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ON APPROXIMATE SOLUTION OF DIRICHLET PROBLEM IN THE CASE OF INFINITE PLANE WITH HOLES

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Abstract. A modified version of MFS (the method of fundamental solutions) for the Laplace operator in the plane case is considered. Based on this version, the method of an approximate solution of Dirichlet value problem for infinite plane with holes is given. Numerical examples are considered to illustrate effectiveness and simplicity of the proposed method.

1. Introduction

Let a domain D be the infinite plane $z = x + iy \equiv (x, y)$ with the holes $B_i (i = 1, 2, \cdots, m)$, which are bounded, respectively by closed piecewise smooth contours $S_i (S_i = \sum_{j=1}^{l} S_i^j;\ i = 1, 2, \cdots, m)$ having no multiple points. It is evident that the whole boundary of domain D will be $S = \bigcup_{i=1}^{m} S_i$. Further, suppose that we know the parametric equations of smooth curves S_i^j: $x_i = x_i^j(t)$, $y_i = y_i^j(t)$, where $\alpha_i^j \leq t \leq \beta_i^j$.

We will consider the following Dirichlet boundary problem for the domain D. We need to find a function $u(x, y) \in C^2(D) \bigcap C(\overline{D})$ satisfying the

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conditions:

$$\Delta u(z) = 0 \quad \forall z \equiv (x,y) \in D \tag{1.1}$$

$$u(z) = g(z) \qquad \forall z \in S \tag{1.2}$$

$$u(z) = c + O\Big(\frac{1}{|z|}\Big) \qquad \text{for} \quad |z| \to \infty. \tag{1.3}$$

Here Δ is the two-dimensional Laplace operator, $g(z) \equiv g(x,y)$ is a given continuous function on the contour S and c is the real constant provided $|c| < \infty$. Problems of type (1.1), (1.2), (1.3) occur, e.g.,: in hydromechanics; in elasticity theory; when studying plane fields (electric, thermal and so on) in infinite domains produced by point sources and in other areas of mathematical physics.

It is known [1]–[2] that the classical problem (1.1), (1.2), (1.3) is correct, i.e., the solution exists, is unique and depends continuously on the data. Note that the boundedness of the solution at infinity is an essential condition for its uniqueness (see [1]–[2]).

It can be easily shown that if we fix in advance the value of the constant c, this will be a rather strong restriction. Really, since under conditions (1.1), (1.2), (1.3) for the function $u(x,y)$ the minimax principle is fulfilled (see [1]–[2]), hence the problem (1.1), (1.2), (1.3) with hitherto fixed c may turn out generally unsolvable. To avoid this fact the constant c should be defined from the condition (1.2) while solving the problem (1.1), (1.2), (1.3).

It should be noted that when dealing with the problem of type (1.1), (1,2), (1.3) finite difference methods and variation methods are not applicable directly. For this we must pass to a finite domain by means of conformal mapping and then use these methods (see [3–5]). It should be also noted that the possibility of solving the problems of type (1.1), (1.2), (1.3) is studied theoretically using singular integral equations (Cauchy type integral) in works of N. I. Muskhelishvili and his successors (see [6]).

2. The Modified Version of MFS (the Method of Fundamental Solutions)

It is known (see [7–9]) that the method of Kupradze-Aleksidze (fundamental solutions) can be used in the general case to solve approximately both internal and external boundary value problems (and also the number of connectivity of domain and dimension of space do not meaning). On the basis of theory [7]–[8] the system

$$\big\{\varphi_{i,k}(z)\big\}_{k=1}^{\infty} = \big\{\ln|z - \widetilde{z}_{i,k}|\big\}_{k=1}^{\infty}, \quad z \in S, \quad (i = 1, 2, \cdots, m), \tag{2.1}$$

plays the role of the system of fundamental solutions of the Laplace operator. In (2.1) $\{\widetilde{z_{i,k}}\}_{k=1}^{\infty}$ is a countable set of points lying everywhere densely on the auxiliary closed Liapunov contours $\widetilde{S}_i(\widetilde{S} = \bigcup\limits_{i=1}^{m} \widetilde{S}_i)$, where contours $\widetilde{S}_i$ lie respectively inside the finite domains B_i and $\min \rho(S_i, \widetilde{S}_i) > 0$, where

