

## ESTIMATES FOR THE ACOUSTIC POTENTIALS AND THEIR APPLICATION

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ABSTRACT. We consider the acoustic potentials  $A^\gamma$ ,  $0 < \operatorname{Re} \gamma < n+1$ , which are complex powers with negative real part of the Helmholtz operator  $I + \Delta$ ,  $\Delta$  being the Laplace operator in  $\mathbb{R}^n$ . We construct some convex sets on the  $(1/p, 1/q)$ -plane for which the operator  $A^\gamma$  is bounded from  $L_p$  into  $L_q$  and point out the domains, where it is not bounded. We also establish some  $L_p \rightarrow L_{q_1} + L_{q_2}$ -estimates for the operator  $A^\gamma$  and apply them in order to describe the range  $A^\gamma(L_p)$ ,  $0 < \operatorname{Re} \gamma < 2$ , in terms of the operator inverse to  $A^\gamma$ .

### 1. INTRODUCTION

We investigate the acoustic potentials  $A^\gamma$  of the complex order  $\gamma$ ,  $0 < \operatorname{Re} \gamma < n+1$ , which are complex powers with negative real part of the Helmholtz operator

$$I + \Delta,$$

where  $\Delta$  is the Laplace operator in  $\mathbb{R}^n$ . These potentials are defined via Fourier transform (on sufficiently nice functions  $\varphi(t)$ ) by the equality

$$(\widehat{A^\gamma \varphi})(\xi) = (|\xi|^2 - 1 - i0)^{-\gamma/2} \widehat{\varphi}(\xi) \quad (1.1)$$

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and admit the following integral representation:

$$(A^\gamma \varphi)(x) = \int_{\mathbb{R}^n} h_\gamma(|y|) \varphi(x - y) dy, \quad (1.2)$$

where

$$h_\gamma(|y|) = \zeta_{n,\gamma} |y|^{\frac{\gamma-n}{2}} H_{\frac{n-\gamma}{2}}^{(1)}(|y|), \quad (1.3)$$

$\zeta_{n,\gamma} = 2^{(\gamma-n)/2} \pi^{1-n/2} \frac{i}{\Gamma(\gamma/2)}$ ,  $H_\nu^{(1)}(z)$  being the Hankel function of the first kind.

We study a principal problem of obtaining the  $L_p \rightarrow L_q$ -estimates for the operators (1.2). Making use of the oscillation of the Hankel function at infinity, it is possible to construct some convex sets on the  $(1/p, 1/q)$ -plane for which the operator (1.2) is bounded from  $L_p$  into  $L_q$ . This was done in [8], where the authors treated the case of real  $\gamma$ . We deal with the case of complex  $\gamma$  which proved to be more difficult. We suggest a new approach to the  $L_p \rightarrow L_q$ -estimates for potential-type operators of some general form, which, even in the case of real  $\gamma$ , allows us to obtain results which prove to be stronger than the results established in [8] (see Remark 2.1). This new approach covers potential-type operators of the form

$$(S_a^\alpha \varphi)(x) = \int_{|y| \geq A} a(|y|) e^{i|y|} |y|^{\alpha-n} \varphi(x - y) dy, \quad 0 < \operatorname{Re} \alpha < n, \quad (1.4)$$

where the function  $a(r)$ , which is referred to as ‘‘characteristic’’, is sufficiently smooth on  $(A, \infty]$  and such that  $a(\infty) \neq 0$ ,  $A$  being an arbitrary number,  $0 < A < \infty$ . We observe that this approach can also be applied to a wider class of operators containing, in particular, the Bochner–Riesz operators and Strichartz–type potentials over  $\mathbb{R}^n$  with oscillating characteristics (see [7]), which are known to be the operators of essentially different nature. Such applications will be given in other papers.

We also obtain some results related to boundedness of the operator  $A^\gamma$  from  $L_p$  into  $L_{q_1} + L_{q_2}$ , which are applied to describe the range  $A^\gamma(L_p)$ ,  $0 < \operatorname{Re} \gamma < 2$ , by means of the operator inverse to  $A^\gamma$ . In this connection, we note that although the inversion of potentials  $A^\gamma \varphi$ ,  $\varphi \in L_p$ , was constructed more than ten years ago (see [22]), the range  $A^\gamma(L_p)$  has never been described. Meanwhile, the importance of description of the classes  $A^\gamma(L_p)$  makes no doubt; it is explained by the fact that these classes can be regarded as the natural domains of the positive powers  $(I + \Delta)^{\gamma/2}$ ,  $\operatorname{Re} \gamma > 0$ . It should be emphasized that we managed to describe these ranges just due to the results on boundedness of the operator  $A^\gamma$  from  $L_p$  into  $L_{q_1} + L_{q_2}$ .

Currently there are many papers on inverting potential-type operators and characterization of their ranges, mainly in the case when potentials

have the form

$$(K_a^\alpha \varphi)(x) = \int_{\mathbb{R}^n} \frac{a(y)}{|y|^{n-\alpha}} \varphi(x-y) dy, \quad 0 < \operatorname{Re} \alpha < n, \quad (1.5)$$

with characteristic  $a(y)$  in various classes (see the books [17], [18], the surveying papers [9], [10], [16] and bibliography therein). The first results are due to S. G. Samko, who constructed the inversion of the Riesz potential operator  $K^\alpha$  and described both the range  $K^\alpha(L_p)$  and more general closely related function spaces  $L_{p,r}^\alpha(\mathbb{R}^n)$  by means of hypersingular integrals (HSI), see [13], [14]. He also first applied the method of HSI to invert the potentials (1.5) with homogeneous characteristics  $a(y) = a(y/|y|)$  in the elliptic case (see [15]). Nevertheless, the investigation of potentials with oscillating kernels are at the very beginning (see [4], [10] for some special-type potentials).

The contents of the paper are as follows. In Section 2 we formulate our main results (Theorems 2.1 and 2.2) together with some comments. Section 3 contains necessary preliminaries. Section 4 can be regarded as a background for the proof of Theorems 2.1 and 2.2. We prove some statements there aimed to obtaining the  $L_p \rightarrow L_q$ -estimates for the operator (1.4). We first dwell on the case  $a(r) \equiv 1$  (Theorem 4.1) and then pass to a more general situation (Theorem 4.2). We note that Theorems 4.1 and 4.2 are of special interest themselves because, as was mentioned above, they can be applied to obtain the  $L_p \rightarrow L_q$ -estimates for potential-type operators in a wider class. Therefore we prove them for all  $\alpha$ ,  $0 < \operatorname{Re} \alpha < n$ , although to prove Theorem 2.1, we could restrict ourselves to the case  $(n-1)/2 < \operatorname{Re} \alpha < n$  only. Section 5 is devoted to the proof of Theorems 2.1 and 2.2.

## 2. THE MAIN RESULTS AND SOME COMMENTS

Throughout the paper, the symbol  $(A, B, \dots, K)$  denotes an open polygon in  $\mathbb{R}^2$  with the vertices at the points  $A, B, \dots, K$ ;  $[A, B, \dots, K]$  stands for its closure;  $\mathcal{L}(A)$  is the  $\mathcal{L}$ -characteristic of the operator  $A$ , that is, the set of all pairs  $(1/p, 1/q)$  for which  $A$  is bounded from  $L_p$  into  $L_q$ .

We introduce the following points on the plane:

$$\begin{aligned} A &= \left(1, 1 - \frac{\operatorname{Re} \alpha}{n}\right), & A' &= \left(\frac{\operatorname{Re} \alpha}{n}, 0\right), \\ B &= \left(1 - \frac{(n-1)(n-\operatorname{Re} \alpha)}{n(n+1)}, 1 - \frac{\operatorname{Re} \alpha}{n}\right), \\ B' &= \left(\frac{\operatorname{Re} \alpha}{n}, \frac{(n-1)(n-\operatorname{Re} \alpha)}{n(n+1)}\right), \\ C &= \left(\frac{3}{2} - \frac{2\operatorname{Re} \alpha}{n-1}, \frac{3}{2} - \frac{2\operatorname{Re} \alpha}{n-1}\right), & C' &= \left(\frac{2\operatorname{Re} \alpha}{n-1} - \frac{1}{2}, \frac{2\operatorname{Re} \alpha}{n-1} - \frac{1}{2}\right), \end{aligned}$$

$$\begin{aligned}
D &= \left( \frac{\operatorname{Re} \alpha + 1}{n+1}, \frac{n - \operatorname{Re} \alpha}{n+1} \right), \quad E = (1, 0), \quad F = \left( \frac{1}{2}, \frac{1}{2} \right), \\
G &= \left( 1 - \frac{(n - \operatorname{Re} \alpha)(n-1)}{n(n+3)}, 1 - \frac{\operatorname{Re} \alpha}{n} \right), \\
G' &= \left( \frac{\operatorname{Re} \alpha}{n}, \frac{(n - \operatorname{Re} \alpha)(n-1)}{n(n+3)} \right), \\
H &= \left( 1 - \frac{\operatorname{Re} \alpha}{n}, 1 - \frac{\operatorname{Re} \alpha}{n} \right), \quad H' = \left( \frac{\operatorname{Re} \alpha}{n}, \frac{\operatorname{Re} \alpha}{n} \right), \\
K &= \left( \frac{2(\operatorname{Re} \alpha + 1)}{n+1} - \frac{1}{2}, \frac{1}{2} \right), \quad K' = \left( \frac{1}{2}, \frac{3}{2} - \frac{2(\operatorname{Re} \alpha + 1)}{n+1} \right), \\
O &= (1, 1), \quad O' = (0, 0),
\end{aligned}$$

