.2}$$

$$T^\alpha_\varepsilon f = \varepsilon^{-\alpha} \int\limits_{R^n} q_\alpha(y) f(x - \varepsilon y) dy,$$

where

$$\widehat{q}_\alpha(\xi) = |\xi|^\alpha \widehat{k}(\xi) \tag{1.3}$$

$k(x)$ being any fixed averaging kernel, that is,

$$\int\limits_{R^n} k(y)\, dy = 1, \tag{1.4}$$

satisfying some general assumption, namely

$$k(y) \in L_1(R^n) \cap I^\alpha(L_1) \tag{1.5}$$

and which can be chosen by our will. We note that the relation (1.3) means that, $q_\alpha = D^\alpha k$.

Our goal in this paper is to develop the technique of approximative inverse operators for the same purposes in the case of weighted spaces $L_p(\rho)$, $\rho(x) \in A_{p,q}$.

2. Preliminaries

We shall use the following notation: $\langle f, \omega \rangle = \int\limits_{R^n} f(x)\overline{\omega(x)}dx$; $\widehat{f}(\xi) = \int\limits_{R^n} f(t)e^{i\xi\cdot t}dt$ is the Fourier transform of a function f; $(F^{-1}f)(x) = \tilde{f}(x) = \frac{1}{(2\pi)^n}\int\limits_{R^n} f(\xi)e^{-ix\cdot\xi}d\xi$ is the inverse Fourier transform; $L_p(\rho) = \big\{f(x) : \int\limits_{R^n}\rho(x)|f(x)|^p dx < \infty\big\}$, $1 \le p < \infty$; S is the Schwartz class of rapidly decreasing smooth functions; $\Psi = \{\psi \in S :\ (D^k\psi)(0) = 0,\ |k| = 0, 1, \dots\}$ is the Lizorkin class; $\Phi = \{\varphi \in S : \widehat{\phi} \in \Psi\}$ is its Fourier dual (see [1,2]). The class Φ is invariant with respect to the Riesz potential I^α.

Let A_p be the Muckenhoupt class (see [3]) of weight $\rho(x)$, satisfying the condition

$$\frac{1}{|Q|}\int\limits_Q \rho(x)dx\left(\frac{1}{|Q|}\int\limits_Q \rho^{-\frac{1}{p-1}}(x)dx\right)^{p-1} \le c < \infty,$$

$1 < p < \infty$, then $A_{p,q} \subset A_p$. We need the imbedding

$$L_p(\rho) \subset L_1((1+|x|)^{-n}), \quad \rho(x) \in A_p, \tag{2.1}$$

(see [5]) and the evident relation

$$\rho^{-\frac{1}{p-1}}(x) \in A_{p'}, \ \text{ if } \ \rho(x) \in A_p. \tag{2.2}$$

We shall also use the weighted Hölder inequality

$$\int\limits_{R^n} |f(x)g(x)|dx \le \|f\|_{L_p(\rho)}\|g\|_{L_{p'}(\rho^{1-p'})} \tag{2.3}$$

(see [16], formula (1.38)).

The end of proof is denoted by. □

3. The main results

Let $A_{p,q}$ be the class of weights $\rho(x)$, satisfying the following condition

$$\left(\frac{1}{|Q|}\int\limits_Q \rho^{\frac{q}{p}}(x)dx\right)^{\frac{p}{q}}\left(\frac{1}{|Q|}\int\limits_Q \rho^{-\frac{1}{p-1}}x)dx\right)^{p-1} \le c < \infty,$$

$$1 < p < \infty, \quad 1 < q < \infty,$$

where Q is an arbitrary n-dimensional cube, $|Q|$ being its measure (see [4]).

3.1. Assumption on the choice of the identity approximation kernel. In what follows, $k(x)$ is any kernel, satisfying condition (1.4), such that $k(x)$ itself and the corresponding kernel $q_\alpha(x)$ defined in (1.3) via Fourier transform, admit radial non-increasing integrable dominants (we note that the relation (1.5) is valid under these assumptions).

As an example of the kernel, satisfying the above assumption, one can take the kernel, defoned via Fourier transforms by

$$\widehat{k}(\xi) = e^{-|\xi|}$$

so that $k(x) = P(x, 1)$, where $P(x, t)$ is the Poisson kernel and then

$$q_\alpha(x) = \frac{\Gamma(n+\alpha)}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})} F\Big(\frac{n+\alpha}{2}, \frac{n+\alpha+1}{2}; \frac{n}{2}; -|x|^2\Big), \tag{3.1}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function, see [13]–[17].