ρ is the distance between S_i and $\widetilde{S}_i$. It is known that the system (2.1) is linearly independent and complete not only in the space $L_2(S)$, but also in $C(S)$. Theoretically, with the aid of system (2.1), the boundary function $g(z)$ can be approximated to within any accuracy, but it is inconvenient for an approximate solution of problem (1.1), (1.2), (1.3). Indeed, when using the MFS, the approximate solution is sought in the form

$$u_N(z) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \ln \left| z - \widetilde{z}_{i,k} \right|, \quad z \in \overline{D},$$

where $N = N_1 + N_2 + \cdots + N_m$, the points $\widetilde{z}_{i,k} (k = 1, 2, \cdots, N_i)$ are situated "uniformly" on the auxiliary contour $\widetilde{S}_i$, and $a_k^{(N_i)}$ are the coefficients of expansion of the function $g(z)$ into a series with respect to the first N_i functions of system (2.1). It is obvious that $\lim u_N(z) = \infty$ as $|z| \to \infty$, which means that condition (1.3) for the solution to be unique is not fulfilled.

Remark 1. In the sequel (see [9]) while solving approximately the boundary value problem, under contour $\widetilde{S}_i$ we will mean the Jordan contour which represents the boundary of the plane figure $\widetilde{B}_i (\widetilde{B}_i \subset B_i)$. The figure $\widetilde{B}_i$ is similar to B_i, oriented in the same way and they have one and same the "center" of gravity. As for the values $\rho(S_i, \widetilde{S}_i)$ and N_i, they can be chosen during the numerical realization of the algorithm, taking into account an *a posteriori* estimates of the accuracy of the results.

Remark 2. If we seek the solution to the problem (1.1), (1.2), (1.3) in the form

$$u_N(z) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \ln \left| z - \widetilde{z}_{i,k} \right| + c_N,$$

under the condition $\sum\limits_{i=1}^{m} \sum\limits_{k=1}^{N_i} a_k^{(N_i)} = 0$, where c_N is a real constant and $|c_N| < \infty$, then it can be easily proved that $u_N(\infty) = c_N$. However while finding the constants $a_k^{(N_i)} (k = 1, 2, \cdots, N_i;\ i = 1, 2, \cdots, m)$ and c_N some considerable difficulties arise which are connected with investigation of the questions about solvability of obtained systems and conditionality of its matrix.

In order to construct the desired system of functions $\{\psi_{i,k}(z)\}_{k=1}^{\infty}$, $(i = 1, 2, \cdots, m)$, we pass from the infinite multiply connected (namely m–connected) domain D to the finite multiply connected domain D^* bounded by S^* $(S^* = \bigcup\limits_{i=1}^{m} S_i^*)$, using the elementary conformal mapping

$$z^* = z_0 + \frac{R^2}{z - z_0}, \tag{2.2}$$

where z_0 is "the center" of either domain (hole) from domains B_i, and R is the real constant which is the radius of the circumference G centered at the point z_0. In the sequel we will call this domain the basic hole and we denote it with B_1. It is obvious that there is no necessity for the domain B_1 to contain the circumference $G(z_0; R)$ entirely. It should be mentioned that theoretically it does not mean which of domains B_i $(i = 1, 2, \cdots, m)$ will be in the role of the basic domain (hole). However while numerical realization that hole (if such exists) should be taken as basic which lies in the "center" with respect to others. This will give possibility to avoid big numbers (i.e., to reduce calculation error). In the case of the conformal mapping (2.2) the situation is as follows: the whole plane z is mapped onto the whole plane z^* and conversely. In particular, $z = \infty$ transforms into the point $z^* = z_0 \in D^*$; the infinite domain D becomes the finite m-connected domain D^*, while the contours S and $\widetilde{S}$ transform without changing their differential properties, into the contours S^* and $\widetilde{S}^*$, respectively; the contour $\widetilde{S}_1^*$ (the image of auxiliary contour $\widetilde{S}_1$) will contain the contour S_1^* and the rest contours $\widetilde{S}_i^*$ $(i \neq 1)$ will be respectively situated inside B_i^* (which are the images of the domains $B_i (i \neq 1)$); $\min \rho(S_i^*, \widetilde{S}_i^*) > 0$ sinse $\min \rho(S_i, \widetilde{S}_i) > 0$.