In order to formulate the results on boundedness of the operator (1.2) from  $L_p$  into  $L_q$  (and the corresponding results for the auxiliary operators (1.4) and (4.41)), we introduce the following sets (see Pictures 1 and 2 for the cases  $(n-1)/2 < \operatorname{Re} \alpha < n$  and  $0 < \operatorname{Re} \alpha \leq (n-1)/2$ , respectively):

$$\begin{aligned}
&\mathcal{L}_1(\alpha, n) = \\
&= \begin{cases} (A', B', B, A, E) \cup (A, E] \cup (A', E), & \frac{n}{2} \leq \operatorname{Re} \alpha < n, \\ (A', G', K', K, G, A, E) \cup (A, E] \cup (A', E), & \frac{n-1}{2} < \operatorname{Re} \alpha < \frac{n}{2}, \\ (A', G', F, G, A, E) \cup (A, E] \cup (A', E), & \alpha = \frac{n-1}{2}, \\ (A', G', F, G, A, E) \cup (A, E] \cup (A', E) \cup \{F\}, & \operatorname{Re} \alpha = \frac{n-1}{2}, \operatorname{Im} \alpha \neq 0, \\ (A', G', C', C, G, A, E) \cup (A, E] \cup (A', E) \cup (C', C), & \\ \frac{n(n-1)}{2(n+1)} < \operatorname{Re} \alpha < \frac{n-1}{2}, & \\ [A', H', H, A, E] \setminus ([A', H'] \cup [A, H]), & 0 < \operatorname{Re} \alpha \leq \frac{n(n-1)}{2(n+1)}, \end{cases}
\end{aligned}$$

if  $n \geq 3$  or  $n = 2$  and  $\operatorname{Im} \alpha \neq 0$ ;

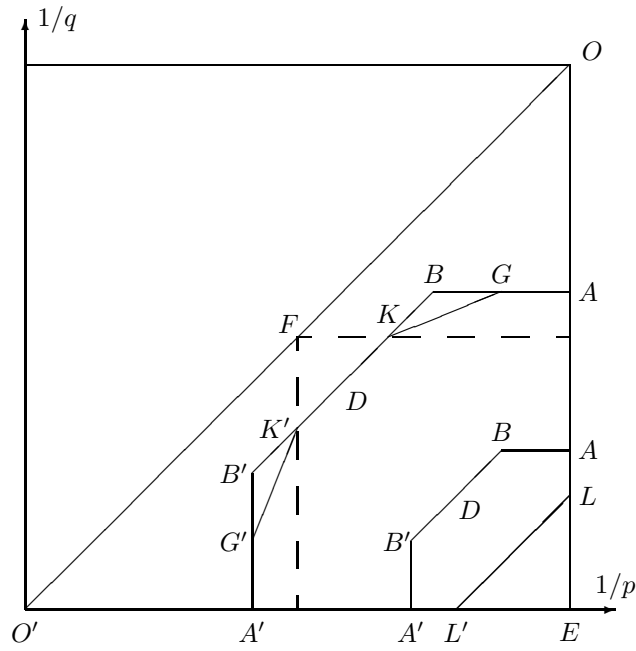
$$\mathcal{L}_1(\alpha, n) = \begin{cases} (A', B', B, A, E) \cup (A, E] \cup (A', E), & \frac{1}{2} \leq \alpha < 2, \quad \alpha \neq 1, \\ (A', B', B, A, E) \cup (B, B') \cup (A, E] \cup (A', E), & \alpha = 1, \\ [A', H', H, A, E] \setminus ([A', H'] \cup [A, H]), & 0 < \alpha < \frac{1}{2}, \end{cases}$$

if  $n = 2$  ( $\operatorname{Im} \alpha = 0$ );

$\mathcal{L}_1^*(\alpha, n) = \mathcal{L}_1(\alpha, n) \cup \{D\}$  if  $(n-1)/2 < \alpha < n$  and  $\mathcal{L}_1^*(\alpha, n) = \mathcal{L}_1(\alpha, n)$  otherwise;

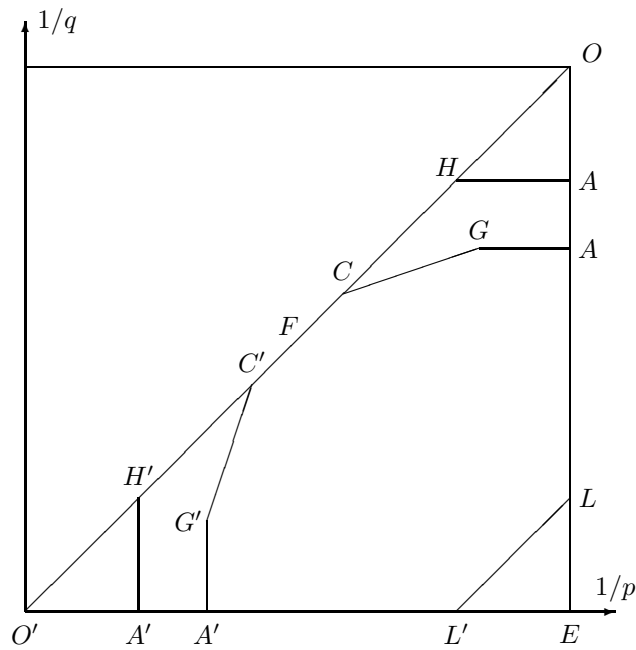
$$\mathcal{L}_2(\gamma, n) = [O, L, L', O'] \setminus (\{L'\} \cup \{L\}),$$

where  $L = (1, 1 - \operatorname{Re} \gamma/n)$ ,  $L' = (\operatorname{Re} \gamma/n, 0)$ .



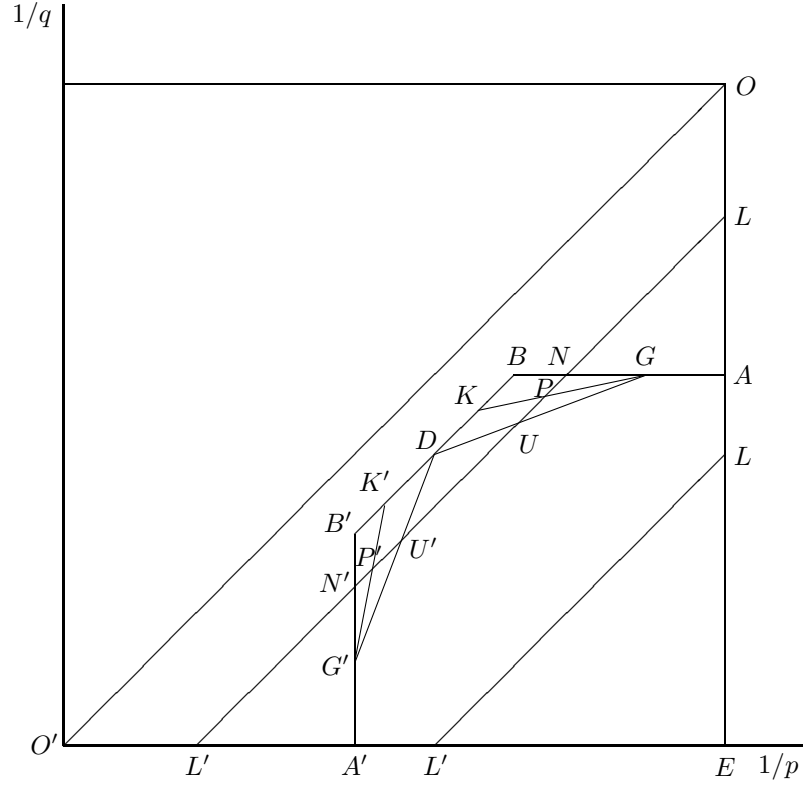
Picture 1.

A smaller pentagon  $(A', B', B, A, E)$  corresponds to the case  $n/2 < \text{Re}\alpha < n$ , when it lies inside the square  $1/2 \leq 1/p \leq 1, 1/2 \leq 1/q \leq 1$ .



Picture 2.

The following theorem provides the  $L_p \rightarrow L_q$ -estimates for the operator  $A^\gamma$  and establishes some negative results (see also Picture 3).



Picture 3.