Another example given below has an advantage in the sense that both $k(x)$ and $q_\alpha(x)$ are elementary functions:

$$q_\alpha(x) = \frac{(-1)^m}{\gamma_n(2m-\alpha)} \Delta^m \Big(\frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}-m}}\Big), \tag{3.2}$$

where m is an integer such that $2m > \alpha$. This kernel is preferable because is much simpler in comparison with (3.1). The kernel (3.2) corresponds to the averaging kernel

$$k(x) = \frac{c}{(1+|x|^2)^{m+\frac{n-\alpha}{2}}}, \qquad c = \frac{\Gamma(m+\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}}\Gamma(m-\frac{\alpha}{2})},$$

see details in [16]–[17].

The following two statements adjoin to Theorems 7.35 and 7.36 in [15] where α was real and instead of the approximative operator T^α there was used the hypersingular operator D^α. It is supposed that the operator T^α is defined by any kernel $k(x)$ satisfying the above **Assumption on identity approximation kernel**.

Theorem 3.1. *Let* $f = I^\alpha\varphi$, $0 < Re\alpha < n$, $\varphi \in L_p(\rho)$, $\rho(x) \in A_{p,q}$, $1 < p < n/Re\alpha$, $\frac{1}{q} = \frac{1}{p} - \frac{Re\alpha}{n}$. *Then*

$$T^\alpha I^\alpha \varphi = \varphi, \tag{3.3}$$

where T^α *is the operator* (1.2) *with the kernel* $q_\alpha(x)$ *of the form* (1.3), *the limit in* (1.2) *being taken in the* $L_p(\rho)$*-norm or almost everywhere.*

The next theorem gives a description of the range $I^\alpha[L_p(\rho)]$.

Theorem 3.2. *Let α and $\rho(x)$ be the same as in Theorem* (3.1)*. A function $f(x)$ belongs to the space $I^\alpha[L_p(\rho)]$ if and only if*

1) $f(x) \in L_q(\rho^{\frac{q}{p}})$;

2) *one of the following conditions is fulfilled:*

a) $T^\alpha f \in L_p(\rho)$, *where T^α is operator* (1.2) *with the limit in* (1.2) *taken in the $L_p(\rho)$-norm,*

b) $\sup\limits_{\varepsilon>0} \|T^\alpha_\varepsilon f\|_{L_p(\rho)} < \infty$.

4. Some auxiliary statements

The following statements play an important role in the proof of Theorems 3.1 and 3.2.

Theorem 4.1 ([4]). *Let $0 < \alpha < n$ and $\rho(x) \in A_{p,q}$, $p < n/\alpha$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then the operator I^α is bounded from $L_p(\rho)$ into $L_q(\rho^{\frac{q}{p}})$.*

Denote $k_\varepsilon(x) = \varepsilon^{-n}k(x/\varepsilon)$.

Theorem 4.2 ([5; 15, Theorem 7.31]). *a) Let $k(x)$ have a non-increasing radial dominant $b(|x|) \in L_1$ and $f \in L_p(\rho)$, $\rho(x) \in A_p$. Then*

$$\sup_{\varepsilon>0} |(k_\varepsilon * f)(x)| \le c\|b\|_1 (Mf)(x), \tag{4.1}$$

where $(Mf)(x)$ is the Hardy-Littlewood maximal function.

b) If in addition $\int\limits_{R^n} k(x)dx = 1$, then

$$(k_\varepsilon * f)(x) \to f(x)$$

as $\varepsilon \to 0$ in the $L_p(\rho)$-norm and almost everywhere.

Theorem 4.3. *The Lizorkin class Φ is dense in $L_p(\rho)$, $\rho(x) \in A_p$.*

Theorem 4.3, which is of special interest itself, was proved in [15], Theorem 7.34. Here we give another proof based on the estimate (4.1).

Proof. Since the class C_0^∞ is dense in $L_p(\rho)$, $\rho(x) \in A_p$ (see [15], Theorem 7.32), it suffices to approximate a function $f \in C_0^\infty$ in the norm of $L_p(\rho)$ by functions in Φ. Let

$$\psi_N(\xi) = \mu(N|\xi|)\widehat{f}(\xi),$$

where $\mu(r) \in C^\infty([0,\infty))$ is such that $\mu(r) = 1$ for $r \ge 2$, $\mu(r) = 0$ for $r \le 1$ and $0 \le \mu(r) \le 1$. We set $f_N(x) = \widetilde{\psi_N}(x)$ (then $f_N \in \Phi$) and represent $f_N(x)$ as follows:

$$f_N(x) = f(x) - (K_N f)(x),$$
$$(K_N f)(x) = \int\limits_{R^n} k(|t|) f(x - Nt)dt,$$

where $k(|t|) = F^{-1}(1-\mu(|\xi|))(t)$. It remains to prove that

$$\lim_{N\to\infty} \int\limits_{R^n} \rho(x)|(K_N f)(x)|^p dx = 0. \tag{4.2}$$

The equality (4.2) is justified by the application of Lebesgue dominated theorem, which is applicable in view of (4.1) due to the fact, that the operator M is bounded in $L_p(\rho)$, $\rho(x) \in A_p$ (see [3]); the relation $(K_N f)(x) \to 0$, $x \in R^n$ as $N \to \infty$ is evident. □