From (2.2) we have

$$z = z_0 + \frac{R^2}{z^* - z_0}, \tag{2.3}$$

consequently it is easy to see that if mapping (2.2) defines the domain D^* with boundary S^* , then with the help of mapping (2.3) the problem (1.1), (1.2), (1.3) transforms into Dirichlet problem for the domain D^*. Indeed, for the conformal mapping (2.3) the function $u(z)$ becomes the function $u^*(z^*) = u(z_0 + \frac{R^2}{z^*-z_0})$ which is harmonic in $D^* \backslash \{z_0\}$, continuous in $\overline{D}^* \backslash \{z_0\}$ and bounded in the neighborhood of the point z_0 (i.e., z_0 is a removable singular point). However, by virtue of the theorem on the elimination of singularities of a harmonic function (see [1], [2], [3]), the function $u^*(z^*)$ is harmonic in the domain D^* and continuous on $\overline{D}^*$. Thus, using mapping (2.2) we actually reduce problem (1.1), (1.2), (1.3) to the problem

$$\Delta u^*(z^*) = 0, \quad \forall z^* = (x^*, y^*) \in D^*, \tag{2.4}$$

$$u^*(z^*) = g^*(z^*), \quad \forall z^* \in S^*, \tag{2.5}$$

where $g^*(z^*) = g(z_0 + \frac{R^2}{z^*-z_0})$.

From (2.2) we obtain

$$\left|(z^*(z))'\right| = \Big(\frac{R}{|z - z_0|}\Big)^2,$$

and therefore, by choosing the point z_0 and radius R, in applied problems a strong compression or extension does not take place at boundary points when mapping (2.2) is used. This means that under mapping (2.2) the countable sets of points $\{z_{i,k}\}_{k=1}^{\infty}$, $\{\widetilde{z}_{i,k}\}_{k=1}^{\infty}$ $(i = 1, 2, \cdots, m)$ lying everywhere densely on the contours S and $\widetilde{S}$ respectively, transforms into the countable sets of points $\{S_{i,k}^{*}\}_{k=1}^{\infty}$, $\{\widetilde{z}_{i,k}^{*}\}_{k=1}^{\infty}$ lying everywhere densely on the contours S^{*} and $\widetilde{S}^{*}$ respectively and conversely. Thus if we consider the system of functions

$$\big\{\varphi_{i,k}^{*}(z^{*})\big\}_{k=1}^{\infty} \equiv \big\{\varphi^{*}(z^{*}, \widetilde{z}_{i,k}^{*}\big\}_{k=1}^{\infty} = \big\{ \ln |z^{*} - \widetilde{z}_{i,k}^{*}|\big\}_{k=1}^{\infty}, \qquad (2.6)$$

where the points $\{\widetilde{z}_{i,k}^{*}\}_{k=1}^{\infty}$ lie everywhere densely on the contour $\widetilde{S}^{*}$, it can be used for an approximate solution of problem (2.4), (2.5), since, by virtue of the theory (see [7,8]), the system of fundamental solutions (2.6) is linearly independent and complete in the spaces $L_2(S^{*})$ and $C(S^{*})$. If we insert the values z^{*} and $\widetilde{z}_{i,k}^{*}$ from (2.2) into system (2.6), then it is obvious that for the resulting system of fundamental solutions

$$\big\{\psi_{i,k}(z)\big\}_{k=1}^{\infty} \equiv \big\{\psi(z, \widetilde{z}_{i,k})\big\}_{k=1}^{\infty} = \Big\{ \ln \Big| \frac{R^2(\widetilde{z}_{i,k} - z)}{(z - z_0)(\widetilde{z}_{i,k} - z_0)} \Big| \Big\}_{k=1}^{\infty}, \qquad (2.7)$$

of Laplace operator following theorem is valid.