**Theorem 2.1.** I. Let  $0 < \operatorname{Re} \gamma < n + 1$ . Then

$$\mathcal{L}(A^\gamma) \supset \mathcal{L}_1^* \left( \frac{\gamma + n - 1}{2}, n \right) \cap \mathcal{L}_2(\gamma, n) \quad (2.1)$$

if  $0 < \operatorname{Re} \gamma < n$ ,

$$\mathcal{L}(A^\gamma) \supset \mathcal{L}_1^* \left( \frac{\gamma + n - 1}{2}, n \right) \quad (2.2)$$

if  $n < \operatorname{Re} \gamma < n + 1$  or  $\operatorname{Re} \gamma = n$  and  $\operatorname{Im} \gamma \neq 0$ ,

$$\mathcal{L}(A^\gamma) \supset \mathcal{L}_1^* \left( \frac{\gamma + n - 1}{2}, n \right) \setminus \{E\} \quad (2.3)$$

if  $\gamma = n$ .

II. The set  $\mathcal{L}(A^\gamma)$  does not contain the points:

1) lying on the segment  $[A, H]$  and above it;

- 2) lying on the segment  $[A', H']$  and to the left of it;
  - 3) lying above the straight line  $B'B$  for  $0 < \gamma < n + 1$ ;
  - 4) of the set  $[L', L, E] \setminus (L', L)$  for  $0 < \operatorname{Re} \gamma < n$ .
- In the case  $\gamma = n$ , the point  $\{E\}$  does not belong to  $\mathcal{L}(A^\gamma)$ .

*Remark 2.1.* We note that the main theorem from [8], related to the case of real  $\gamma$ ,  $0 < \gamma < n + 1$ , does not answer the question about boundedness of the operator  $A^\gamma$  from  $L_p$  into  $L_q$  if

- a)  $(1/p, 1/q) \in ([D, B, G] \cup [D, B', G']) \setminus ([B', G'] \cup [B, G] \cup \{D\})$ ,  $n \geq 3$ ,  $(n-1)/2 \leq \gamma < n+1$ ;
- b)  $(1/p, 1/q) \in ([D, B, N, U] \cup [D, B', N', U']) \setminus ([B, N] \cup [B', N'] \cup \{D\})$ ,  $n \geq 3$ ,  $0 < \gamma < (n-1)/2$ ;
- c)  $(1/p, 1/q) \in (B', B) \setminus \{D\}$ ,  $n = 2$ ,  $\gamma \neq 1$

(see Picture 3).

Theorem 2.1, which covers the case of complex  $\gamma$  as well, gives the positive answer to this question in the following cases:

- $(1/p, 1/q) \in (D, B, G) \cup (D, B', G')$  for  $n \geq 3$  and  $(n-1)/2 \leq \gamma < n+1$ ;
- $(1/p, 1/q) \in (D, U', N', B') \cup (N', U') \cup (U', D) \cup (D, U, N, B) \cup (N, U) \cup (U, D)$  for  $n \geq 3$  and  $1 \leq \gamma < (n-1)/2$ ;
- $(1/p, 1/q) \in (P', K', D, U') \cup (P', U') \cup (U', D) \cup (P, K, D, U) \cup (P, U) \cup (U, D)$  for  $n \geq 3$  and  $0 < \gamma < 1$ .

Our next result is related to description of the range  $A^\gamma(L_p)$  in terms of the operator inverse to  $A^\gamma$ . In [22], the authors keeping in mind (1.1), applied the method of approximative inverse operators (AIO) to construct the inversion of potentials  $f = H^\gamma \varphi$ ,  $0 < \operatorname{Re} \gamma < n + 1$ ,  $\varphi \in L_p$ ,  $1 \leq p < 2/(n + \operatorname{Re} \gamma - 1)$  (see also [17], Subsection 11, and [9], [16]). Within the framework of AIO's method the inversion was constructed in the form

$$(T^\gamma f)(x) = \lim_{\varepsilon \rightarrow 0}^{(L_{p,\mu})} \left( F^{-1} \left( \frac{(|\xi|^2 - 1)^{n+\gamma/2}}{(|\xi|^2 + (\varepsilon + i)^2)^n} e^{-\varepsilon|\xi|^2} \right) * f \right)(x). \quad (2.4)$$

Here  $L_{p,\mu} = \{f(x) : (1 + |x|^2)^{\mu/2} f(x) \in L_p\}$ ,  $\mu < -\frac{n-1}{2}p$ ; the limit in (2.4) can be replaced by the almost everywhere limit.

Turning to the description of the range  $A^\gamma(L_p)$ , we denote

$$= \begin{cases} \frac{1}{p_0(\gamma)} = \\ \frac{1}{2n} \left( \operatorname{Re} \gamma + n - 1 + \frac{(\operatorname{Re} \gamma - 1)(n^2 - 1)(\operatorname{Re} \gamma + 2)}{2(n(-n^2 + n - 1) + (\operatorname{Re} \gamma + n - 1)(n^2 + 6n + 1))} \right) \\ \quad \text{if } 0 < \operatorname{Re} \gamma < \frac{n+1}{2n}, \\ \frac{\operatorname{Re} \gamma(3n-1)}{n(n+1)} \quad \text{if } \frac{n+1}{2n} \leq \operatorname{Re} \gamma < 2. \end{cases}$$

We note that the number  $1/p_0(\gamma)$  is the abscissa of the point, where the straight line  $1/q = (n-1)/(2n)$  meets the straight lines  $B'B$  and  $K'G'$  in the cases  $(n+1)/(2n) \leq \operatorname{Re} \gamma < 2$  and  $0 < \operatorname{Re} \gamma < (n+1)/(2n)$ , respectively.

The following theorem is valid.

**Theorem 2.2.** *Let  $0 < \operatorname{Re} \gamma < 2$ ,  $1/p \in (1/p_0(\gamma), 1]$ . Then*

$$A^\gamma(L_p) = \{f \in L_{q_1} + L_{q_2} : T^\gamma f \in L_p\}, \quad (2.5)$$

where  $T^\gamma$  is the operator (2.4) and  $q_1, q_2$  are such numbers that  $2n/(n+1) < q_1'$  and  $q_2' \leq \infty$  and the operator  $A^\gamma$  is bounded from  $L_p$  into  $L_{q_1} + L_{q_2}$  (see Remark 4.4 below).

### 3. PRELIMINARIES

**3.1. Notation.**  $S^{n-1}$  is the unit sphere in  $\mathbb{R}^n$ ;  $\chi_d(y)$  is the characteristic function of the ball  $|y| \leq d$ ,  $0 < d < \infty$ ;  $\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)}g(x) dx$ ; the symbol

$\overline{A}$  denotes the operator with the kernel which is complex conjugate to that of the operator  $A$ ;  $W_\varepsilon \varphi$  is the Gauss–Weierstrass integral;  $S$  is the Schwartz class of rapidly decreasing smooth functions. The end of proof is denoted by.

**3.2. Uniform asymptotic expansion for the Bessel function  $J_\nu(z)$ .** Let  $\Omega = \{z \in \mathbb{C} : |z| > \eta, |\arg z| < \theta\}$ , where  $\eta > 0$  and  $\theta \in (0, \pi/2)$ . Representing  $J_\nu(z)$  as a linear combination of the Hankel functions  $H_{\pm\nu}^{(1)}(z)$  and  $H_{\pm\nu}^{(2)}(z)$  (we take  $+\nu$ , if  $\nu > -1/2$  and  $-\nu$  otherwise) and applying the results from [21, p. 220], we arrive at the equality

$$J_\nu(z) = \left(\frac{\pi z}{2}\right)^{-1/2} \left[ e^{-iz} \left( \sum_{j=0}^m C_{j,-}^{(\nu)} z^{-j} + R_{m,-}^{(\nu)}(z) \right) + e^{iz} \left( \sum_{j=0}^m C_{j,+}^{(\nu)} z^{-j} + R_{m,+}^{(\nu)}(z) \right) \right], \quad (3.1)$$

where  $C_{0,\pm}^{(\nu)} = \frac{1}{2} e^{\mp(i\pi/4)(2\nu+1)}$ .

*Remark 3.1.* The remainders  $R_{m,\pm}^{(\nu)}(z)$  are analytic in  $\Omega$  and  $R_{m,\pm}^{(\nu)}(z) = O(|z|^{-m-1})$ , as  $|z| \rightarrow \infty$ . Then  $(d/dz)^k R_{m,\pm}^{(\nu)}(z) = O(|z|^{-m-1-k})$ ,  $|z| \rightarrow \infty$ , in any closed sector  $\Omega_0 \subset \Omega$  (see [12, p. 21]).

**3.3. Asymptotic expansion of some integrals containing an oscillating exponent.** Let  $\omega$  be a fixed smooth function on  $(0, \infty)$  such that  $0 \leq \omega(x) \leq 1$ ,  $\omega(x) = 0$  if  $x \leq 1$  and  $\omega(x) = 1$  if  $x \geq 2$ . The following lemma is valid.