5. Proof of the main results

5.1. *Proof of Theorem* 3.1. We base ourselves on the equality

$$(T^\alpha_\varepsilon I^\alpha \varphi)(x) = \frac{1}{\varepsilon^n} k\Big(\frac{x}{\varepsilon}\Big) * \varphi, \tag{5.1}$$

$\phi \in L_p(\rho)$, $\rho(x) \in A_{p,q}$. This equality is verified via Fourier transform for $\varphi \in \Phi$. It is extended by boundedness to the whole $L_p(\rho)$ by virtue of Theorem 4.3, since the operators on both sides of (5.1) are bounded from $L_p(\rho)$ into $L_q(\rho^{\frac{q}{p}})$. This is evident for the operator on the right-hand side. Boundedness of the operator $T^\alpha_\varepsilon I^\alpha$ follows from Theorem 4.1 and the fact, that T^α_ε is bounded in $L_q(\rho^{\frac{q}{p}})$ (we observe that $\rho^{\frac{q}{p}}(x) \in A_q$). The last statement follows from (4.1) (in view of our assumption on $q_\alpha(x)$). It remains to pass to the limit as $\varepsilon \to 0$ in (5.1), in the $L_p(\rho)$-norm or almost everywhere, which is possible by the statement b) of Theorem 4.2. As a result, we arrive at (3.1). □

5.2. *Proof of Theorem* 3.2. The “only if” part of Theorem 3.2 follows from Theorem 4.1 and the equality (5.1). Passing to the “if” part, first, we consider the part a), that is, we assume that $f \in L_q(\rho^{\frac{q}{p}})$ and $T^\alpha f \in L_p(\rho)$, the limit in (1.2) being understood in the norm of $L_p(\rho)$. For $\omega \in \Phi$ we have

$$\begin{aligned} \langle I^\alpha T^\alpha f, \omega\rangle = \langle T^\alpha f, I^\alpha \omega\rangle = \\ = \langle \overset{(L_p(\rho))}{\lim_{\varepsilon\to 0}} T^\alpha_\varepsilon f, I^\alpha\omega\rangle = \lim_{\varepsilon\to 0}\langle T^\alpha_\varepsilon f, I^\alpha \omega\rangle = \\ = \lim_{\varepsilon\to 0}\langle f, T^\alpha_\varepsilon I^\alpha\omega\rangle = \lim_{\varepsilon\to 0}\langle f, \frac{1}{\varepsilon^n}k(\frac{x}{\varepsilon}) * \omega\rangle = \langle f, \omega\rangle. \end{aligned} \tag{5.2}$$

The third of these equalities follows from the fact, that the convergence in $L_p(\rho)$ implies that in the space of distributions Φ'. The fourth equality is justified by Fubini theorem, applicable in view of (2.3). The last equality is justified with the aid of (2.3).

Thus, we arrived at the formula

$$\langle f, \omega\langle = \langle I^\alpha T^\alpha f, \omega\langle, \quad \omega \in \Phi. \tag{5.3}$$

Since two locally integrable functions (in S'), which agree as Φ'-distributions, may differ from each other only by a polynomial, we obtain

$$f(x) = (I^\alpha T^\alpha f)(x) + P(x),$$

where $P(x)$ is a polynomial. But because of (2.1) we have $P(x) \equiv 0$.

Let now $f \in L_q(\rho^{\frac{q}{p}})$ and $\sup\limits_{\varepsilon>0} \|T^\alpha_\varepsilon f\|_{L_p(\rho)} < \infty$. Making use of weak compactness of the space $L_p(\rho)$, $p > 1$, we choose such a sequence $T^\alpha_{\varepsilon_k} f$ that

$$w - \overset{(L_p(\rho))}{\lim_{\varepsilon_k \to 0}} T^\alpha_{\varepsilon_k} f = \varphi \in L_p(\rho),$$

where the symbol $w - \overset{(L_p(\rho))}{\lim}$ denotes the weak limit in $L_p(\rho)$. Replacing $\overset{(L_p(\rho))}{\lim}$ by $w - \overset{(L_p(\rho))}{\lim}$ in the chain of equalities (5.2), we also arrive at (5.3). As above, we derive from here that $f(x) = (I^\alpha T^\alpha f)(x)$. $\square$

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