Theorem 1. 1_0. *$\Delta\psi_{i,k}(z) = 0$, $\forall z \in D$;*

2_0. *The system $\{\psi_{i,k}(z)\}_{k=1}^{\infty}(i = 1, 2, \cdots, m)$ is linearly independent and complete not only in the space $L_2(S)$, but also in $C(S)$.*

3_0. *$\lim\limits_{\psi_{i,k}}(z)$ is finite as $|z| \to \infty$.*

Since, in our case $g(z) \in C(S)$, therefore on the basis of property 2_0 for the arbitrary $\varepsilon > 0$ there exist such natural numbers $N_i^0(\varepsilon)$ and system of coefficients $a_k^{(N_i)}$ $(i = 1, 2, \cdots, m,\ k = 1, 2, \cdots, N_i)$, that if $N_i \geq N_i^0$, then

$$\max_{z\in S} \Big| g(z) - \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \psi_{i,k}(z) \Big| < \varepsilon.$$

If we introduce notation

$$u_N(z) \equiv u_N(x, y) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \psi_{i,k}(z),$$

where $N = N_1 + N_2 + \cdots + N_i$, then on the basis of minimax principle we obtain that $\max\limits_{z\in\overline{D}} |u(z) - u_N(z)| < \varepsilon$, where $u(z)$ is exact solution to the problem (1.1), (1.2), (1.3), i.e., $u_N(z)$ converges uniformly to $u(z)$ in $\overline{D}$ for $N \to \infty$. Actually, we have the sequence of harmonic functions $\{u_N(z)\}$ in $\overline{D}$, which converges uniformly to $u(z)$ in $\overline{D}$. It should be noted that when solving the problem (2.4), (2.5) with the help of the system (2.6) the conditions of the first theorem of Harnack [5] are fulfilled. On this basis it is evident that analogous theorem takes place in solving of the problem (1.1), (1.2), (1.3) using the system (2.7). Namely, following theorem is valid.

Theorem 2. *The sequence of harmonic functions $\{u_N(z)\}$ not only converges uniformly in $\overline{D}$ to $u(z)$, but the sequence of derivatives of $u_N(z)$ converges as well to derivatives of $u(z)$ uniformly in D, i.e.,*

$$\lim_{N\to\infty} u_N(z) = u(z), \quad z \in \overline{D},$$
$$\lim_{N\to\infty} \frac{\partial^n u_N(x,y)}{\partial x^p \partial y^{n-p}} = \frac{\partial^n u(x,y)}{\partial x^p \partial y^{n-p}}, \quad (x,y) \in D, \tag{2.8}$$

where n is fixed natural number, p is integer and $n \geq p \geq 0$.

Under $N \to \infty$ we mean that all $N_i \to \infty$. For visualization we show correctness of the equality (2.8), when $n = 1$.

From (2.2) and (2.3) we have

$$\begin{cases} x^* = x^*(x,y) \\ y^* = y^*(x,y) \end{cases}, \qquad \begin{cases} x = x(x^*,y^*) \\ y = y(x^*,y^*) \end{cases},$$

therefore

$$\begin{aligned} \frac{\partial u_N(x,y)}{\partial x} &= \frac{\partial u_N^*(x^*,y^*)}{\partial x} = \frac{\partial u_N^*(x^*,y^*)}{\partial x^*}\,\frac{\partial x^*}{\partial x} + \frac{\partial u_N^*(x^*,y^*)}{\partial y^*}\,\frac{\partial y^*}{\partial x}, \\ \frac{\partial u(x,y)}{\partial x} &= \frac{\partial u^*(x^*,y^*)}{\partial x} = \frac{\partial u^*(x^*,y^*)}{\partial x^*}\,\frac{\partial x^*}{\partial x} + \frac{\partial u^*(x^*,y^*)}{\partial y^*}\,\frac{\partial y^*}{\partial x}. \end{aligned} \tag{2.9}$$

Since in (2.9) $u^*(x^*,y^*)$ is exact solution to the problem (2.4), (2.5) and $u_N^*(x^*,y^*)$ is the approximate solution to the problem (2.4), (2.5) (with the aid of system (2.6)), hence in terms of the theorem of Harnack we have

$$\begin{aligned} &\frac{\partial_N(x^*,y^*)}{\partial x^*} \to \frac{\partial u^*(x^*,y^*)}{\partial x^*}, \ \frac{\partial u_N^*(x^*,y^*)}{\partial y^*} \to \\ &\to \frac{\partial u^*(x^*,y^*)}{\partial y^*}, \ \text{ as } \ N \to \infty, \end{aligned} \tag{2.10}$$

for the arbitrary point (x^*,y^*) from the domain D^*. It is evident, in terms of (2.10) from (2.9) we have

$$\frac{\partial u_N(x,y)}{\partial x} \to \frac{\partial u(x,y)}{\partial x}, \ \text{ as } \ N \to \infty,$$

for $\forall (x,y) \in D$. Q.E.D.