**Lemma 3.1 ([5]).** *Let  $\lambda \in \mathbb{C}$ . For  $\varepsilon > 0$  set*

$$J_{\lambda,\varepsilon}(t) = (2\pi)^{-n} \int_0^\infty \omega(s) s^\lambda e^{ist - \varepsilon s} ds.$$

Then  $J_{\lambda,\varepsilon}(t)$  converges, as  $\varepsilon$  tends to zero, uniformly in  $|t| \geq \delta$  for every  $\delta > 0$ , and the limit  $J_\lambda(t) = \lim_{\varepsilon \rightarrow 0} J_{\lambda,\varepsilon}(t)$  has the following estimates

$$J_\lambda(t) = \begin{cases} A_\lambda(t+i0)^{-\lambda-1} + \tilde{J}_\lambda(t), & \lambda \neq -1, -2, \dots, \\ t^{-\lambda-1}(A'_\lambda + A''_\lambda \ln(t+i0) + \tilde{J}_\lambda(t)), & \lambda = -1, -2, \dots, \end{cases} \quad (3.2)$$

where  $\tilde{J}_\lambda(t)$  is smooth on  $\mathbb{R}$  and  $A_\lambda = (2\pi)^{-n} e^{\frac{i\pi}{2}(\lambda+1)} \Gamma(\lambda+1)$ ,

$$A'_\lambda = (2\pi)^{-n} \frac{(\psi(-\lambda) + i\frac{\pi}{2}) e^{-\frac{i\pi}{2}(\lambda+1)}}{(-\lambda-1)!}, \quad A''_\lambda = -(2\pi)^{-n} \frac{e^{-\frac{i\pi}{2}(\lambda+1)}}{(-\lambda-1)!},$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Moreover,

$$J_\lambda(t) = O(|t|^{-M}), \quad \text{as } t \rightarrow \infty, \quad \text{for every } M > 0.$$

We note that formula (3.2) gives the asymptotic behaviour of  $J_\lambda(t)$  as  $t \rightarrow 0$ .

### 3.4. $L_p$ -boundedness of a certain convolution operator with the oscillating kernel.

**Lemma 3.2 ([19, p. 392]).** *Let  $\psi(x)$  be a smooth function on  $\mathbb{R}^n$  with a compact support which vanishes in some neighbourhood of the origin. Then the operator*

$$(G_\lambda \varphi)(x) = \int_{\mathbb{R}^n} e^{i\lambda|x-y|} \psi(x-y) \varphi(y) dy, \quad \lambda > 0,$$

admits the estimate

$$\|G_\lambda \varphi\|_p \leq A \lambda^{-n/p'} \|\varphi\|_p,$$

where  $1 \leq p \leq \frac{2(n+1)}{n+3}$ .

## 4. SOME AUXILIARY STATEMENTS

**4.1.  $L_p \rightarrow L_q$ -estimates for some potential-type operators with oscillating kernels.** Here we study mapping properties of the operator (1.4) We first dwell on the model case of the operator

$$(S^\alpha \varphi)(x) = \int_{|t| \geq A} |t|^{\alpha-n} e^{i|t|} \varphi(x-t) dt, \quad 0 < \operatorname{Re} \alpha < n. \quad (4.1)$$

(its kernel is denoted by  $k^\alpha(|t|)$ ). The following theorem provides the  $L_p \rightarrow L_q$ -estimates for this operator (see also Pictures 1 and 2 for the cases  $(n-1)/2 < \operatorname{Re} \alpha < n$  and  $0 < \operatorname{Re} \alpha \leq (n-1)/2$ , respectively).

**Theorem 4.1.** *Let  $0 < \operatorname{Re} \alpha < n$ .*

I. *The imbedding*

$$\mathcal{L}(S^\alpha) \supset \mathcal{L}_1(\alpha, n) \quad (4.2)$$

*holds.*

II. *The set  $\mathcal{L}(S^\alpha)$  does not contain the points lying:*

- 1) *on the segment  $[A, H]$  and above it;*
- 2) *on the segment  $[A', H']$  and to the left of it;*
- 3) *above the straight line  $B'B$  in the case  $(n-1)/2 < \alpha < n$ ;*
- 4) *on the segment  $[O', O]$  if  $\alpha = (n-1)/2$ .*

*Proof.* To prove the statement I, we first establish the estimate

$$\|S^\alpha \varphi\|_q \leq C \|\varphi\|_p, \quad \varphi \in S, \quad (4.3)$$

with the constant  $C$  not depending on  $\varphi$ , where

$$(1/p, 1/q) \in \mathcal{L}_1(\alpha, n). \quad (4.4)$$

We decompose  $S^\alpha \varphi$  into

$$(S^\alpha \varphi)(x) = \sum_{\ell=0}^{\infty} (S_\ell^\alpha \varphi)(x), \quad x \in \mathbb{R}^n,$$

where

$$(S_\ell^\alpha \varphi)(x) = 2^{(\alpha-n)\ell} \int_{|y| \geq A} U_\alpha\left(\frac{|y|}{2^\ell}\right) e^{i|y|} \varphi(x-y) dy, \quad 0 < \operatorname{Re} \alpha < n.$$

The function  $U_\alpha(r)$ , supported on the segment  $[A/2, 2A]$ , is defined as follows:  $U_\alpha(r) = r^{\alpha-n}(\eta(r) - \eta(2r))$ , where  $\eta(r) \in C^\infty(0, \infty)$  is such that  $0 \leq \eta(r) \leq 1$ ,  $\eta(r) = 1$  if  $r \leq A$ , and  $\eta(r) = 0$  if  $r \geq 2$ . We have

$$\|S^\alpha \varphi\|_q \leq \sum_{\ell=0}^{\infty} \|S_\ell^\alpha \varphi\|_q. \quad (4.5)$$

To estimate the series on the right-hand side of (4.5), we need the following lemmas. (For the rest of the proof of Theorem 4.1, the same letter  $C$  will be used to denote various constants not depending on  $\ell$ , which are not necessarily the same at each occurrence).

**Lemma 4.1.** *Let  $k_\ell^\alpha(|t|)$  be the kernel of  $S_\ell^\alpha$ . The following estimates hold:*

$$1) \quad |\hat{k}_\ell^\alpha(|\xi|)| \leq C 2^{-M\ell} \begin{cases} 1 & \text{if } |\xi| \leq 1/2, \\ (1 + |\xi|)^{-M} & \text{if } |\xi| \geq 2, \end{cases} \quad (4.6)$$

for every  $M > 0$ ;

$$2) \quad |\hat{k}_\ell^\alpha(|\xi|)| \leq C 2^{(\operatorname{Re} \alpha - (n-1)/2)\ell} \quad \text{if } 1/2 < |\xi| < 2. \quad (4.7)$$

*Proof.* By the Bochner formula we have

$$\hat{k}_\ell^\alpha(|\xi|) = \frac{2^{(\alpha-n/2+1)\ell}(2\pi)^{n/2}}{|\xi|^{(n-2)/2}} \int_0^\infty \rho^{n/2} U_\alpha(\rho) e^{i2^\ell \rho} J_{\frac{n-2}{2}}(\rho|\xi|2^\ell) d\rho.$$

Let  $|\xi| \leq 1/2$ . Applying the formula (7) from [2, p. 92], after the interchange of the order of integration and integration by parts  $k = [\operatorname{Re} \alpha] + M + 1$  times in the inner integral, we arrive at the equality

$$\begin{aligned} \hat{k}_\ell^\alpha(|\xi|) &= 2^{\ell(\alpha-k)} \frac{(2\pi)^{(n-1)/2}}{\Gamma((n-1)/2)} \int_{-1}^1 \frac{(1-t^2)^{(n-3)/2} dt}{(i(1+|\xi|t))^k} \times \\ &\quad \times \int_0^\infty e^{i2^\ell \rho(1+|\xi|t)} \left(\frac{d}{d\rho}\right)^k (\rho^{n-1} U_\alpha(\rho)) d\rho, \end{aligned}$$

which yields (4.6).

Let  $|\xi| > 1/2$ . Making use of the formula (3.1) with  $m = 0$ , we have

$$\begin{aligned} \hat{k}_\ell^\alpha(|\xi|) &= 2^{(\alpha-(n-1)/2)\ell+(n+1)/2} \left( \frac{C_0^+}{|\xi|^{(n-1)/2}} I_{0,+}^{\alpha,\ell}(|\xi|) + \frac{C_0^-}{|\xi|^{(n-1)/2}} I_{0,-}^{\alpha,\ell}(|\xi|) + \right. \\ &\quad \left. + \frac{C_1^+}{|\xi|^{(n-1)/2}} I_{1,+}^{\alpha,\ell}(|\xi|) + \frac{C_1^-}{|\xi|^{(n-1)/2}} I_{1,-}^{\alpha,\ell}(|\xi|) \right), \end{aligned} \quad (4.8)$$

where

$$I_{0,\pm}^{\alpha,\ell}(|\xi|) = \int_0^\infty \rho^{(n-1)/2} U_\alpha(\rho) e^{i2^\ell \rho(1\pm|\xi|)} d\rho, \quad (4.9)$$

$$I_{1,\pm}^{\alpha,\ell}(|\xi|) = \int_0^\infty \rho^{(n-1)/2} U_\alpha(\rho) e^{i2^\ell \rho(1\pm|\xi|)} R_{0,\pm}^{(\frac{n-2}{2})} (2^\ell \rho|\xi|) d\rho. \quad (4.10)$$

From (4.8)–(4.9) we derive (4.7).