Thus, for approximate solution of the problem (1.1), (1.2), (1.3), we built one modified version of MFS and on the basis of the stated Theorem 2 the approximate solution $u_N(z)$ is to be found in the form

$$u_N(z) \equiv u_N(x,y) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \ln \left| \frac{R^2(\widetilde{z}_{i,k} - z)}{(z - z_0)(\widetilde{z}_{i,k} - z_0)} \right|, \tag{2.11}$$

where the auxiliary points (simulation sources) $\widetilde{z}_{i,k}(i = 1, 2, \cdots, m;\ k = 1, 2, \cdots N_i)$ are situated "uniformly" on the contours $\widetilde{S}_i$.

As for the coefficients $a_k^{(N_i)}$, they can be found (see [7,8,9]) from the system of linear algebraic equations of the form

$$\sum_{i=1}^{m}\sum_{k=1}^{N_i} a_k^{(N_i)}\psi(z_{r,j},\widetilde{z}_{i,k}) = g(z_{r,j}), \tag{2.12}$$

where the collocation points $z_{r,j}$ ($r = 1,2,\cdots,m$; $j = 1,2,\cdots,N_i$) lie "uniformly" on the contours S_r.

From (2.11) for the approximate value of constant c we have

$$c_N = \lim_{z\to\infty} u_N(z) = u_N(\infty) = \sum_{i=1}^{m}\sum_{k=1}^{N_i} a_k^{(N_i)} \ln\Big|\frac{R^2}{\widetilde{z}_{i,k}-z_0}\Big|,$$

or $|c_N| < \infty$. In paper [10] is considered the case when $m = 1$ and the hole B_1 is of a cut-type.

3. Numerical Examples

As the given numerical experiments show, while solving the problem (1.1), (1.2), (1.3) with the help of the offered method when $m \geq 2$ the following should be taken into account.

In the given modified version of MFS (as in the general MFS) it is evident that total number of auxiliary and collocation points depends on accuracy of solution of the problem (1.1), (1.2), (1.3) (and conversely), i.e., $N = N(\varepsilon)$. Since when constructing the approximate solution $u_N(z)$ in the form (2.11) in order to find the coefficients $a_k^{(N_i)}$, the system (2.12) of N-th order is being solved, therefore, on the other hand, the accuracy of the solution to the problem (1.1), (1.2), (1.3) is being limited by technical possibilities of the computers. In order to achieve possibly high accuracy of the solution to the problem considering the possibilities of the given computer, when $m \geq 2$, the optimal (in the sense of approximation of the boundary function) auxiliary contour $\widetilde{S}_i$ should be found separately for each contour S_i (for the same boundary function $g(z)$) and then the numbers N_i should be chosen, the system (2.12) formed and solved. The indicated process will be described for Example 1 in more details. In the exemples in the role of the domain D is taken the infinit plane z with five holes, i.e., considered the case when $m = 5$. In the role of the boundary function $g(z)$ were taken the functions: 1) $g_1(z) = \ln|z - z_0^*| - \ln|z| + 1$, $z \in S$, where the fixed point $z_0^* \overline{\in} D$; 2) $g_2(z) = \frac{|z|-1}{|z|+1}$, $z \in S$. In all examples the origin $z = 0$ was inside the basic domain (hole) B_1 and $z_0 = 0$. It is evident, for $g(z) = g_1(z)$,the exact solution to the problem (1.1), (1.2), (1.3) for $u(\infty) = 1$, is the following function $u(z) = \ln|z - z_0^*| - \ln|z| + 1$, $z \in \overline{D}$. In all examples considered below the coefficients $a_k^{(N_i)}$ of the approximate solution (2.11) were found by the collocation method from system (2.12).

In the Tables 1 and 3 $\varepsilon_i (i = 1, 2, \cdots, 5)$ and ε_1^* are *a posteriori* estimates of maximal error of the approximate solution $u_L^{(i)}(z)$ to the problem (1.1), (1.2), (1.3), whereas the infinite plane z has only one hole of the type $B_i (i = 1, 2, \cdots, 5)$ (see below); ε_{max} and β are the *a posteriori* estimats of maximal and mean square error of the approximate solution to the problem (1.1), (1.2), (1.3) for $m = 5$ on the contour S; i.e.,

$$\varepsilon_i = \max_{z \in S_i} \big| g(z) - u_L^{(i)}(z) \big| \approx \max \big| g(\tau_{i,j}) - u_L^{(i)}(\tau_{i,j}) \big|,$$

$$\varepsilon_{max} = \max_{z \in S} \big| g(z) - u_N(z) \big| \approx \max \big\{ \big| g(\tau_{i,j}) - u_N(\tau_{i,j}) \big| \big\}_{i=1}^{m},$$

$$\beta = \Big\{ \sum_{i=1}^{m} \sum_{j=1}^{M} \big[g(\tau_{i,j}) - u_N(\tau_{i,j}) \big]^2 \Big\}^{1/2} / (Mm),$$

where the points $\tau_{i,j}$ $(j = 1, 2, \cdots, M)$ lie "uniformly" on the contours S_i, respectively; $\gamma = |u(\infty) - u_N(\infty)| = |1 - u_N(\infty)|$. As for the approximate solutions $u_L^{(i)}$, they are sought in the form

$$u_L^{(i)}(z) = \sum_{k=1}^{L} a_{ik}^{(L)} \psi(z, \widetilde{z}_{i,k}) \equiv \sum_{k=1}^{L} a_{i,k}^{(L)} \ln \Big| \frac{R^2(\widetilde{z}_{i,k} - z)}{(z - z_0^i)(\widetilde{z}_{i,k} - z_0^i)} \Big|,$$

where the point z_0^i is the "center" of the domain B_i and the points $\widetilde{z}_{i,k}$ $(k = 1, 2, \cdots, L)$ are situated "uniformly" on the contour $\widetilde{S}_i$. The coefficients $a_{i,k}^{(L)}$ can be found (see [7], [8], [9]) from the system of linear algebraic equations of the form

$$\sum_{k=1}^{L} a_{i,k}^{(L)} \psi(z_{i,r}, \widetilde{z}_{i,k}) = g(z_{i,r}),$$

where the collocation points $z_{i,r}$ $(r = 1, 2, \cdots, L)$ lie "uniformly" on the contour S_i. In the numerical experiments were taken $M = 10000$ and calculations were performed with double accuracy.

The auxiliary points, the collocation points and the points for *a posteriori* estimates lie "uniformly" with respect to the parameter t on the contours $\widetilde{S}_i$ and S_i $(i = 1, 2, \cdots, 5)$, respectively.

Example 1. In the role of the boundary function $g(z)$ is taken $g(z) = g_1(z)$ and in the role of the domain D is taken the infinite plane z with the holes B_i $(i = 1, 2, \cdots, 5)$, which is situated in the following form. The domain (hole) B_1 is the square

$$B_1: \ |x| < 5, \quad |y| < 5;$$

the domain B_2 is the interior of the ellipse

$$S_2: x = 5 \cos t + 200, \quad y = \sin t + 100, \quad 0 \le t \le 2\pi;$$

the domain B_3 is the interior of Pascal's limacon

$$S_3 : z = R^*(\zeta + a\zeta^2) - 20 - 100i,$$

where $\zeta = e^{it}$ $(0 \le t \le 2\pi)$, $R^* = 2$, $a = 0.5$ and $i = \sqrt{-1}$ the domain B_4 is the interior of the Booth's lemniscate

$$S_4 : z = \frac{\zeta}{1 - a\zeta^2} - 100 + 200i,$$

where $\zeta = e^{it}$ $(0 \le t \le 2\pi)$ and $a = 0.5$; the domain B_5 is the interior of the epicycloid

$$S_5 : z = R^*\Big(\zeta + \frac{\zeta^4}{4}\Big) + 50 - 100i,$$

where $\zeta = e^{it}$ $(0 \le t \le 2\pi)$ and $R^* = 2$.