Integrating the integrals (4.9) and (4.10) by parts  $M + \lceil \operatorname{Re} \alpha - (n-1)/2 \rceil + 1$  times and making some evident estimations, we obtain (4.6) for  $|\xi| \geq 2$  (when evaluating the integral containing  $R_{0,\pm}^{(\frac{n-2}{2})}$ , we essentially use Remark 3.1).  $\square$

Together with Lemma 3.2, the following statement is a crucial point in the proof of statement I.

**Lemma 4.2.** *Let  $\ell \geq 1$ . Then*

$$\|S_\ell^\alpha \varphi\|_2 \leq C 2^{\ell(\operatorname{Re} \alpha - n/2)} \|\varphi\|_p, \quad \varphi \in S, \quad (4.11)$$

where  $1 \leq p \leq 2(n+1)/(n+3)$ .

*Proof.* By the Parseval equality we have

$$\begin{aligned} & \|S_\ell^\alpha \varphi\|_2^2 = \\ & = (2\pi)^{-n} \left( \int_{|\xi| \leq 1/2} + \int_{1/2 < |\xi| < 2} + \int_{|\xi| \geq 2} \right) |\hat{k}_\ell^\alpha(|\xi|)|^2 |\hat{\varphi}(\xi)|^2 d\xi \equiv J_1 + J_2 + J_3. \end{aligned}$$

With the aid of Lemma 4.1 we get

$$\begin{aligned} J_1 & \leq C 2^{\ell(\operatorname{Re} \alpha - n/2)} \int_{|\xi| \leq 1/2} |\hat{\varphi}(\xi)|^2 d\xi \leq C 2^{\ell(\operatorname{Re} \alpha - n/2)} \|\varphi\|_p, \\ J_3 & \leq C 2^{\ell(\operatorname{Re} \alpha - n/2)} \int_{|\xi| \geq 2} (1 + |\xi|)^{-[\operatorname{Re} \alpha - n/2]} |\hat{\varphi}(\xi)|^2 d\xi \leq \\ & \leq C 2^{\ell(\operatorname{Re} \alpha - n/2)} \|\varphi\|_p. \end{aligned}$$

To evaluate  $J_2$ , we make use of the following restriction theorem for the Fourier transform:

$$\int_{S^{n-1}} |\hat{\varphi}(\sigma)|^2 d\sigma \leq C \|\varphi\|_p^2, \quad \varphi \in S, \quad 1 \leq p \leq \frac{2(n+1)}{n+3} \quad (4.12)$$

(see [20] for  $1 \leq p < \frac{2(n+1)}{n+3}$  and [19, p. 386] in the case  $p = \frac{2(n+1)}{n+3}$ ). From (4.7) and (4.12) we have

$$\begin{aligned} J_2 & \leq C \int_{1/2}^2 \rho^{n-1} |\hat{k}_\ell^\alpha(\rho)|^2 d\rho \int_{S^{n-1}} |\hat{\varphi}(\rho\sigma)|^2 d\sigma \leq \\ & \leq C \|\varphi\|_p^2 \int_{1/2}^2 |\hat{k}_\ell^\alpha(\rho)|^2 d\rho \leq C 2^{2\ell(\operatorname{Re} \alpha - n/2)} \|\varphi\|_p^2. \end{aligned}$$

Gathering the above estimates, we arrive at (4.11).  $\square$

The next lemma provides  $L_p$ -estimates for the operator  $S^\alpha$ . It will be essentially used in the proof of (4.3) in the case  $\operatorname{Re} \alpha < (n-1)/2$ .

**Lemma 4.3.** *The estimate (4.4) with  $p = q$  is valid in the following cases:*

- 1)  $0 < \operatorname{Re} \alpha < n(n-1)/(2(n+1))$ ,  $n/(n - \operatorname{Re} \alpha) < p < n/\operatorname{Re} \alpha$ ;
- 2)  $n(n-1)/(2(n+1)) < \operatorname{Re} \alpha < (n-1)/2$ ,  
 $2(n-1)/(3(n-1) - 4\operatorname{Re} \alpha) < p < 2(n-1)/(4\operatorname{Re} \alpha - n + 1)$ .

*Proof.* Evidently,

$$\mathcal{L}(S_0^\alpha) = [O', O, E]. \quad (4.13)$$

Noting that

$$(S_\ell^\alpha \varphi)(x) = 2^{\alpha\ell} (G_{2^\ell} \varphi(2^\ell y)) \left( \frac{x}{2^\ell} \right), \quad \ell \geq 1,$$

by virtue of Lemma 3.2 we get

$$\|S_\ell^\alpha \varphi\|_p \leq C 2^{\ell(\operatorname{Re} \alpha - n/p')} \|\varphi\|_p, \quad \ell \geq 1, \quad (4.14)$$

where  $1 \leq p \leq 2(n+1)/(n+3)$ . Applying (4.14) to each summand on the right-hand side of (4.5), except for the first one, in view of (4.13), we obtain the statement of Lemma 4.3 in the case 1).

Consider the case 2). Taking  $p = 2(n+1)/(n+3)$  in (4.14), we have

$$\|S_\ell^\alpha \varphi\|_{\frac{2(n+1)}{n+3}} \leq C 2^{\ell(\operatorname{Re} \alpha - n(n-1)/(2(n+1)))} \|\varphi\|_{\frac{2(n+1)}{n+3}}. \quad (4.15)$$

Moreover,

$$\|S_\ell^\alpha \varphi\|_2 \leq C 2^{\ell(\operatorname{Re} \alpha - (n-1)/2)} \|\varphi\|_2 \quad (4.16)$$

by Lemma 4.1. Interpolating between (4.15) and (4.16), we obtain

$$\|S_\ell^\alpha \varphi\|_p \leq C 2^{\ell(\operatorname{Re} \alpha + (n-1)/(2p) - 3(n-1)/4)} \|\varphi\|_p, \quad \frac{2(n+1)}{n+3} \leq p \leq 2. \quad (4.17)$$

From (4.5), (4.13), and (4.17) we deduce the statement of lemma in the case 2).  $\square$

**Lemma 4.4.** *In some neighbourhood of  $S^{n-1}$  the symbol  $\hat{k}^\alpha(|\xi|)$  of the operator  $S^\alpha$  can be represented as follows:*

$$\begin{aligned} \hat{k}^\alpha(|\xi|) &= C_\alpha (1 - |\xi| + i0)^{\frac{n-1}{2} - \alpha} + u(|\xi|) \quad \text{if } \alpha - \frac{n+1}{2} \neq -1, -2, \dots, \\ \hat{k}^\alpha(|\xi|) &= (1 - |\xi|)^{\frac{n-1}{2} - \alpha} (C'_\alpha + C''_\alpha \ln(1 - |\xi| + i0)) + v(|\xi|) \\ &\quad \text{if } \alpha - \frac{n+1}{2} = -1, -2, \dots, \end{aligned}$$

where  $u(|\xi|)(v(|\xi|)) = o(|1 - |\xi||^{\frac{n-1}{2} - \alpha})$ , as  $|\xi| \rightarrow 1$ ,

$$\begin{aligned} C_\alpha &= (2\pi)^{-n} e^{\frac{i\pi}{2}(\alpha - \frac{n-1}{2})} \Gamma\left(\alpha - \frac{n-1}{2}\right), \\ C'_\alpha &= (2\pi)^{-n} \frac{\psi(\frac{n+1}{2} - \alpha) + \frac{i\pi}{2}}{(\frac{n-1}{2} - \alpha)!} e^{\frac{i\pi}{2}(\frac{n-1}{2} - \alpha)}, \\ C''_\alpha &= -(2\pi)^{-n} \frac{e^{\frac{i\pi}{2}(\frac{n-1}{2} - \alpha)}}{(\frac{n-1}{2} - \alpha)!}. \end{aligned}$$

Moreover,  $\hat{k}^\alpha(|\xi|)$  is bounded outside of the mentioned neighbourhood.

The statement of lemma is derived from Lemma 3.1. It will be used in order to get information about boundedness of the operator  $S^\alpha$  in  $L_2$ .

Returning to the proof of (4.3) under assumption (4.4), we note that  $k^\alpha(|t|) \in L_q$  if  $0 \leq 1/q < 1 - \operatorname{Re} \alpha/n$  and the statement of Sobolev theorem is

valid for the operator  $S^\alpha$ . From here by the convexity and duality arguments we deduce (4.3) for the points  $(1/p, 1/q)$ ,

$$(1/p, 1/q) \in [A', A, E] \setminus (\{A\} \cup \{A'\}). \quad (4.18)$$

Proceeding, we first consider the case  $n/2 \leq \operatorname{Re} \alpha < n$  ( $n \geq 2$ ). Taking into account inequality (4.11) and the evident estimate

$$\|S_\ell^\alpha \varphi\|_\infty \leq C 2^{\ell(\operatorname{Re} \alpha - n)} \|\varphi\|_1,$$

by virtue of the Riesz–Thorin theorem we get

$$\|S_\ell^\alpha \varphi\|_q \leq C 2^{\ell(t(\operatorname{Re} \alpha - n/2) + (1-t)(\operatorname{Re} \alpha - n))} \|\varphi\|_p, \quad (4.19)$$

where  $1/p = 1 - t/p'_0$ ,  $p_0 \in \left[1, \frac{2(n+1)}{n+3}\right]$ ,  $1/q = t/2$ ,  $t \in [0, 1]$ . The exponent  $\sigma = tn/2 + \operatorname{Re} \alpha - n$  is negative if  $t < 2(n - \operatorname{Re} \alpha)/n$ , which implies  $1/p > 1 - (n - \operatorname{Re} \alpha)(n - 1)/(n(n + 1))$  and  $1/q < 1 - \operatorname{Re} \alpha/n$ . Applying (4.19) to each summand on the right-hand side of (4.5), except for the first one, in view of (4.13), we obtain (4.3) if

$$(1/p, 1/q) \in (B, E]. \quad (4.20)$$

Let next  $n/2 > \operatorname{Re} \alpha > (n - 1)/2$ . The interpolation between (4.16) and (4.14) with  $p = \frac{2(n+1)}{n+3}$  yields

$$\|S_\ell^\alpha \varphi\|_q \leq C 2^{\ell(t(n(1/p_0 - 1/2) + \operatorname{Re} \alpha - (n+1)/2) + (1-t)(\operatorname{Re} \alpha - n))} \|\varphi\|_p, \quad (4.21)$$

$$\ell \geq 1,$$

where  $t \in [0, 1]$ ,  $1/p = t/p_0 + 1 - t$ ,  $1/q = t/p_0$ ,  $p_0 = 2(n + 1)/(n + 3)$ . Applying (4.21) to the right-hand side of (4.5), we obtain (4.3) if

$$(1/p, 1/q) \in (G, E]. \quad (4.22)$$

Making use of (4.16) and (4.19) with  $t = 1$  and interpolating, we get

$$\|S_\ell^\alpha \varphi\|_2 \leq C 2^{\ell(t(\operatorname{Re} \alpha - (n-1)/2) + (1-t)(\operatorname{Re} \alpha - n/2))} \|\varphi\|_p, \quad (4.23)$$

where  $t \in [0, 1]$ ,  $1/p = t/2 + (1 - t)(n + 3)/(2(n + 1))$ . Further application of (4.23) to the right-hand side of (4.5) yields (4.3) if

$$(1/p, 1/q) \in (K, E]. \quad (4.24)$$

Now we are in a position to carry out the final interpolation in the case  $(n - 1)/2 < \operatorname{Re} \alpha < n$ . Interpolating between the points of the set  $[A', A, E] \setminus (\{A'\} \cup \{A\})$  and the points  $(1/p, 1/q)$  satisfying (4.20) and (4.22), (4.24) in the cases  $n/2 \leq \operatorname{Re} \alpha < n$  and  $(n - 1)/2 < \operatorname{Re} \alpha < n/2$ , respectively, we arrive at (4.3) under the assumption (4.4), except for the case  $n = 2$  and  $1/2 < \alpha \leq 1$ .

Let now  $0 < \operatorname{Re} \alpha < (n-1)/2$ . In the case  $n(n-1)/(2(n+1)) < \operatorname{Re} \alpha < (n-1)/2$  the application of Lemma 4.3 (statement 2)) yields (4.3) if

$$(1/p, 1/q) \in (C', C). \quad (4.25)$$

Moreover, as in the case  $(n-1)/2 < \operatorname{Re} \alpha < n/2$  considered above, we also have (4.3) if the point  $(1/p, 1/q)$  satisfies (4.22).

In view of Lemma 4.3 (statement 1)) inequality (4.3) holds in the case  $0 < \operatorname{Re} \alpha \leq n(n-1)/(2(n+1))$  if

$$(1/p, 1/q) \in (H', H). \quad (4.26)$$

Owing to (4.18), (4.22), (4.25), and (4.26), the interpolation similar to that carried out in the case  $(n-1)/2 < \operatorname{Re} \alpha < n$ , also yields (4.3) under assumption (4.4), except for the case when  $n = 2$  and  $1/3 < \alpha < 1/2$ .

If  $\operatorname{Re} \alpha = (n-1)/2$  and  $\operatorname{Im} \alpha \neq 0$  ( $n \geq 2$ ), then  $S^\alpha$  is bounded in  $L_2$ , since its symbol is a 2-multiplier in accordance with Lemma 4.4.

Therefore, altogether we have the statement I, except for the case when  $n = 2$  and  $1/3 < \alpha \leq 1$ ,  $\alpha \neq 1/2$ . Direct analysis of the proof of the main result from [3] shows the validity of (4.3) in the mentioned case as well.

Thus we have proved (4.3) under the condition (4.4). This enables us to extend the operator  $S^\alpha$  to the whole space  $L_p$  to a bounded operator from  $L_p$  into  $L_q$  if this condition is fulfilled. As is easily verified, this extension coincides with the right-hand side of (4.1).

Statement I of Theorem 4.1 has been proved completely.

Turning to the proof of statement II, item 1), we construct the counter-example involving the function  $\chi_{1/10}(y)$ ;  $\chi_{1/10} \in L_p$ ,  $1 \leq p \leq \infty$ . It suffices to prove that  $(S^\alpha \chi_{1/10})(x) \notin L_{n/(n-\operatorname{Re} \alpha)}$ . We have

$$\begin{aligned} (S^\alpha \chi_{1/10})(x) &= \int_{|t| < 1/10} \frac{e^{i|x-t|}}{|x-t|^{n-\alpha}} dt - \\ &- \int_{|t| < 1/10} \frac{(1 - \chi_1(|x-t|))e^{i|x-t|}}{|x-t|^{n-\alpha}} dt. \end{aligned} \quad (4.27)$$

Evidently, the second summand on the right-hand side of (4.27) belongs to  $L_{n/(n-\operatorname{Re} \alpha)}$ .

We wish to prove that  $J(x) \equiv \int_{|t| < 1/10} \frac{e^{i|x-t|}}{|x-t|^{n-\alpha}} dt \notin L_{n/(n-\operatorname{Re} \alpha)}$  and thus come to the desired conclusion. We have

$$|J(x)| \geq \left| \int_{|t| < 1/10} \frac{dt}{|x-t|^{n-\alpha}} \right| - \int_{|x| < 1/10} \frac{|t| dt}{|x-t|^{n-\operatorname{Re} \alpha}}.$$

Let us evaluate the right-hand side of this inequality from below as  $|x| \rightarrow \infty$ . After the change of variable  $y = |x|\omega_x(\tau)$ , where  $\omega_x(\tau)$  is a rotation in

$\mathbb{R}^n$  such that  $x' = \omega_x(e_1)$ ,  $e_1 = (1, 0, \dots, 0)$ , we get

$$\begin{aligned} & \left| \int_{|t| < 1/10} \frac{dt}{|x-t|^{n-\alpha}} \right| \geq \\ & \geq |x|^{\operatorname{Re} \alpha} \int_{|\tau| < 1/(10|x|)} \frac{\cos(\operatorname{Im} \alpha \ln |\tau - e_1|) d\tau}{|\tau - e_1|^{n-\operatorname{Re} \alpha}} \geq \\ & \geq A|x|^{\operatorname{Re} \alpha} \int_{|\tau| < 1/(10|x|)} \frac{d\tau}{|\tau - e_1|^{n-\operatorname{Re} \alpha}}, \end{aligned} \quad (4.28)$$

where the constant  $A$  is close to 1.

Besides this,

$$\int_{|t| < 1/10} \frac{|t|}{|x-t|^{n-\operatorname{Re} \alpha}} dt \leq \frac{|x|^{\operatorname{Re} \alpha}}{10} \int_{|\tau| < 1/(10|x|)} \frac{d\tau}{|\tau - e_1|^{n-\operatorname{Re} \alpha}}. \quad (4.29)$$

From (4.28) and (4.29) we derive that

$$\begin{aligned} |J(x)| & \geq |x|^{\operatorname{Re} \alpha} \left( A - \frac{1}{10} \right) \int_{|\tau| < 1/(10|x|)} \frac{d\tau}{|\tau - e_1|^{n-\operatorname{Re} \alpha}} \geq \\ & \geq \frac{C}{|x|^{n-\operatorname{Re} \alpha}} \notin L_{n/(n-\operatorname{Re} \alpha)}. \end{aligned}$$

Statement 2) follows from 1) by duality.

To prove 3), we have to use the counter-example constructed in [3, p. 231–232].