In the role of the auxiliary contours $\widetilde{S}_i$ $(i = 1, 2, \cdots, 5)$ are taken the following contours: $\widetilde{S}_1$ is the boundary of the squere

$$\widetilde{B}_1 : |x| < 2.5, \quad |y| < 2.5; \widetilde{S}_2 : \widetilde{x} = 4.95 \cos t + 200; \quad y = 0.7 \sin t + 100,$$

where $0 \le t \le 2\pi$;

$$\widetilde{S}_3 : \widetilde{z} = R^*(\widetilde{\zeta} + a\widetilde{\zeta}^2) - 20 - 100i,$$

where $\widetilde{\zeta} = 0.9e^{it}$ $(0 \le t \le 2\pi)$, $R^* = 2$ and $a = 0.5$;

$$\widetilde{S}_4 : \widetilde{z} = \frac{\widetilde{\zeta}}{1 - a\widetilde{\zeta}^2} - 100 + 200i$$

where $\widetilde{\zeta} = 0.8e^{it}$ $(0 \le t \le 2\pi)$ and $a = 0.5$;

$$\widetilde{S}_5 : \widetilde{z} = R^*\Big(\widetilde{\zeta} + \frac{\widetilde{\zeta}^4}{4}\Big) + 50 - 100i,$$

where $\widetilde{\zeta} = 0.9e^{it}$ $(0 \le t \le 2\pi)$ and $R^* = 2$.

Remark 3. As it was noted above, selection of the auxiliary contours $\widetilde{S}_i$ for each contour S_i should be done separately, i.e., independently from each other. Evidently, this can be done in two ways: directly for the given contours S_i or using parallel transfer. For example in the case of contour S_2 we can first find the auxiliary contour $\widetilde{S}_2^0$ for the contour $S_2^0 : x = 5 \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ and then by the help of parallel transfer obtain to contour $\widetilde{S}_2$.

In the Table 1 the results of approximate solution to the problem (1.1), (1.2), (1.3) are given for the case, when the infinite plane z has only one hole of the type B_i^0 $(i = 1, 2, \cdots, 5)$ with boundary S_i^0; L is the number of the auxiliary points and collocation points situated "uniformly" on the contours $\widetilde{S}_i^0$ and S_i^0, respectively and $z_0^* = (1, 0)$.

Table 1

L	ε_1	ε_2	ε_3	ε_4	ε_5
100	$0.2E-10$	$0.6E-03$	$0.3E-05$	$0.1E-07$	$0.2E-05$
200	$0.9E-14$	$0.1E-06$	$0.4E-10$	$0.5E-14$	$0.3E-10$
300	$0.6E-14$	$0.2E-10$	$0.4E-14$	$0.3E-14$	$0.5E-14$
400	$0.4E-14$	$0.8E-14$	$0.1E-14$	$0.1E-14$	$0.4E-14$

In the Table 2 the results of approximate solution to the problem (1.1), (1.2), (1.3) are given for domain D in the two cases: 1) $z_0^* = (1.0)$, i.e., $z_o^* \in B_1$; 2) $z_0^* = (200, 100)$, i.e., $z_0^* \in B_2$. The numbers N_i $(i = 1, 2, \cdots, 5)$ were chosen from the Table 1. It should be noted that, when $z_0^* \in B_1$, function $u(z) = g_1(z)$ is the exact solution to the problem (1.1), (1.2), (1.3) and it is evident it has not singularity outside of the domain B_1. Respectively, the approximate solution $u_N(z)$ of this problem would have no singularity outside of the domain B_1. It means that in this case intensities $2\pi a_k^{(N_i)}$ of the simulation sources ((see [9]), which are situated in the points $\widetilde{z}_{ik}(i = 2, 3, 4, 5;\ k = 1, 2, \cdots, N_i)$ should practically be equal to zero and this takes place while solving system (2.12). It is evident, in this case, if $\max|u_N(z) - u(z)| < \varepsilon$ as $z \in S_1$, then on the basis of the minimax principle this condition takes place automatically on the contours S_i $(i = 2, \cdots, 5)$. Analogously, if $z_0^* \in B_2$, then function $u(z) = g_1(z)$ is harmonic outside of the domain $B_1 \cup B_2$ and respectively the coefficients $a_k^{(N_i)} \approx 0$ $(i = 3, 4, 5;\ k = 1, 2, \cdots, N_i)$, which takes place while numerical realization.