Statement 4) follows from Lemma 4.4.  $\square$

*Remark 4.1.* We note that

$$\{D\} \in \mathcal{L}(S^\alpha)$$

in the case  $(n-1)/2 < \alpha < n$  (see [10], [11]). Then

$$\mathcal{L}(S^\alpha) \supset \mathcal{L}_1^*(\alpha, n).$$

The following observation is essentially used in the passage to the general case of the operator (1.4).

*Remark 4.2.* We observe that for every  $\alpha$ ,  $1 < \operatorname{Re} \alpha < n$ , there exists a neighbourhood  $\Omega(\alpha, n)$  of the set  $\mathcal{L}(S^\alpha)$  such that

$$\Omega(\alpha, n) \cap [O', O, E] \subset \mathcal{L}_1^*(\alpha - 1, n).$$

This is a consequence of the definition of  $\mathcal{L}_1^*(\alpha, n)$  and the statements I and II (items 1) and 2)) of Theorem 4.1.

*Remark 4.3.* We note that in the case of real  $\alpha$ ,  $0 \leq \alpha < (n-1)/2$  and  $n/2 \leq \alpha < n$ , the statement I of Theorem 4.1 was partially proved by the authors jointly with E. E. Urnysheva.

Passing to the case of the operator (1.4), we assume that the characteristic  $a(r)$  is such that the function  $a^*(r) = a(r^{-1})$ ,  $0 < r < A^{-1}$ ,  $a^*(r) = \lim_{r \rightarrow 0} a(r^{-1})$ , is continuously differentiable up to the order  $[\operatorname{Re} \alpha] + 2$  on the interval  $[0, A^{-1})$  and  $a^*(0) \neq 0$ . The following theorem describes mapping properties of the operator  $S_a^\alpha$ . It plays a crucial role in the proof of Theorem 2.1. It also answers a natural question: whether the  $\mathcal{L}$ -characteristics  $\mathcal{L}(S_a^\alpha)$  and  $\mathcal{L}(S^\alpha)$  coincide or not?

**Theorem 4.2.** *Let  $0 < \operatorname{Re} \alpha < n$ . Then*

$$\mathcal{L}(S_a^\alpha) = \mathcal{L}(S^\alpha) \quad (4.30)$$

(this means that the statements of Theorem 4.1 are valid for the operator  $S_a^\alpha$  as well).

*Proof.* Let first  $\operatorname{Re} \alpha \neq 1, 2, \dots$ . We base ourselves on the following equality, which is obtained by the application of Taylor's formula to  $a^*(r)$ ,  $r \in [0, A^{-1})$ :

$$(S_a^\alpha \varphi)(x) = \sum_{k=0}^{m-1} \frac{(a^*)^{(k)}(0)}{k!} (S^{\alpha-k} \varphi)(x) + (T_m^\alpha \varphi)(x), \quad m = [\operatorname{Re} \alpha] + 1, \quad (4.31)$$

where

$$(T_m^\alpha \varphi)(x) = \int_{|y| \geq A} \frac{e^{i|y|} R_m(|y|^{-1})}{|y|^{n-\alpha}} \varphi(x-y) dy,$$

$$R_m(r^{-1}) = \frac{r^{-m}}{(m-1)!} \int_0^1 (1-u)^{m-1} (a^*)^{(m)}(ur^{-1}) du.$$

Since the kernel of  $T_m^\alpha$  belongs to  $L_1 \cap L_\infty$ , we have

$$\mathcal{L}(T_m^\alpha) = [O', O, E]. \quad (4.32)$$

Besides this, the following imbedding holds:

$$\mathcal{L}^*(\alpha, n) \subset \mathcal{L}^*(\alpha-1, n), \quad 1 < \operatorname{Re} \alpha < n, \quad (4.33)$$

which follows directly from the definition of the set  $\mathcal{L}_1^*(\alpha, n)$  (see Section 2). Now (4.2) and (4.33) imply the imbedding

$$\mathcal{L}(S^{\alpha-k}) \supset \mathcal{L}(\alpha, n), \quad k = 0, 1, \dots, m-1. \quad (4.34)$$

Then equality (4.30) follows from (4.31), (4.32), and (4.34) in view of Remark 4.2.

The case  $\operatorname{Re} \alpha = \ell$ ,  $\ell = 1, 2, \dots, n-1$ , is considered similarly on the base of equality

$$(S_a^\alpha \varphi)(x) = \sum_{k=0}^{\ell-1} \frac{(a^*)^{(k)}(0)}{k!} (S^{\alpha-k} \varphi)(x) + \frac{(a^*)^{(\ell)}(0)}{\ell!} (R_0^\alpha \varphi)(x) + (T_{\ell+1}^\alpha \varphi)(x),$$

where

$$(R_0^\alpha \varphi)(x) = \int_{|y| \geq A} \frac{e^{iy|}}{|y|^{n-i \operatorname{Im} \alpha}} \varphi(x-y) dy.$$

We only have to note that

$$\mathcal{L}(R_0^\alpha) = [O', O, E] \setminus (\{O'\} \cup \{O\}). \quad (4.35)$$

Indeed, since  $[O', O, E] \setminus [O', O] \subset \mathcal{L}(R_0^\alpha)$  and  $\{O'\}, \{O\} \notin \mathcal{L}(R_0^\alpha)$  (the relation  $\{O'\} \notin \mathcal{L}(R_0^\alpha)$  is evident, then  $\{O\} \notin \mathcal{L}(R_0^\alpha)$  by duality), equality (4.35) will follow from the estimate

$$\|R_0^\alpha \varphi\|_p \leq C \|\varphi\|_p, \quad \varphi \in S, \quad 1 < p < \infty, \quad (4.36)$$

with the constant  $C$  not depending on  $\varphi$ .

To prove (4.36), we split  $(R_0^\alpha \varphi)(x)$  into

$$(R_0^\alpha \varphi)(x) = \sum_{j=0}^{\infty} (R_{0,j}^\alpha \varphi)(x), \quad (4.37)$$

where

$$(R_{0,j}^\alpha \varphi)(x) = 2^{j(i \operatorname{Im} \alpha - n)} \int_{|y| \geq A} U_\alpha \left( \frac{|y|}{2^j} \right) e^{iy|} \varphi(x-y) dy.$$

Evidently,

$$\mathcal{L}(R_{0,0}^\alpha) = [O', O, E]. \quad (4.38)$$

Besides this, the following estimate is valid:

$$\|R_{0,j}^\alpha \varphi\|_p \leq C 2^{-jn/p'} \|\varphi\|_p, \quad j \geq 1, \quad (4.39)$$

which is proved in just the same way as (4.14). Then (4.38) and (4.39) yield (4.36) by (4.37).  $\square$

**4.2. On the  $\mathcal{L}$ -characteristic of the operator with the acoustic kernel truncated at infinity.** In order to prove Theorem 2.1, we need the following result for the operator

$$(A^{\gamma,0}\varphi)(x) = \int_{|y|\leq A} h_\gamma(|y|)\varphi(x-y)dy, \quad 0 < \operatorname{Re} \gamma < n+1, \quad (4.40)$$

where  $h_\gamma(|y|)$  was defined in (1.3).

**Theorem 4.3.** *Let  $0 < \operatorname{Re} \gamma < n+1$ , then*

$$\mathcal{L}(A^{\gamma,0}) = \mathcal{L}_2(\gamma, n) \quad \text{if } 0 < \operatorname{Re} \gamma < n,$$

$$\mathcal{L}(A^{\gamma,0}) = [O', O, E] \setminus \{E\} \quad \text{if } \gamma = n,$$

$$\mathcal{L}(A^{\gamma,0}) = [O', O, E] \quad \text{if } \operatorname{Re} \gamma = n \text{ and } \operatorname{Im} \gamma \neq 0 \text{ or } n < \operatorname{Re} \gamma < n+1.$$

The statement of theorem is easily derived from the well-known asymptotic relations for the function  $H_\nu^{(1)}(z)$ ,  $z \rightarrow 0$  (see, for instance, [1, p. 180]).

**4.3. On  $L_p \rightarrow L_{q_1} + L_{q_2}$ -estimates for the operator  $A^\gamma$ .** The following theorem is essentially used in the proof of Theorem 2.2.

**Theorem 4.4.** *Let  $0 < \operatorname{Re} \gamma < n+1$ ,  $1 \leq p < 2n/(n + \operatorname{Re} \gamma - 1)$ . Then the operator  $A^\gamma$  is bounded from  $L_p$  into  $L_{q_1} + L_{q_2}$ , where  $q_1$  and  $q_2$  are such that*

- 1)  $(1/p, 1/q_1) \in \mathcal{L}_1^*\left(\frac{\gamma+n-1}{2}, n\right)$  and  $(1/p, 1/q_2) \in \mathcal{L}_2(\gamma, n)$   
if  $0 < \operatorname{Re} \gamma < n$ ;
- 2)  $(1/p, 1/q) \in \mathcal{L}_1^*\left(\frac{\gamma+n-1}{2}, n\right)$  and  $p \leq q_2 \leq \infty$   
if  $n < \operatorname{Re} \gamma < n+1$  or  $\operatorname{Re} \gamma = n$  and  $\operatorname{Im} \gamma \neq 0$ ;
- 3) the conditions of item 2) are fulfilled, except for the case  $p = 1$ , in which we assume that  $1 \leq q_2 < \infty$ , if  $\gamma = n$ .