Table 2

$z_0^* = (1,0)$							
N_1	N_2	N_3	N_4	N_5	ε_{max}	β	γ
100	100	100	100	100	$0.2E-10$	$0.4E-11$	$0.7E-15$
200	200	200	200	200	$0.7E-13$	$0.2E-13$	$0.2E-14$
200	300	200	200	200	$0.8E-13$	$0.2E-13$	$0.1E-13$
200	400	300	200	300	$0.2E-13$	$0.5E-14$	$0.3E-14$
$z_0^* = (200,100)$							
100	100	100	100	100	$0.8E-03$	$0.1E-03$	$0.8E-06$
100	200	200	200	200	$0.3E-07$	$0.6E-08$	$0.1E-09$
200	300	200	200	200	$0.5E-12$	$0.1E-12$	$0.7E-14$
200	400	300	200	300	$0.5E-13$	$0.1E-13$	$0.6E-15$

Example 2. In the role of function $g(z)$ was taken function $g(z) = g_2(z)$ and for the domain D two cases were considered. In the first case the domain D has previous form and in the second case only the domain B_1 is replaced with the domain B_1^*, which represents the interior of ellipse $S_1^* : x^* = \cos t$, $y^* = 10 \sin t$, $0 \le t \le 2\pi$. In the first case in the role of contour S_1^* was taken boundary of squere $\widetilde{S}_1 : \widetilde{z} = 0.999z$, where $z \in S_1$ and the remainder contours were the same. In the second case in the role of contour $\widetilde{S}_1^*$ was taken ellipse $\widetilde{S}_1^* x = 0.2 \cos t$, $y = 9.95 \sin t$ $(0 \le t \le 2\pi)$. The results of numerical experiments are given in the Tables 3 and 4.

Table 3

L	ε_1	ε_2	ε_3	ε_4	ε_5	ε_1^*
100	$0.1E-01$	$0.2E-04$	$0.5E-06$	$0.5E-09$	$0.1E-05$	$0.7E-04$
200	$0.6E-02$	$0.4E-08$	$0.9E-11$	$0.3E-14$	$0.4E-08$	$0.9E-07$
300	$0.4E-02$	$0.3E-11$	$0.2E-13$	$0.3E-14$	$0.3E-10$	$0.6E-09$
400	$0.2E-02$	$0.4E-14$	$0.1E-14$	$0.2E-14$	$0.6E-11$	$0.6E-11$
500	$0.1E-02$	$0.4E-14$	$0.1E-14$	$0.2E-14$	$0.3E-14$	$0.5E-13$
1000	$0.8E-03$	$0.1E-14$	$0.1E-14$	$0.1E-14$	$0.1E-14$	$0.2E-14$

Table 4

N_1	N_2	N_3	N_4	N_5	ε_{max}	β	c_N
100	100	100	100	100	$0.1E-01$	$0.8E-02$	0.9324
200	200	200	200	200	$0.5E-02$	$0.3E-03$	0.9333
400	300	200	200	300	$0.2E-02$	$0.1E-02$	0.9337
1000	100	100	100	100	$0.6E-03$	$0.2E-03$	0.9339
N_1^*	N_2	N_3	N_4	N_5	ε_{max}	β	c_N
100	100	100	100	100	$0.7E-04$	$0.1E-04$	0.9246
200	200	200	200	200	$0.8E-07$	$0.1E-07$	0.9246
400	300	300	300	300	$0.6E-11$	$0.1E-11$	0.9246
500	400	300	300	500	$0.5E-13$	$0.7E-14$	0.9246

4. Comments

1. The Tables 1 and 3 have only auxiliary character for the Tables 2 and 4, respectively. These tables help us to find the numbers N_i and the contours $\widetilde{S}_i$ when solving the problem (1.1), (1.2), (1.3). At this principle of selection of the numbers N_i and the contours $\widetilde{S}_i$ we use economically the possibilities of computer, in order to achieve higher accuracy of solution of the problem (1.1), (1.2), (1.3). Besides, at this significantly less machine time is needed than without using such selection.

2. In the Example 2, when the contour S_1 represents the boundary of square $B_1 : |x| < 5$, $|y| < 5$, in the role of auxiliary contour is taken contour $\widetilde{S}_1 : \widetilde{z} = 0.999z$, where $z \in S_1$. In order to achieve the accuracy, $\varepsilon_1 = 0.8E - 03$, $L = 1000$ auxiliary points are taken (see the Table 3). It is explained by the fact that if a domain has re-entrant angles (see [8]) and the boundary function $g(z)$ is not of type of the function $g_1(z)$, then to improve the conditionality of the system matrix (2.12) the contour $\widetilde{S}_1$ should approach S. In this case in order to obtain good results, number L should be increased.

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