*Proof.* In view of the equality

$$h_\gamma(|y|) = \chi_A(|y|)h_\gamma(|y|) - (1 - \chi_A(|y|))h_\gamma(|y|) \equiv h_{\gamma,0}(|y|) + h_{\gamma,\infty}(|y|),$$

we represent  $A^\gamma\varphi$  as

$$\begin{aligned} (A^\gamma\varphi)(x) &= (h_{\gamma,0} * \varphi)(x) + (h_{\gamma,\infty} * \varphi)(x) \equiv \\ &\equiv (A^{\gamma,0}\varphi)(x) + (A^{\gamma,\infty}\varphi)(x). \end{aligned} \quad (4.41)$$

Making use of the following integral representation

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{i(z-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-t} t^{\nu-1/2} \left(1 + \frac{it}{2z}\right)^{\nu-1/2} dt$$

(see [6, p. 165]), we obtain

$$h_{\gamma,\infty}(|y|) = \zeta_{n,\gamma} |y|^{\frac{\gamma-n-1}{2}} e^{i|y|} a(|y|), \quad (4.42)$$

where

$$a(|y|) = (1 - \chi_A(|y|)) \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{i\pi(\gamma-n-1)}{2}}}{\gamma(\frac{n-\gamma+1}{2})} \times \\ \times \int_0^\infty e^{-t} t^{\frac{n-\gamma-1}{2}} \left(1 + \frac{it}{2|y|}\right)^{\frac{n-\gamma-1}{2}} dt. \quad (4.43)$$

Since the characteristic (4.43) satisfies the conditions of Theorem 4.2, we have

$$\mathcal{L}(A^{\gamma,\infty}) = \mathcal{L}(S_a^{\frac{n+\gamma-1}{2}}). \quad (4.44)$$

Now, to prove Theorem 4.3, we only have to refer to Theorems 4.2 and 4.3 applicable to the operators  $A^{\gamma,\infty}$  and  $A^{\gamma,0}$ , respectively.  $\square$

*Remark 4.4.* If  $1/p > 1/p_0(\gamma)$ , then, in accordance with Theorem 4.4, there exist such numbers  $q_1$  and  $q_2$  that  $2n/(n+1) < q'_1, q'_2 \leq \infty$  and the operator  $A^\gamma$  is bounded from  $L_p$  into  $L_{q_1} + L_{q_2}$ .

## 5. PROOF OF THE MAIN RESULTS

**5.1.** *Proof of Theorem 2.1.* The statements of Theorem 2.1 follow from (4.41) in view of (4.44) and Theorems 4.2 and 4.3.  $\square$

**5.2.** *Proof of Theorem 2.2.* The imbedding

$$A^\gamma(L_p) \subset \{f \in L_{q_1} + L_{q_2} : T^\gamma f \in L_p\} \quad (5.1)$$

follows from Remark 4.4 and the above result on the inversion of potentials  $f = A^\gamma \varphi$ ,  $\varphi \in L_p$ , via AIO (see Section 2). We prove the imbedding inverse to (5.1).

Let  $f = f_1 + f_2$ , where  $f_1 \in L_{q_1}$ ,  $f_2 \in L_{q_2}$  ( $q_1$  and  $q_2$  were described in the formulation of Theorem 2.2), and  $T^\gamma f \in L_p$ . For  $\omega \in S$  we have:

$$\langle A^\gamma T^\gamma f, \omega \rangle = \langle T^\gamma f, \overline{A^\gamma \omega} \rangle = \left\langle \lim_{\varepsilon \rightarrow 0}^{(L_{p,\mu})} T_\varepsilon^\gamma f, \overline{A^\gamma \omega} \right\rangle = \lim_{\varepsilon \rightarrow 0} \langle T_\varepsilon^\gamma f, \overline{A^\gamma \omega} \rangle = \\ = \lim_{\varepsilon \rightarrow 0} \langle f, \overline{T_\varepsilon^\gamma A^\gamma \omega} \rangle = \lim_{\varepsilon \rightarrow 0} (\langle f_1, \overline{T_\varepsilon^\gamma A^\gamma \omega} \rangle + \langle f_2, \overline{T_\varepsilon^\gamma A^\gamma \omega} \rangle). \quad (5.2)$$

The third of equalities (5.2) follows from the fact that convergence in the  $L_{p,\mu}$ -norm implies that in  $S'$ .

Taking into account the equality

$$\begin{aligned} (\overline{T_\varepsilon^\gamma A^\gamma \omega})(x) &= (W_\varepsilon \omega)(x) + \\ &+ \sum_{k=1}^{\ell} (-\varepsilon(-2i + \varepsilon))^k (\varepsilon - i)^{\ell-2k} (G_{2k}((\varepsilon - i)|t|) * W_\varepsilon \omega)(x), \quad \omega \in S, \end{aligned}$$

where  $G_{2k}(z|t) = \frac{2^{(2-n-2k)/2} K_{n/2-k}(z|t)}{\pi^{n/2} \Gamma(k)(z|t)^{n/2-k}}$ ,  $z = \varepsilon + i$  and  $K_\nu(z)$  is the McDonald function (see [17, Subsection 11]), we obtain

$$\begin{aligned} \langle f_j, \overline{T_\varepsilon^\gamma} A^\gamma \omega \rangle &= \langle f_j, W_\varepsilon \omega \rangle + \\ &+ \sum_{k=1}^{\ell} C_\ell^k \langle f_j, (\varepsilon(2i - \varepsilon))^k (\varepsilon - i)^{\ell-2k} (G_{2k}((\varepsilon - i)|t) * W_\varepsilon \omega) \rangle, \end{aligned} \quad (5.3)$$

$j = 1, 2$ .

Further we have

$$\begin{aligned} (I_{\varepsilon,k} \omega)(x) &\equiv (\varepsilon(2i - \varepsilon))^k (\varepsilon - i)^{\ell-2k} (G_{2k}((\varepsilon - i)|t) * W_\varepsilon \omega)(x) = \\ &= (\varepsilon(2i - \varepsilon))^k (\varepsilon - i)^{\ell-2k} \left( \int_{|t| < \frac{1}{\sqrt{1+|\varepsilon|^2}}} + \int_{|t| > \frac{1}{\sqrt{1+|\varepsilon|^2}}} \right) \times \\ &\times G_{2k}((\varepsilon - i)|t) * (W_\varepsilon \omega)(x - t) dt = \\ &= (I_{\varepsilon,k}^1 \omega)(x) + (I_{\varepsilon,k}^2 \omega)(x), \quad k = 1, 2, \dots, \ell. \end{aligned}$$

We make use of the equalities

$$\lim_{\varepsilon \rightarrow 0}^{(L_s)} \overline{(I_{\varepsilon,k}^1)}(x) = 0, \quad 1 \leq s \leq \infty, \quad (5.4)$$

$$\lim_{\varepsilon \rightarrow 0}^{(L_r)} \overline{(I_{\varepsilon,k}^2)}(x) = 0, \quad 2n/(n+1) < r \leq \infty \quad (5.5)$$

(see [17, Subsection 11]). Applying the Hölder inequality, we obtain

$$\begin{aligned} |\langle f_j, I_{\varepsilon,k}^1 \omega \rangle| &\leq \|f_j\|_{q'_j} \|I_{\varepsilon,k}^1 \omega\|_{q'_j}, \quad j = 1, 2, \\ |\langle f_j, I_{\varepsilon,k}^2 \omega \rangle| &\leq \|f_j\|_{q'_j} \|I_{\varepsilon,k}^2 \omega\|_{q'_j}, \quad j = 1, 2, \end{aligned}$$

where  $2n/(n+1) < q'_j \leq \infty$ . From here we derive that

$$\lim_{\varepsilon \rightarrow 0} \langle f_j, I_{\varepsilon,k} \omega \rangle = 0, \quad j = 1, 2; \quad k = 1, 2, \dots, \ell, \quad (5.6)$$

by virtue of (5.4) and (5.5).

Since  $\langle f_j, W_\varepsilon \omega \rangle \rightarrow \langle f_j, \omega \rangle$ , as  $\varepsilon \rightarrow 0$ , in view of (5.2), (5.3), and (5.6), we obtain the equality

$$\langle A^\gamma T^\gamma f, \omega \rangle = \langle f, \omega \rangle, \quad \omega \in S,$$

which yields  $f(x) = (A^\gamma T^\gamma f)(x)$  for almost all  $x \in \mathbb{R}^n$ . This means that

$$A^\gamma(L_p) \supset \{f : f \in L_{q_1} + L_{q_2}, T^\gamma f \in L_p\}. \quad (5.7)$$

Now equality (2.4) follows from (5.1) and (5.7).  $\square$

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